

THE FIRST ℓ^2 -BETTI NUMBER AND GROUPS ACTING ON TREES

INDIRA CHATTERJI¹, SAM HUGHES² AND PETER KROPHOLLER²

¹ *Université de Nice, Parc Valrose, 06000 Nice, France* (indira.chatterji@math.cnrs.fr)

² *School of Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, UK* (sam.hughes@maths.ox.ac.uk; p.h.kropholler@soton.ac.uk)

(Received 22 January 2021; first published online 25 October 2021)

Abstract We generalize results of Thomas, Allcock, Thom–Petersen, and Kar–Niblo to the first ℓ^2 -Betti number of quotients of certain groups acting on trees by subgroups with free actions on the edge sets of the graphs.

Keywords: Cohomology of groups; group actions on trees; presentations of groups

2020 *Mathematics subject classification:* Primary 20E08; 20J06;
Secondary 57M05

1. Introduction

The ℓ^2 -Betti numbers were introduced by Atiyah as dimensions of heat kernels of certain operators on Riemannian manifolds. The modern formulation assigns ℓ^2 -Betti numbers $b_i^{(2)}(G)$ to arbitrary groups G . We refer the reader to Lück’s account where the history can be found in the introduction of [12]. Technical results about ℓ^2 -Betti numbers that we need can be found in chapters 6 and 8 of loc. cit. The ℓ^2 -Euler characteristic $\chi^{(2)}(G)$ is defined to be the alternating sum of these Betti numbers when this series is absolutely convergent. Let \mathfrak{C} denote the class of groups F such that

- $\sum_{i \geq 0} b_i^{(2)}(G)$ is finite (this being the condition for absolute convergence),
- $b_1^{(2)}(F) = b_2^{(2)}(F) = 0$, and
- either $\chi^{(2)}(F) = 0$ or F is finite.

Note that \mathfrak{C} contains all ℓ^2 -acyclic groups (i.e. the groups for which $b_i^{(2)} = 0$ for all $i > 0$) and, in particular, it contains all amenable groups. Relevant background on ℓ^2 -cohomology is included in §2. In this note, we prove the following theorem.

© The Author(s), 2021. Published by Cambridge University Press on Behalf of The Edinburgh Mathematical Society. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

Theorem 1.1. *Let F be a group acting simplicially and cocompactly on a simplicial tree, with vertex and edge stabilizers in \mathfrak{C} , let N be a subgroup normally generated by m elements, intersecting the vertex stabilizers trivially, and let G denote F/N . Then $\chi^{(2)}(F)$ is defined and setting $k := \chi^{(2)}(F) + m$ the following conclusions hold:*

- (i) *If $k \leq 0$, then G is infinite.*
- (ii) *If $k < 0$, then $b_1^{(2)}(G) \geq -k > 0$.*
- (iii) *If G is finite, then $k > 0$ and $|G| \geq \frac{1}{k}$.*

Note that the hypotheses of this theorem guarantee that N acts freely on the specified tree and, in particular, N is necessarily a free group. Note also that, according to [2, Corollary 1.4], if $b_1^{(2)}(G) > 0$ then G has no commensurated infinite amenable subgroup and according to [3, Corollary 6] does not have property (T). If we also have $b_2^{(2)}(G) = 0$, then G is in the class \mathcal{D}_{reg} by [15, Lemma 2.8]. We refer the reader to [4] for background on property (T) and to [15, Definition 2.6] for the definition of the class \mathcal{D}_{reg} . Acylindrically hyperbolic groups form a large class of groups admitting coarsely proper actions on hyperbolic metric spaces. The class is a generalization of relative hyperbolicity including many Artin groups, mapping class groups, and $\text{Out}(F_n)$. By the main result of Osin’s paper [13], we have the following corollary.

Corollary 1.2. *Let G , F and N be as in Theorem 1.1. Assume that G is finitely presented, (virtually) indicable and that $\chi^{(2)}(F) + m < 0$. Then G is (virtually) acylindrically hyperbolic.*

The simplest way in which the indicability hypothesis may arise is through *stable letters*: Let T denote the F -tree of Theorem 1.1. Let K denote the (necessarily normal) subgroup generated by the vertex stabilizers. Then there is a subgroup $E \leq F$ that complements K and all such subgroups are free of uniquely determined rank. Such a subgroup may be referred to as a *subgroup of stable letters of the action*. The group G has an infinite cyclic quotient when $N \cap E$ has infinite index in E , in other words, when there is a stable letter that is faithfully represented in G , and in this case, G is indicable.

Recall that a group G is C^* -simple if the reduced group C^* -algebra, denoted $C_r^*(G)$, has exactly two norm closed 2-sided ideals, 0, and the algebra $C_r^*(G)$ itself. By [5, Corollary 6.7], we obtain the following.

Corollary 1.3. *Let G , F and N be as in Theorem 1.1. Then G is C^* -simple if and only if it has trivial amenable radical.*

These two corollaries highlight the relation between the first ℓ^2 -Betti number and other areas of geometric group theory. It is an interesting matter for further research to find out whether they lead to new examples.

Theorem 1.1 has some historical pedigree. It originally began life as a result about quotients of free groups due to Thomas (see Theorem 1.4(i)) and was proved using combinatorial methods [16]. The result was generalized by Allcock to incorporate a bound on the rank of the abelianisation of the quotient group [1]. The introduction of ℓ^2 -cohomology

came when Peterson–Thom [14, Theorem 3.6] and Kar–Niblo [11] independently linked the inequality of Thomas to the first ℓ^2 -Betti number. These discoveries are summarized in the following result.

Theorem 1.4 (Thomas [16], Allcock [1], Peterson–Thom [14], Kar–Niblo [11]). *Let G be a group with a presentation*

$$\langle x_1, \dots, x_n \mid r_1^{k_1}, \dots, r_m^{k_m} \rangle$$

in which the elements r_i have order k_i when interpreted in G .

- (i) *If $n - \sum_{i=1}^m \frac{1}{k_i} \geq 1$ then G is infinite.*
- (ii) *If G is finite then $|G| \geq \frac{1}{1 - n + \sum_{i=1}^m \frac{1}{k_i}}$.*
- (iii) *If $n - \sum_{i=1}^m \frac{1}{k_i} > 1$ then G is non-amenable.*

Deduction of Theorem 1.4 from Theorem 1.1. Let G be a group with a presentation

$$G = \langle x_1, \dots, x_n \mid r_1^{k_1}, \dots, r_m^{k_m} \rangle.$$

Adding m fresh generators y_1, \dots, y_m , we can give the following alternative presentation of the same group:

$$G \cong \langle x_1, \dots, x_n, y_1, \dots, y_m \mid y_1^{k_1}, \dots, y_m^{k_m}, r_1 y_1^{-1}, \dots, r_m y_m^{-1} \rangle.$$

Let F be the group with presentation

$$F = \langle x_1, \dots, x_n, y_1, \dots, y_m \mid y_1^{k_1}, \dots, y_m^{k_m} \rangle$$

and let N be the subgroup of F normally generated by $r_1 y_1^{-1}, \dots, r_m y_m^{-1}$. Then F is a free product of cyclic groups which has a natural action on a simplicial tree T (the Bass–Serre tree). In particular, it is virtually free and has Euler characteristic $\chi(F) = \sum_{i=1}^m \frac{1}{k_i} - n - m + 1$ (as explained in [6, Chapter IX.7]). This is equal to $\chi^{(2)}(F)$ by [12, Remark 6.81]. The condition that the r_i have order k_i in the original presentation ensures that N does not meet any of the finite subgroups of F , so is torsion-free and acts freely on T . Applying Theorem 1.1 with these choices of F, N, G and T yields Theorem 1.4. \square

Throughout this paper, for a group or subgroup G , we will adopt the convention that $\frac{1}{|G|}$ be interpreted as zero if G is infinite.

Finally, we compute the first ℓ^2 -Betti number for certain groups acting on trees leading to Theorem 1.5 below, which is proved in §4. This generalizes a result of Lück [7], which covers the case of an amalgamated free product, and a result of Tsouvalas [17, Corollary 3.7]. Tsouvalas assumes the vertex stabilisers are either residually finite or virtually torsion-free and the edge stabilisers are finite. Here we replace both of these assumptions with Lück’s less restrictive assumption that the first ℓ^2 -Betti numbers of the edge stabilisers vanish. So, for example, the theorem applies to fundamental groups of graphs of \mathfrak{C} -groups.

Theorem 1.5. *Let F be a group acting simplicially on a simplicial tree and let V and E denote sets of representatives of F -orbits of vertices and edges. Assume for each $e \in E$ that $b_1^{(2)}(F_e) = 0$, then we have*

$$b_1^{(2)}(F) = \sum_{v \in V} \left(b_1^{(2)}(F_v) - \frac{1}{|F_v|} \right) + \sum_{e \in E} \frac{1}{|F_e|} + \frac{1}{|F|}.$$

2. Background on ℓ^2 -homology

Let G be a group. Then both G and the complex group algebra $\mathbb{C}G$ act by left multiplication on the Hilbert space $\ell^2 G$ of square-summable sequences. The group von Neumann algebra $\mathcal{N}G$ is the ring of G -equivariant bounded operators on $\ell^2 G$. The regular elements of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}G$ can be identified with the *ring of affiliated operators*, and is denoted by $\mathcal{U}G$. One has the inclusions $\mathbb{C}G \subseteq \mathcal{N}G \subseteq \ell^2 G \subseteq \mathcal{U}G$ and it is also known that $\mathcal{U}G$ is a self-injective ring which is flat over $\mathcal{N}G$. For more details concerning these constructions, we refer the reader to [12] and especially to Theorem 8.22 of § 8.2.3 therein. The *von Neumann dimension* and the basic properties we need can be found in [12, § 8.3]. Now let Y be a G -CW complex as defined in [12, Definition 1.25 of § 1.2]. The ℓ^2 -homology groups of Y are then defined to be the equivariant homology groups $H_i^G(Y; \mathcal{U}G)$, and we have

$$b_i^{(2)}(Y) = \dim_{\mathcal{U}G} H_i^G(Y; \mathcal{U}G).$$

The ℓ^2 -Betti numbers of a group G are then defined to be the ℓ^2 -Betti numbers of EG , that is to say

$$b_i^{(2)}(G) := b_i^{(2)}(EG). \tag{1}$$

By [12, Theorem 6.54(8)], the zeroth ℓ^2 -Betti number of G is equal to $1/|G|$. Moreover, if G is finite then $b_n^{(2)}(G) = 0$ for $n \geq 1$.

Let $C_*(Y; \mathcal{U}G)$ denote the standard cellular chain complex of Y with coefficients in $\mathcal{U}G$. We have the formula

$$\dim_{\mathcal{U}G} C_i(Y; \mathcal{U}G) = \sum_{\sigma} \frac{1}{|G_{\sigma}|}$$

where σ runs through a set of orbit representatives of i -dimensional cells in Y . Suppose that the ℓ^2 -Euler characteristic of Y is defined. Standard arguments of homological algebra give the connection between two Euler characteristic computations (for the details see [12, Lemma 6.80(1)]):

$$\chi^{(2)}(Y) = \sum_i (-1)^i b_i^{(2)}(Y) = \sum_i (-1)^i \dim_{\mathcal{U}G} C_i(Y; \mathcal{U}G) = \sum_i (-1)^i \sum_{\sigma} \frac{1}{|G_{\sigma}|}. \tag{2}$$

We will need the following lemma for the proofs in the next section. One should think of it as a mild generalization of Theorem 6.54(2) in [12].

Lemma 2.1 (Comparison with the Borel construction up to rank). *Let X be a G -CW complex. Suppose for all $x \in X$ the isotropy group G_x is finite or $b_p^{(2)}(G_x) = 0$ for all $0 \leq p \leq n$, then*

$$b_p^{(2)}(X) = b_p^{(2)}(EG \times X) \quad \text{for } 0 \leq p \leq n.$$

Proof. It suffices to prove that the von Neumann dimensions of the kernel and cokernel of the map

$$\text{pr}_p : H_p^G(EG \times X; \mathcal{U}G) \rightarrow H_p^G(X; \mathcal{U}G)$$

induced by the projection $EG \times X \rightarrow X$ are zero for $0 \leq p \leq n$. Here $EG \times X$ carries the diagonal action of G . By an identical argument to [12, Theorem 6.54(2)], it suffices to prove for each isotropy subgroup $H \leq G$ and $0 \leq p \leq n$ the kernel and cokernel of the map $\text{pr}_p : H_p^H(EH; \mathcal{U}H) \rightarrow H_p^H(*; \mathcal{U}H)$ have dimension equal to zero. If H is finite this follows from [12, Theorem 6.54(8a)], and is immediate if $b_p^{(2)}(H) = 0$ for all $0 \leq p \leq n$. \square

3. The main theorem

To prove Theorem 1.1, one needs the following method of computing the ℓ^2 -Euler characteristic of a group acting on a tree analogous to Chiswell’s result [9] for rational Euler characteristic.

Proposition 3.1 (Chatterji–Mislin [8]). *Let F be a group acting on a tree and let V and E denote sets of representatives of F -orbits of vertices and edges. If the ℓ^2 -Euler characteristic of each vertex and edge group is finite, then*

$$\chi^{(2)}(F) = \sum_{v \in V} \chi^{(2)}(F_v) - \sum_{e \in E} \chi^{(2)}(F_e).$$

Proof of Theorem 1.1. There is a cocompact action of F on a tree T with vertex and edge stabilisers in the class \mathfrak{C} . Let V and E denote the vertex and edge sets. Let \bar{T} denote the quotient graph T/N and write \bar{V} and \bar{E} for its vertex and edge sets. Now $G = F/N$ acts cocompactly on \bar{T} with vertex and edge stabilizers in \mathfrak{C} . The augmented chain complex of T is the short exact sequence

$$0 \rightarrow \mathbb{Z}E \rightarrow \mathbb{Z}V \rightarrow \mathbb{Z} \rightarrow 0$$

of $\mathbb{Z}F$ -modules. Restricting to the action of N this short exact sequence leads to a long exact sequence for the homology of N . It is straightforward to identify $H_0(N; \mathbb{Z}V)$ with $\mathbb{Z}\bar{V}$ and $H_0(N; \mathbb{Z}E)$ with $\mathbb{Z}\bar{E}$, so that the tail end of the sequence takes the form

$$H_1(N; \mathbb{Z}) \rightarrow \mathbb{Z}\bar{E} \rightarrow \mathbb{Z}\bar{V} \rightarrow \mathbb{Z} \rightarrow 0. \tag{3}$$

Let $\{r_i : i = 1, \dots, m\}$ denote a normal generating set for N . Choose a vertex v_0 in T to be a fixed basepoint. For $1 \leq i \leq m$, consider the geodesic from v_0 to v_0r_i . In the quotient graph \bar{T} , this geodesic descends to a loop because v_0 and v_0r_i become identified in \bar{T} . Now 2-discs can be glued to each loop. By adjoining free G -orbits of 2-discs equivariantly,

we can build a 2-complex Y with an action of G , whose 1-skeleton is \overline{T} , and which has augmented cellular chain complex

$$\mathbb{Z}G^m \rightarrow \mathbb{Z}\overline{E} \rightarrow \mathbb{Z}\overline{V} \rightarrow \mathbb{Z} \rightarrow 0. \tag{4}$$

By construction the map $\mathbb{Z}G^m \rightarrow \mathbb{Z}\overline{E}$ factors through a surjection $\mathbb{Z}G^m \rightarrow H_1(N; \mathbb{Z})$. Therefore, the exactness of (3) ensures the exactness of (4). It follows that Y is 1-acyclic.

Let V_0 and E_0 be sets of orbit representatives of vertices and edges in Y . Now, applying Proposition 3.1 then (2), we have that

$$\begin{aligned} \chi^{(2)}(F) + m &= \sum_{v \in V_0} \frac{1}{|G_v|} - \sum_{e \in E_0} \frac{1}{|G_e|} + m \\ &= b_0^{(2)}(Y) - b_1^{(2)}(Y) + b_2^{(2)}(Y). \end{aligned}$$

Lemma 2.1 with $n = 2$, yields

$$\begin{aligned} \chi^{(2)}(F) + m &= b_0^{(2)}(EG \times Y) - b_1^{(2)}(EG \times Y) + b_2^{(2)}(EG \times Y) \\ &\geq b_0^{(2)}(EG \times Y) - b_1^{(2)}(EG \times Y). \end{aligned}$$

Applying [12, Theorem 6.54(1a)] to the projection $f : EG \times Y \rightarrow EG$ with $n = 2$ (note that we are using the fact Y is 1-acyclic), we obtain $b_i^{(2)}(EG \times Y) = b_i^{(2)}(EG)$ for $i = 0, 1$. Recalling (1), we therefore have

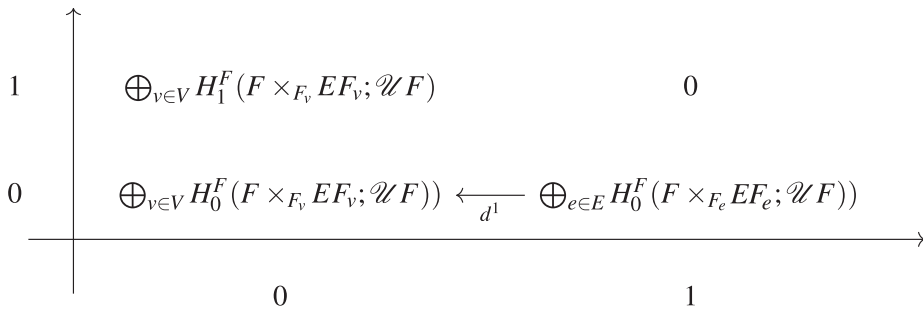
$$\chi^{(2)}(F) + m \geq b_0^{(2)}(G) - b_1^{(2)}(G).$$

Let $k = \chi^{(2)}(F) + m$. If $k \leq 0$, then $b_0^{(2)}(G) - b_1^{(2)}(G) \leq 0$ and so G is infinite, this proves (i). Now, assume $k < 0$. In this case, G is infinite and therefore $b_0^{(2)}(G) = 0$. It follows that $b_1^{(2)}(G) \geq -k > 0$, this proves (ii).

If G is finite, then $b_0^{(2)}(G) = \frac{1}{|G|}$, $b_1^{(2)}(G) = 0$, and $k > 0$. In particular, $k \geq \frac{1}{|G|} > 0$ and (iii) follows. □

4. On the ℓ^2 -invariants for certain groups acting on trees

Proof of Theorem 1.5. Let V and E denote sets of representatives of F -orbits of vertices and edges for the action of F on the tree. We consider the relevant part of the E^1 -page for the F -equivariant spectral sequence (see Chapter VII.9 of [6]) applied to the tree:



If F is finite then $b_1^{(2)}(F) = 0$, so d^1 is injective and $E_{1,0}^2 = 0$. The result follows from the fact $E_{0,1}^1 = 0$.

Now, assume F is infinite, then d^1 is surjective since $b_0^{(2)}(F) = 0$. Thus,

$$\dim_{\mathcal{U}F}(\text{Ker}(d^1)) = \sum_{e \in E} b_0^{(2)}(F_e) - \sum_{v \in V} b_0^{(2)}(F_v).$$

Now, the spectral sequence obviously collapses on the E^2 -page and $E_{0,1}^1 = E_{0,1}^2$. Since von Neumann dimension is additive over short exact sequences, we have

$$\begin{aligned} b_1^{(2)}(F) &= \dim_{\mathcal{U}F}(\text{Ker}(d^1)) + \dim_{\mathcal{U}F}(E_{0,1}^2) \\ &= \left(\sum_{e \in E} b_0^{(2)}(F_e) - \sum_{v \in V} b_0^{(2)}(F_v) \right) + \sum_{v \in V} b_1^{(2)}(F_v), \end{aligned}$$

and the result follows. \square

Acknowledgements. This paper contains material from the second author's PhD thesis [10]. The second author wishes to thank Ian Leary for his tireless and good humoured role as thesis adviser. The second author was supported by the Engineering and Physical Sciences Research Council grant number 2127970. We are also indebted to Kevin Li and to the anonymous referee for a number of helpful comments and corrections.

References

1. D. ALLCOCK, Spotting infinite groups, *Math. Proc. Cambridge Philos. Soc.* **125**(1) (1999), 39–42.
2. U. BADER, A. FURMAN AND R. SAUER, Weak notions of normality and vanishing up to rank in L^2 -cohomology, *Int. Math. Res. Not. IMRN* **12**, (2014), 3177–3189.
3. M. E. B. BEKKA AND A. VALETTE, Group cohomology, harmonic functions and the first L^2 -Betti number, *Potential Anal.* **6**(4) (1997), 313–326.
4. B. BEKKA, P. DE LA HARPE AND A. VALETTE, Kazhdan's property (T), Vol. 11, *New Mathematical Monographs* (Cambridge University Press, Cambridge, 2008).
5. E. BREUILLARD, M. KALANTAR, M. KENNEDY AND N. OZAWA, C^* -simplicity and the unique trace property for discrete groups, *Publ. Math. Inst. Hautes Études Sci.* **126** (2017), 35–71.
6. K. S. BROWN, *Cohomology of groups*, Vol. 87, Graduate Texts in Mathematics (Springer-Verlag, New York, 1982).
7. N. P. BROWN, K. J. DYKEMA AND K. JUNG, Free entropy dimension in amalgamated free products, *Proc. Lond. Math. Soc. (3)* **97**(2) (2008), 339–367. With an appendix by Wolfgang Lück.
8. I. CHATTERJI AND G. MISLIN, Hattori-Stallings trace and Euler characteristics for groups, in *Geometric and homological methods in group theory*, pp. 256–271, London Math. Soc. Lecture Note Ser., Vol. 358 (Cambridge University Press, Cambridge, 2009).
9. I. M. CHISWELL, Exact sequences associated with a graph of groups, *J. Pure Appl. Algebra* **8**(1) (1976), 63–74.
10. S. HUGHES, Equivariant cohomology, lattices, and trees, *PhD thesis* (2021).
11. A. KAR AND G. NIBLO, Some non-amenable groups, *Pub. Mat.* **56**(1) (2012), 255–259.

12. W. LÜCK, *L^2 -invariants: theory and applications to geometry and K -theory*, Vol. 44, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in mathematics and related areas. 3rd Series. A Series of Modern Surveys in Mathematics]* (Springer-Verlag, Berlin, 2002).
13. D. OSIN, On acylindrical hyperbolicity of groups with positive first ℓ^2 -Betti number, *Bull. Lond. Math. Soc.* **47**(5) (2015), 725–730.
14. J. PETERSON AND A. THOM, Group cocycles and the ring of affiliated operators, *Invent. Math.* **185** (2011), 561–592.
15. A. THOM, Low degree bounded cohomology and L^2 -invariants for negatively curved groups, *Groups Geom. Dyn.* **3**(2) (2009), 343–358.
16. R. M. THOMAS, Cayley graphs and group presentations, *Math. Proc. Cambridge Philos. Soc.* **103**(3) (1988), 385–387.
17. K. TSOUVALAS, Euler characteristics on virtually free products, *Comm. Algebra* **46**(8) (2018), 3397–3412.