# Finite quotients of generic free-by-cyclic groups

01 June 2023 16:19 j.w/ Monika Kudlicka

**Def$^n$:** $G$ a f.g r.f. group

$\mathcal{F}_G := \{G/N \mid N \trianglelefteq_f G\}/\cong$

**Def$^n$:** $\hat{G} \cong \varprojlim_{N \trianglelefteq_f G} G/N$

e.g.: $\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$ functorial so $\psi: H \to G$ gives $\hat{\psi}: \hat{H} \to \hat{G}$

**Thm** [Dixon–Formanek–Poland–Ritter '82]: $G$ f.g. r.f. Then $\hat{G}$ is determined up to iso by $\mathcal{F}_G$.

**Thm:** If $\hat{G} \cong \hat{H}$, then the lattice of f.i. subgroups of $G$ and $H$ are iso. (including indices and normality)

Let $M$ be a 3-mnfld possibly with bndry. Let $G = \pi_1 M$.
The following props are determined by $\hat{G}$ amongst 3-mnflds:

- [Wilkes, Wilton–Zalesskii]
  - JSJ decomposition
- [Wilton–Zalesskii]:
  - Thurston geometry

Let $\mathcal{C}$ be a class of groups.
A f.g r.f. group $G$ is <u>almost</u> profinitely rigid amongst $\mathcal{C}$ if there are at most finitely many f.g. r.f. $H \in \mathcal{C}$ s.t.

- $\mathcal{F}_H = \mathcal{F}_G$ (or $\hat{H} \cong \hat{G}$)
- $H \ncong G$.

Back to 3-mnflds:

- [Wilkes]: If $G$ is Seifert fibred then $G$ is a.p.r amongst 3-mnfld groups
- [Wilkes]: $G = \pi_1(\Sigma \times_{\varphi} S^1)$, then $G$ is a.p.r amongst 3-mnflds
- [Liu '23]: If $G$ is $\pi_1(M^3_{type})$ then $G$ is a.p.r. amongst 3-mnfld groups.

Right board

<u>How to prove Liu (Sketch):</u>

- Apply Agol, Wise to obtain a fibered cover
  $\Sigma \to M^3 \xrightarrow{\phi} S^1$
- Show $\hat{\pi_1} M^3$ detects the Thurston polytope:
  ↳ Depends on hard theorems of [Gruel–Wilton '11] and [Bachin–Papini '17]
- Suppose $\hat{H} \cong \hat{G}$
- Get $\hat{\phi}$ fibered characters

$$\begin{tikzcd} & \tilde{G} \arrow[d] & \tilde{H} \arrow[d] \\ G \arrow[ur, "\phi"] \arrow[r] & \mathbb{Z} \arrow[r, "\cong"] & \hat{\mathbb{Z}} \end{tikzcd}$$

we do not have [F.–V.]

- Show $\ker \hat{\psi} \cong \ker \hat{\varphi}$
- Show <u>stretch factors</u> of $\psi$ and $\varphi$ are equal.
  translation distance $\lambda(\psi), \lambda(\varphi)$ of $\psi, \varphi$ on Teichmüller Space when viewed as mapping classes
  ↳ This uses some serious homological algebra
  This is a problem as dynamics of $\psi, \varphi$ are different for us. Metric on $CV_n$ is not symm.
- Lemma: $\exists$ finitely many p.s.A. on $\Sigma_{g,b}$ with stretch factor at most $C \geq 0$

We need fully irr. to be true as no Nielsen–Thurston decomposition $\square$

Right board

**Def$^n$:** $G$ is free-by-cyclic if $\exists$ a s.e.s $1 \to F_n \to G \xrightarrow{\varphi} \mathbb{Z} \to 1$

**Thm** [H.–K.]: Let $G = F_n \rtimes_\varphi \mathbb{Z}$. If $\varphi$ is irr. and $b_1(G) = 1$ then $G$ is a.p.r amongst irr. $F_n$-by-$\mathbb{Z}$s.

How to show stretch factors are equal

Step 1: Show twisted Alexander polynomials are equal. $\Delta^{\psi,\rho}_n(t) = \Delta^{\varphi,\rho}_n(t)$
↳ Uses work of Ueki
These are "rough" characteristic polynomials of $\psi_*: H_n(\ker\hat{\psi}; \mathbb{C}[t,t^{-1}]) \to H_n(\ker\hat{\psi}; \mathbb{C}[t,t^{-1}])$

Step 2: Show twisted Reidemeister torsions are equal $\tau^{\psi,\rho}_n(t) = \tau^{\varphi,\rho}_n(t)$

- Uses $\tau^{\psi,\rho}(t) \doteq \prod_{i \geq 0} \Delta^{\psi,\rho}_i(t)^{(-1)^i}$

$Nm(\psi) \leq \# \mathrm{Fix}(\hat{\psi})$

Step 3: <u>Nielsen numbers</u> are equal

- These measure the fixed point indices of the homeo $f: \Sigma \to \Sigma$ — these are constant on orbits on $\Sigma$ and hence on conj classes of $G$.
- Relate them to class functions on finite quotients (sums of irr. characters) and the Reidemeister torsion.
- Uses work of [Jiang '96]
  + Conjugacy Separability [Minasyan–Zalesskii]

Step 4: $\lambda(\psi) = \limsup_{m \to \infty} Nm(\psi)^{1/m}$

Hence $\lambda(\psi) = \lambda(\varphi)$ ◈

In our case we get
$\Delta^{\psi,\rho}_n(t) \cdot \Delta^{\psi,\rho}_n(t^{-1}) \doteq \Delta^{\varphi,\rho}_n(t) \cdot \Delta^{\varphi,\rho}_n(t^{-1})$

In our case we get a pair of invariants
$\{\tau^{\psi,\rho}(t), \tau^{\psi,\rho}(t^{-1})\} \doteq \{\tau^{\varphi,\rho}(t), \tau^{\varphi,\rho}(t^{-1})\}$
Note we had to calculate $\Delta^{\psi,\rho}_0(t)$ for all f.a. gen groups!!

This step is a fight for us as we get a pair of numbers
$\{Nm(\psi), Nm(\psi^{-1})\} = \{Nm(\varphi), Nm(\varphi^{-1})\}$

After much work have to solve
$\begin{cases} Nm(\psi) + Nm(\psi^{-1}) = Nm(\varphi) + Nm(\varphi^{-1}) \\ Nm(\psi) Nm(\psi)^{-1} = Nm(\varphi) Nm(\varphi^{-1}) \end{cases}$

Here we use Hagen–Wise calculation

This works ✓ :)

---

**Thm 2** [H.–Kudlicka]: Let $G = F_n \rtimes_\psi \mathbb{Z}$, $\psi$ generic. Then a.a.s $G$ is a.p.r. amongst $F_n$-by-$\mathbb{Z}$ groups.

(let $\mathrm{Out}(F_n) = \langle \Sigma \rangle$ finite.
$\psi_\sigma = \sigma(\psi)$ $\sigma \in \Sigma$
$P$ holds a.a.s. if $\frac{|\{\sigma \in \Sigma \mid \sigma \text{ satisfies } P\}|}{|\Sigma|} \to 1$ as $C \to \infty$)

Lemma: $\psi$ Generic $\Rightarrow$ hyp., fully irr., $b_1(G) = 1$

**Thm 3** [H.–K.]: Let $G = F_n \rtimes_\varphi \mathbb{Z}$.
$\hat{G}$ detects $\varphi$ atoroidal, $\varphi$ fin. order and $\varphi$ super irr.

Combining first Thm + Lemma and third Thm implies 2nd Thm //