

ON THE EQUIVARIANT K - AND KO -HOMOLOGY OF SOME SPECIAL LINEAR GROUPS

SAM HUGHES

ABSTRACT. We compute the equivariant KO -homology of the classifying space for proper actions of $SL_3(\mathbb{Z})$ and $GL_3(\mathbb{Z})$. We also compute the Bredon homology and equivariant K -homology of the classifying spaces for proper actions of $PSL_2(\mathbb{Z}[\frac{1}{p}])$ and $SL_2(\mathbb{Z}[\frac{1}{p}])$ for each prime p . Finally, we prove the Unstable Gromov-Lawson-Rosenberg Conjecture for a large class of groups whose maximal finite subgroups are odd order and have periodic cohomology.

1. INTRODUCTION

There has been considerable interest in the Baum-Connes conjecture, which states that for a group Γ a certain ‘assembly map’, from the equivariant K -homology of the classifying space for proper actions $\underline{E}\Gamma$ to the topological K -theory of the reduced group C^* -algebra, is an isomorphism [3]. The Baum-Connes conjecture is known to hold for several families of groups, including word-hyperbolic groups, $CAT(0)$ -cubical groups and groups with the Haagerup property. An excellent survey can be found in [2].

Conjecture 1.1 (The Baum-Connes Conjecture). *Let Γ be a discrete group, then the following assembly map is an isomorphism*

$$\mu : K_*^\Gamma(\underline{E}\Gamma) \rightarrow K_*^{\text{top}}(C_r^*(\Gamma)).$$

There is also a ‘real’ Baum-Connes conjecture which predicts that an assembly map from the equivariant KO -homology of $\underline{E}\Gamma$ to the topological K -theory of the real group C^* -algebra is an isomorphism. It is known that the two conjectures are equivalent, that is, one of the assembly maps is an isomorphism if and only if the other is [4].

Conjecture 1.2 (The Real Baum-Connes Conjecture). *Let Γ be a discrete group, then the following assembly map is an isomorphism*

$$\mu_{\mathbb{R}} : KO_*^\Gamma(\underline{E}\Gamma) \rightarrow KO_*^{\text{top}}(C_r^*(\Gamma)).$$

In spite of the interest, to date there have been very few computations of K^Γ - and KO^Γ -homology. Indeed, for K^Γ -homology there are complete calculations for one relator groups [34], NEC groups [31], some Bianchi groups and hyperbolic reflection groups [29, 37, 38], some Coxeter groups [16, 45, 46], Hilbert modular groups [43], $SL_3(\mathbb{Z})$ [44], and $PSL_4(\mathbb{Z})$ [10]. Explicit assembly maps have also been computed for solvable Baumslag-Solitar groups [36], lamplighter groups of finite groups [18] and certain wreath products [35, 30]. For KO^Γ -homology the author is aware of two complete computations; the first, due to Davis and Lück, on a family of Euclidean crystallographic groups [14], and the second, due to Mario Fuentes-Rumí, on simply connected graphs of cyclic groups of odd order and of some Coxeter groups [17].

In this paper we compute the equivariant K -homology of $SL_2(\mathbb{Z}[\frac{1}{p}])$ and the equivariant KO -homology of $SL_3(\mathbb{Z})$. We give the relevant background and the connection to Bredon homology in Section 2.

Date: December 31, 2020.

The calculation for KO^Γ -homology is of particular interest because it is (to the author's knowledge) the first computation of KO_*^Γ for a property (T) group. For background on property (T) the reader may consult the monograph [5]. This interest stems from the fact that property (T) is a strong negation of the Haagerup property which implies Baum-Connes [21]. Moreover, the (real) Baum-Connes conjecture is still open for $SL_n(\mathbb{Z})$ when $n \geq 3$. We note that there are counterexamples for the Baum-Connes conjecture for groupoids constructed from $SL_3(\mathbb{Z})$ and more generally a discrete group with property (T) for which the assembly map is known to be injective [22].

Theorem 1.3 (Theorem 3.2). *Let $\Gamma = SL_3(\mathbb{Z})$, then for $n = 0, \dots, 7$ we have*

$$KO_n^\Gamma(\underline{E}\Gamma) = \mathbb{Z}^8, \quad \mathbb{Z}_2^8, \quad \mathbb{Z}_2^8, \quad 0, \quad \mathbb{Z}^8, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Applying a Künneth type theorem [45, Theorem 3.6] to the isomorphism $GL_3(\mathbb{Z}) \cong SL_3(\mathbb{Z}) \times \mathbb{Z}_2$ on the level of Bredon homology, we obtain the following result for $GL_3(\mathbb{Z})$.

Corollary 1.4 (Theorem 3.3). *Let $\Gamma = GL_3(\mathbb{Z})$, then for $n = 0, \dots, 7$ we have*

$$KO_n^\Gamma(\underline{E}\Gamma) = \mathbb{Z}^{16}, \quad \mathbb{Z}_2^{16}, \quad \mathbb{Z}_2^{16}, \quad 0, \quad \mathbb{Z}^{16}, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

We also consider $\Gamma = PSL_2(\mathbb{Z}[\frac{1}{p}])$ or $SL_2(\mathbb{Z}[\frac{1}{p}])$, for p a prime, computing the equivariant K -homology groups $K_n^\Gamma(\underline{E}\Gamma)$. There has been considerable interest in determining homological properties of the groups $SL_2(\mathbb{Z}[\frac{1}{m}])$ and groups related to them [1, 9, 23]. It appears, however, that even with computer based methods the problem of determining the cohomology of $SL_2(\mathbb{Z}[\frac{1}{m}])$ for m a product of 3 primes is out of reach [9]. In Lemma 5.4 we give a short proof of the Baum-Connes conjecture for $SL_2(\mathbb{Z}[\frac{1}{p}])$ and so we obtain the topological K -theory of the reduced group C^* -algebra of $SL_2(\mathbb{Z}[\frac{1}{p}])$ as well.

Theorem 1.5 (Theorem 5.6). *Let p be a prime and $\Gamma = PSL_2(\mathbb{Z}[\frac{1}{p}])$, then $K_n^\Gamma(\underline{E}\Gamma)$ is a free abelian group with rank as given in Table 1. Moreover, since the Baum-Connes Conjecture holds for Γ we have $K_*^\Gamma(\underline{E}\Gamma) \cong K_*^{top}(C_r^*(\Gamma))$.*

	$p = 2$	$p = 3$	$p \equiv 1 \pmod{12}$	$p \equiv 5 \pmod{12}$	$p \equiv 7 \pmod{12}$	$p \equiv 11 \pmod{12}$
$n = 0$	7	6	$4 + \frac{1}{6}(p - 7)$	$6 + \frac{1}{6}(p + 1)$	$5 + \frac{1}{6}(p - 1)$	$7 + \frac{1}{6}(p + 7)$
$n = 1$	0	0	3	1	2	0

TABLE 1. \mathbb{Z} -rank of the equivariant K -homology of the classifying space for proper actions of $PSL_2(\mathbb{Z}[\frac{1}{p}])$.

Theorem 1.6 (Theorem 5.7). *Let p be a prime and $\Gamma = SL_2(\mathbb{Z}[\frac{1}{p}])$. Then $K_n^\Gamma(\underline{E}\Gamma)$ is additively isomorphic to the direct sum of two copies of the corresponding equivariant K -homology group of $PSL_2(\mathbb{Z}[\frac{1}{p}])$.*

Finally, we will give a proof of the Unstable Gromov-Lawson-Rosenberg Conjecture for positive scalar curvature for a large class of groups whose torsion subgroups have periodic cohomology. The statement and background concerning this conjecture is given in Section 6. However, we will introduce the following notation now before the theorem statement. We say a group Γ satisfies:

(M) If every finite subgroup is contained in a unique maximal finite subgroup.

- (NM) If M is a maximal finite subgroup of Γ , then the normaliser $N_\Gamma(M)$ of M is equal to M .
 (BC) If Γ satisfies the Baum-Connes conjecture.
 (PFS) If all maximal finite subgroups of Γ are odd order and have periodic cohomology.

A large number of arithmetic groups satisfy the following theorem including many finite index subgroups $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for $p \equiv 11 \pmod{12}$ and Hilbert modular groups. We will detail a number of additional examples in Section 6.

Theorem 1.7 (Theorem 6.3). *Let Γ be a group satisfying (BC), (M), (NM) and (PFS). If $\underline{B}\Gamma$ is finite and has dimension at most 9, then the Unstable Gromov-Lawson-Rosenberg Conjecture holds for Γ .*

In Section 2 we give the relevant background on equivariant K and KO -homology. In Section 3 we give the computations of the equivariant KO -homology for $\mathrm{SL}_3(\mathbb{Z})$ and $\mathrm{GL}_3(\mathbb{Z})$. In Section 4 we provide auxiliary computations of the equivariant K -homology of Fuchsian groups. In Section 5 we compute the equivariant K -homology of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. Finally, in Section 6 we prove the results about the Unstable Gromov-Lawson-Rosenberg Conjecture and give a number of examples of groups satisfying the conjecture.

Acknowledgements. This paper contains material from the author's PhD thesis. The author would like to thank his PhD supervisor Professor Ian Leary for his guidance and support. He would also like to thank Jim Davis for pointing out Remark 6.4, Naomi Andrew, Guy Boyde, and Kevin Li for helpful conversations and the anonymous reviewer whose feedback greatly improved the exposition of this paper. This work was supported by the Engineering and Physical Sciences Research Council grant number 2127970.

2. PRELIMINARIES

In this section we introduce the relevant background from Bredon homology and its interactions with equivariant K - and KO -homology. We follow the treatment given in Mislin's notes [34].

2.1. Classifying spaces for families. Let Γ be a discrete group. A Γ -CW complex X is a CW-complex equipped with a cellular Γ -action. We say the Γ action is proper if all of the cell stabilisers are finite.

Let \mathcal{F} be a family of subgroups of Γ which is closed under conjugation and finite intersections. A *model* for the classifying space $E_{\mathcal{F}}\Gamma$ for the family \mathcal{F} is a Γ -CW complex such that all cell stabilisers are in \mathcal{F} and the fixed point set of every $H \in \mathcal{F}$ is weakly-contractible. This is equivalent to the following universal property: For every Γ -CW complex Y there is exactly one Γ -map $Y \rightarrow E_{\mathcal{F}}\Gamma$ up to Γ -homotopy.

In the case where $\mathcal{F} = \mathcal{FIN}$, the family of all finite subgroups of Γ , we denote $E_{\mathcal{FIN}}(\Gamma)$ by $\underline{E}\Gamma$. We call such a space, the classifying space for proper actions of Γ . Note that if Γ is torsion-free then $\underline{E}\Gamma = E\Gamma$.

2.2. Bredon homology. Let Γ be a discrete group and \mathcal{F} be a family of subgroups. We define the *orbit category* $\mathbf{Or}_{\mathcal{F}}(\Gamma)$ to be the category with objects given by left cosets Γ/H for $H \in \mathcal{F}$ and morphisms the Γ -maps $\phi : \Gamma/H \rightarrow \Gamma/K$. A morphism in the orbit category is uniquely determined by its image $\varphi(H) = \gamma K$ and $\gamma H \gamma^{-1} \subseteq K$; conversely, each such $\gamma \in \Gamma$ defines a G -map.

A (*left*) *Bredon module* is a covariant functor $M : \mathbf{Or}_{\mathcal{F}}(\Gamma) \rightarrow \mathbf{Ab}$, where \mathbf{Ab} is the category of Abelian groups. Consider a Γ -CW complex X and a family of subgroups \mathcal{F} containing all cell stabilisers. Let M be a Bredon module and define the *Bredon chain complex with coefficients in M* as follows:

Let $\{c_\alpha\}$ be a set of orbit representatives of the n -cells in X and let Γ_α denote the stabiliser of the cell α . The n th chain group is then

$$C_n := \bigoplus_{\alpha} M(\Gamma/\Gamma_{c_\alpha}).$$

If $\gamma c'$ is an $(n-1)$ -cell in the boundary of c , then $\gamma^{-1}\Gamma_c\gamma \subseteq \Gamma_{c'}$. This defines a Γ -map $\varphi : \Gamma/\Gamma_c \rightarrow \Gamma/\Gamma_{c'}$, which in turn gives an induced homomorphism $M(\varphi) : M(\Gamma/\Gamma_c) \rightarrow M(\Gamma/\Gamma_{c'})$. Therefore, we obtain a differential $\partial : C_n \rightarrow C_{n-1}$. Taking homology of the chain complex (C_*, ∂) gives the *Bredon homology groups* $H_n^{\mathcal{F}}(X; M)$. A right Bredon module and Bredon cohomology are defined analogously using contravariant functors.

2.3. Equivariant K -homology. The original definition of equivariant K -homology used Kasparov's KK -theory [3]. There is also homotopy theoretic approach using spaces and spectra over the orbit category due to Davis-Lück [12]. We will highlight the details we need.

Let Γ be a discrete group. In the context of the Baum-Connes conjecture we are specifically interested in the case where $X = \underline{E}\Gamma$, $\mathcal{F} = \mathcal{FIN}$ and $M = \mathcal{R}_{\mathbb{C}}$ the complex representation ring. We consider $\mathcal{R}_{\mathbb{C}}(-)$ as a Bredon module in the following way: For $\Gamma/H \in \mathbf{Or}_{\mathcal{F}}(\Gamma)$ set $\mathcal{R}_{\mathbb{C}}(\Gamma/H) := \mathcal{R}_{\mathbb{C}}(H)$, the ring of complex representations of the finite group H . Morphisms are then given by induction of representations.

We note that $\mathcal{R}_{\mathbb{C}}(\Gamma) := H_0^{\mathcal{F}}(\Gamma) = \text{colim}_{\Gamma/H \in \mathbf{Or}(\Gamma)} \mathcal{R}_{\mathbb{C}}(H)$. In the case that Γ has finitely many conjugacy classes of finite subgroups, $\mathcal{R}_{\mathbb{C}}(\Gamma)$ is a finitely generated quotient of $\bigoplus \mathcal{R}_{\mathbb{C}}(H)$, where H runs over conjugacy classes of finite subgroups.

We now exhibit the connection between Bredon homology and $K_*^{\Gamma}(\underline{E}\Gamma)$, the equivariant K -homology of the classifying space for proper actions. Indeed, for each subgroup $H \leq \Gamma$ equivariant K -homology satisfies

$$K_n^{\Gamma}(\Gamma/H) = K_n^{\text{top}}(C_r^*(H)).$$

In the case H is a finite subgroup we have $C_r^*(H) = \mathbb{C}H$, $K_0^{\Gamma}(\Gamma/H) = K_0^{\text{top}}(\mathbb{C}H) = \mathcal{R}_{\mathbb{C}}(H)$, and $K_1^{\Gamma}(\Gamma/H) = K_1^{\text{top}}(\mathbb{C}H) = 0$. The remaining K_n^{Γ} groups are given by 2-fold Bott periodicity. This allows us to view $K_n^{\Gamma}(-)$ as a Bredon module over $\mathbf{Orb}_{\mathcal{FIN}}(\Gamma)$.

We may use an equivariant Atiyah-Hirzebruch spectral sequence to compute the K^{Γ} -homology of a proper Γ -CW-complex X from its Bredon homology.

Theorem 2.1. [34, Page 50] *Let Γ be a group and X a proper Γ -CW complex, then there is an Atiyah-Hirzebruch type spectral sequence*

$$E_{p,q}^2 := H_p^{\mathcal{FIN}}(X; K_q^{\Gamma}(-)) \Rightarrow K_{p+q}^{\Gamma}(X).$$

2.4. Equivariant KO -homology. In this section we summarise the material from [14, Section 9] which we will require for our calculations. Again fixing a discrete group Γ and $\mathcal{F} = \mathcal{FIN}$, we introduce two more Bredon modules, the real representation ring $\mathcal{R}_{\mathbb{R}}(-)$, and the quaternionic representation ring $\mathcal{R}_{\mathbb{H}}(-)$. These are defined on $\mathbf{Or}_{\mathcal{F}}(\Gamma)$ in exactly the same way as the complex representation ring. We have natural transformations between the functors. Indeed for a finite subgroup $H \leq \Gamma$ we have a diagram:

$$\begin{array}{ccccc} & & \nu & & \sigma \\ & & \curvearrowright & & \curvearrowright \\ \mathcal{R}_{\mathbb{R}}(H) & & & \mathcal{R}_{\mathbb{C}}(H) & & \mathcal{R}_{\mathbb{H}}(H) \\ & & \curvearrowleft & & \curvearrowleft \\ & & \rho & & \eta \end{array}$$

Note that the diagram does not commute. For instance let $\mathbf{1} \in \mathcal{R}_{\mathbb{R}}(H)$ denote the trivial representation, then $\rho\nu(\mathbf{1}) = 2 \cdot (\mathbf{1})$.

For a real representation ψ , the *complexification* is $\nu(\psi) = \psi \otimes \mathbb{C}$. For a complex representation ϕ , the *symplectification* is $\sigma(\phi) = \phi \otimes \mathbb{H}$. Going the other way, for an n -dimensional quaternionic representation ω , the *complexification* is $\eta(\omega) = \eta$ considered as $2n$ -dimensional complex representation. Similarly, for an n -dimensional complex representation ϕ , the *realification* is $\rho(\phi) = \phi$ considered as a $2n$ -dimensional real representation. Note that any composition of the x-ification natural transformations with the same source and target is necessarily not the identity.

The situation for the equivariant KO -homology, denoted $KO_*^\Gamma(-)$, is similar to the equivariant K -homology but more complicated. For a subgroup $H \leq \Gamma$ we set $KO_n^\Gamma(\Gamma/H) = KO_n^{\text{top}}(C_r^*(H))$. By [8, Section 1.2], in the case that H is a finite subgroup we have that

$$KO_n^\Gamma(\Gamma/H) = KO_n^{\text{top}}(C_r^*(H)) = \begin{cases} \mathcal{R}_{\mathbb{R}}(H) & n = 0, \\ \mathcal{R}_{\mathbb{R}}(H)/\rho(\mathcal{R}_{\mathbb{C}}(H)) & n = 1, \\ \mathcal{R}_{\mathbb{C}}(H)/\eta(\mathcal{R}_{\mathbb{H}}(H)) & n = 2, \\ 0 & n = 3, \\ \mathcal{R}_{\mathbb{H}}(H) & n = 4, \\ \mathcal{R}_{\mathbb{H}}(H)/\sigma(\mathcal{R}_{\mathbb{C}}(H)) & n = 5, \\ \mathcal{R}_{\mathbb{C}}(H)/\nu(\mathcal{R}_{\mathbb{R}}(H)) & n = 6, \\ 0 & n = 7, \end{cases}$$

with the remaining groups given by 8-fold Bott-periodicity. For X a proper Γ -space, the Atiyah-Hirzebruch spectral sequence from before now takes the form

$$E_{p,q}^2 := H_p^{\mathcal{FLN}}(X; KO_q^\Gamma(-)) \Rightarrow KO_{p+q}^\Gamma(X).$$

2.5. Spectra and homotopy. The section gives an alternative Γ -equivariant homotopy theoretic viewpoint. Now, we consider Γ -equivariant homology theories as functors $\mathbf{E}: \mathbf{Or}_{\mathcal{F}}(\Gamma) \rightarrow \mathbf{Spectra}$. Technically, to avoid functorial problems one must take composite functors through the categories C^* -**Cat** and **Groupoids**. We do not concern ourselves with this complication and refer the reader to [12] and [15].

Instead we will take for granted that there is a composite functor

$$\mathbf{KO}: \mathbf{Or}_{\mathcal{F}}(\Gamma) \rightarrow \mathbf{Spectra}$$

which satisfies $\pi_n \mathbf{KO}(\Gamma/H) = KO_n^{\text{top}}(C_r^*(H))$. When $\mathcal{F} = \mathcal{FLN}$ this perspective gives a homotopy theoretic construction of the (real) Baum-Connes assembly map. Indeed, we have maps

$$B\Gamma_+ \wedge \mathbf{KO} \simeq \underset{\mathbf{Or}_{\mathcal{TRV}}(\Gamma)}{\text{hocolim}} \mathbf{KO} \rightarrow \underset{\mathbf{Or}_{\mathcal{F}}(\Gamma)}{\text{hocolim}} \mathbf{KO} \rightarrow \underset{\mathbf{Or}_{\mathcal{ACL}}(\Gamma)}{\text{hocolim}} \mathbf{KO} \simeq \mathbf{KO}(C_r^*(\Gamma; \mathbb{R})).$$

The assembly map $\mu_{\mathbb{R}}$ is then π_n applied to the composite.

2.6. Group C^* -algebras and KK -theory. In this section we give a brief outline of Kasparov's KK -theory, the material here will not be used elsewhere in the paper. The theory was introduced by Kasparov in [25, 26] in relation to the Novikov Conjecture. The original formulation of the Baum-Connes Conjecture using KK -theory was given in [3].

For a C^* -algebra A define $\mathbf{M}_\infty(A)$ to be the direct limit of sets of $(n \times n)$ -matrices over A as $n \rightarrow \infty$. Similarly, define $\text{GL}_\infty(A)$ to be the direct limit of groups of invertible $(n \times n)$ -matrices over A .

Topological or operator K -theory is a 2-periodic homology theory of unital C^* -algebras denoted $K_*^{\text{top}}(-)$. The zeroth K -group of a unital C^* -algebra A is defined to be the Grothendieck group of the set of projections in $\mathbf{M}_\infty(A)$ up to Murray von Neumann equivalence. The first K -group is defined to be $\text{GL}_\infty(A)/\text{GL}_\infty(A)_0$, where $\text{GL}_\infty(A)_0$ is the path component of the identity.

An alternative formulation is given by Kasparov’s bifunctor $KK(-, -)$. For any two C^* -algebras A and B there is an abelian group $KK(A, B)$. An element of $KK(A, B)$ is a homotopy class of (A, B) -Fredholm bimodules (see [2, Section 3] for the precise definition). The zeroth K -group of A from before is recovered as $KK(\mathbb{C}, A)$ and the first K -group is recovered as $KK(C_0(\mathbb{R}), A)$.

Let Γ be a discrete group. The *reduced C^* -algebra* of Γ , denoted $C_r^*(\Gamma)$, is the norm closure of the algebra of bounded operators on $\ell^2(\Gamma)$ by the left regular representation of Γ . The algebra and its K -groups are intimately related with the theory of elliptic operators on manifolds M with fundamental group Γ . For more information the reader should consult the survey [2] and the references therein.

3. EQUIVARIANT KO -HOMOLOGY OF $SL_3(\mathbb{Z})$

3.1. A classifying space for proper actions. A model for $X = \underline{ESL}_3(\mathbb{Z})$ can be constructed as a $SL_3(\mathbb{Z})$ -equivariant deformation retract of the symmetric space $SL_3(\mathbb{R})/O(3)$. This construction has been detailed several times in the literature ([49, Theorem 2], [20, Theorem 2.4] or [44, Theorem 13]), so rather than detailing it again here, we simply extract the relevant cell complex and cell stabilisers. Specifically, we follow the notation of Sánchez-García [44] and collect the information in Table 2.

Dimension	Cell	Boundary	Stabiliser
3	T_1	$-t_1 + t_2 - t_3 + t_4 - t_5$	$\{1\}$
2	t_1	$e_1 - e_2 - e_4$	\mathbb{Z}_2
	t_2	$e_4 - e_5 + e_6$	$\{1\}$
	t_3	$e_6 - e_7 + e_8$	\mathbb{Z}_2^2
	t_4	$e_1 - e_3 + e_5 + e_8$	\mathbb{Z}_2
	t_5	$e_2 - e_3 + e_6 - e_6 + e_7$	\mathbb{Z}_2
1	e_1	$v_1 - v_2$	\mathbb{Z}_2^2
	e_2	$v_3 - v_1$	D_3
	e_3	$v_5 - v_1$	D_3
	e_4	$v_3 - v_2$	\mathbb{Z}_2
	e_5	$v_4 - v_2$	\mathbb{Z}_2
	e_6	$v_4 - v_3$	\mathbb{Z}_2^2
	e_7	$v_5 - v_3$	D_4
	e_8	$v_5 - v_4$	D_4
0	v_1		$\text{Sym}(4)$
	v_2		D_6
	v_3	-	$\text{Sym}(4)$
	v_4		D_4
	v_5		$\text{Sym}(4)$

TABLE 2. Cell structure and stabilisers of a model for $\underline{ESL}_3(\mathbb{Z})$.

3.2. Proof of Theorem 1.3. The calculation of the equivariant KO -groups will require the following proposition and an analysis of the representation theory of the finite subgroups of $SL_3(\mathbb{Z})$. We remark that one could prove a dozen subtle variations on the theme of the following proposition. However, rather than do this we offer the slogan: “Computations with coefficients in

$KO_n^\Gamma(-)$ can be greatly simplified by looking for chain maps to the Bredon chain complex with coefficients in $\mathcal{R}_\mathbb{C}(-)$.”

Proposition 3.1. *Let Γ be a discrete group, $\mathcal{F} = \mathcal{FLN}$ and suppose X is a proper Γ -CW complex with finitely many Γ orbits of cells in each dimension. Assume that for every cell stabiliser the real, complex and quaternionic character tables are equal, then the Atiyah-Hirzebruch spectral sequence converging to $KO_*^\Gamma(X)$ has E^2 -page isomorphic to*

$$E_{p,q}^2 = H_p^{\mathcal{FLN}}(X; K_0^\Gamma) \otimes KO_q(*) \oplus \text{Tor}_1^{\mathbb{Z}}[H_{p-1}^{\mathcal{FLN}}(X; K_0^\Gamma), KO_q(*)]$$

where for $q = 0, \dots, 7$ we have

$$KO_q(*) = \mathbb{Z}, \quad \mathbb{Z}_2, \quad \mathbb{Z}_2, \quad 0, \quad \mathbb{Z}, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Note that the Tor terms vanish except possibly when $q = 1$ or 2 .

Proof. Since the three character tables are equal, the complexification from $\nu : \mathcal{R}_\mathbb{R} \rightarrow \mathcal{R}_\mathbb{C}$ and the symplectification from $\sigma : \mathcal{R}_\mathbb{C} \rightarrow \mathcal{R}_\mathbb{H}$ are isomorphisms. In the other direction, the complexification from $\eta : \mathcal{R}_\mathbb{H} \rightarrow \mathcal{R}_\mathbb{C}$ and the realification from $\rho : \mathcal{R}_\mathbb{C} \rightarrow \mathcal{R}_\mathbb{R}$ correspond to multiplication by 2. We will now compute each row of the spectral sequence in turn.

q = 0 : We have $E_{p,0}^2 = H_p^{\mathcal{FLN}}(X; KO_0^\Gamma)$ which is exactly equal to $H_p^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{R})$, the result follows from the isomorphism $H_p^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{R}) \cong H_p^{\mathcal{FLN}}(X; K_0^\Gamma) \otimes \mathbb{Z}$ and the vanishing of the Tor group.

q = 1 : The realification $\rho : \mathcal{R}_\mathbb{C} \rightarrow \mathcal{R}_\mathbb{R}$ is multiplication by 2, thus the cokernel of the map

$$\rho_* : C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{C}) \rightarrow C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{R})$$

is the modulo 2 reduction of $C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{R})$. Consider $C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{R})$ as a chain complex of abelian groups. The result follows from the Universal Coefficient Theorem in homology with \mathbb{Z}_2 coefficients applied to the homology of the chain complex $C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{R})$.

q = 2 : The complexification $\eta : \mathcal{R}_\mathbb{H} \rightarrow \mathcal{R}_\mathbb{C}$ is multiplication by 2 and $C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{C})$ is isomorphic to $C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{R})$. The result now follows as in the case $q = 1$.

q = 3 : Immediate since $KO_3^\Gamma(-) = 0$.

q = 4 : Since ν and σ are both isomorphisms, their composition gives an isomorphism of Bredon chain complexes $C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{R}) \cong C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{H})$. The result now follows as in the $q = 0$ case.

q = 5 : Since σ is an isomorphism, the cokernel of the map

$$\sigma_* : C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{C}) \rightarrow C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{H})$$

vanishes. The result follows.

q = 6 : Since ν is an isomorphism, the cokernel of the map

$$\nu_* : C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{R}) \rightarrow C_*^{\mathcal{FLN}}(X; \mathcal{R}_\mathbb{C})$$

vanishes. The result follows.

q = 7 : Immediate since $KO_7^\Gamma(-) = 0$. □

Theorem 3.2 (Theorem 1.3). *Let $\Gamma = \text{SL}_3(\mathbb{Z})$, then for $n = 0, \dots, 7$ we have*

$$KO_n^\Gamma(\underline{\mathbb{E}}\Gamma) = \mathbb{Z}^8, \quad \mathbb{Z}_2^8, \quad \mathbb{Z}_2^8, \quad 0, \quad \mathbb{Z}^8, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Proof. Let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, $\mathcal{F} = \mathcal{FLN}$ and $X = \underline{\mathrm{ESL}}_3(\mathbb{Z})$. We can now complete the calculation for the equivariant KO -homology groups. First, we recap the calculation of the Bredon chain complex with complex representation ring coefficients due to Sánchez-García. We have a chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^{11} \xrightarrow{\partial_2} \mathbb{Z}^{28} \xrightarrow{\partial_1} \mathbb{Z}^{26} \longrightarrow 0$$

where

$$\partial_3 \sim \begin{bmatrix} 1 & \mathbf{0}_{1 \times 10} \end{bmatrix}, \quad \partial_2 \sim \begin{bmatrix} I_{10} & \mathbf{0}_{10 \times 18} \\ \mathbf{0}_{10 \times 1} & \mathbf{0}_{1 \times 18} \end{bmatrix}, \quad \text{and} \quad \partial_1 \sim \begin{bmatrix} I_{18} & \mathbf{0}_{18 \times 8} \\ \mathbf{0}_{10 \times 18} & \mathbf{0}_{10 \times 8} \end{bmatrix}.$$

Therefore, the homology groups of the chain complex are isomorphic to \mathbb{Z}^8 in dimension 0 and to 0 in every other dimension.

Now, the cell stabiliser subgroups of $\mathrm{SL}_3(\mathbb{Z})$ acting on X are isomorphic to $\{1\}$, \mathbb{Z}_2 , \mathbb{Z}_2^2 , D_3 , D_4 , $\mathrm{Sym}(4)$ and D_6 . Each of these satisfies the conditions of the proposition above. This is easily checked by computing the Schur indicators of each of the irreducible characters of each group. Since the Schur indicator equals 1 in every case we conclude the three character tables for each group are equal (see for instance [19, Exercise 3.38]). Applying this to the previous calculation we obtain a single non-trivial column when $p = 0$ in the Atiyah-Hirzebruch spectral sequence and so it collapses trivially. \square

Corollary 3.3 (Corollary 1.4). *Let $\Gamma = \mathrm{GL}_3(\mathbb{Z})$, then for $n = 0, \dots, 7$ we have*

$$KO_n^\Gamma(\underline{\mathrm{E}}\Gamma) = \mathbb{Z}^{16}, \quad \mathbb{Z}_2^{16}, \quad \mathbb{Z}_2^{16}, \quad 0, \quad \mathbb{Z}^{16}, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Proof. First, note that the direct product of \mathbb{Z}_2 with any of the cell stabiliser subgroups of $\mathrm{SL}_3(\mathbb{Z})$ still satisfies the conditions of the Proposition 3.1. Now, we may compute the E^2 -page of the associated Atiyah-Hirzebruch spectral sequence by applying the Künneth formula [45, Theorem 3.6] to the calculation of each row of the E^2 -page for $\underline{\mathrm{ESL}}_3(\mathbb{Z})$. Since the spectral sequence is concentrated in a single column we have isomorphisms

$$KO_n^{\mathrm{GL}_3(\mathbb{Z})}(\underline{\mathrm{E}}\mathrm{GL}_3(\mathbb{Z})) \cong KO_n^{\mathrm{SL}_3(\mathbb{Z})}(\underline{\mathrm{E}}\mathrm{SL}_3(\mathbb{Z})) \otimes KO_n^{\mathbb{Z}_2}(*),$$

from which the result is immediate. \square

4. EQUIVARIANT K -HOMOLOGY OF FUCHSIAN GROUPS

In this section we compute the equivariant K -homology of every finitely generated Fuchsian group, that is, a finitely generated discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. The reason for this apparent detour is that we will later split the groups $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ as amalgamated free products of certain Fuchsian subgroups. Thus, we can use a Mayer-Vietoris type argument to compute their K -homology.

Note that Theorem 4.1(a) was computed in [31] along with a more general result for cocompact NEC groups. Moreover, their integral cohomology was determined by the author in [23]. An introduction to Fuchsian groups is provided by [27].

The computation is made easier by the fact that every finitely generated Fuchsian group is described by piece of combinatorial data called a *signature* [27, Chapter 4.3]. Indeed, a Fuchsian group of *signature* $[g, s; m_1, \dots, m_r]$ has presentation

$$\left\langle a_1, \dots, a_{2g}, c_1, \dots, c_s, d_1, \dots, d_r \mid \prod_{i=1}^g [a_i, a_{g+i}] \prod_{j=1}^r d_j \prod_{k=1}^s c_k = d_1^{m_1} = \dots = d_r^{m_r} = 1 \right\rangle$$

and acts on the hyperbolic plane $\mathbb{R}\mathbf{H}^2$ with a $4g + 2s + 2r$ sided fundamental polygon. The tessellation of the polygon under the group action has $1 + s + r$ orbits of vertices, s of which are on the boundary $\partial\mathbb{R}\mathbf{H}^2$, $2g + s + r$ orbits of edges and 1 orbit of faces. All edge and face stabilisers are trivial. All vertex stabilisers are trivial except for r orbits of vertices, each of which is stabilised by some \mathbb{Z}_{m_j} . Note that if $s = 0$ we say Γ is *cocompact*.

The signature also describes a quotient 2-orbifold which is homeomorphic to a genus g surface with s points removed. The orbifold data is then given by the r marked points, each corresponding to one of the m_j , or equivalently a maximal conjugacy of finite subgroups.

If $r = 0$, we do not write any m_j in the signature. In which case Γ has signature $[g, s;]$, is torsionfree, and isomorphic to either the fundamental group of a genus g surface, or a free group of rank $2g + s - 1$.

Theorem 4.1. *Let Γ be a Fuchsian group of signature $[g, s; m_1, \dots, m_r]$.*

(a) *If $s = 0$ then,*

$$K_n^\Gamma(\underline{E}\Gamma) = K_n(C_r^*(\Gamma)) = \begin{cases} \mathbb{Z}^{2-r+\sum_{j=1}^r m_j} & n \text{ even,} \\ \mathbb{Z}^{2g} & n \text{ odd.} \end{cases}$$

(b) *If $s > 0$ then,*

$$K_n^\Gamma(\underline{E}\Gamma) = K_n(C_r^*(\Gamma)) = \begin{cases} \mathbb{Z}^{1-r+\sum_{j=1}^r m_j} & n \text{ even,} \\ \mathbb{Z}^{2g+s-1} & n \text{ odd.} \end{cases}$$

Proof of (a). Let Γ be a Fuchsian group of signature $[g, s; m_1, \dots, m_r]$ with $s = 0$ and $\mathcal{F} = \mathcal{FIN}$. Since Γ satisfies the Baum-Connes conjecture [21] it is enough to compute the equivariant K -homology. The hyperbolic plane with the induced cell structure of the Γ action is a model for $\underline{E}\Gamma$ (see for instance [32]). Recall that the cell structure has $r + 1$ orbits of vertices, $2g + r$ orbits of edges and exactly 1 orbit of 2-cells. One vertex v_0 is stabilised by the trivial group and for $j = 1, \dots, r$ the vertex v_j is stabilised by \mathbb{Z}_{m_j} . Thus, we have a Bredon chain complex

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^r \mathcal{R}_{\mathbb{C}}(\mathbb{Z}_{m_j}) \right) \xleftarrow{\partial_1} \mathbb{Z}^{2g+r} \xleftarrow{\partial_2} \mathbb{Z} \longleftarrow 0,$$

and substituting in $\mathcal{R}_{\mathbb{C}}(\mathbb{Z}_{m_j}) = \mathbb{Z}^{m_j}$ we obtain

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}^{m_j} \right) \xleftarrow{\partial_1} \mathbb{Z}^{2g+r} \xleftarrow{\partial_2} \mathbb{Z} \longleftarrow 0.$$

We fix the following basis for each chain group: In degree 0 we have generators $x_{j,l}$, for $j = 1, \dots, r$ and $l = 1, \dots, m_j$, and the generator z . In degree 1 we have a_1, \dots, a_{2g} and y_1, \dots, y_r , and in degree 2, the generator w . An easy calculation yields that $\partial_2(w) = 0$, $\partial_1(a_i) = 0$, and $\partial_1(y_j) = \sum_{l=1}^{m_j} x_{j,l} - z$. Thus,

$$H_n^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}}) = \begin{cases} \mathbb{Z}^{1+\sum_{j=1}^r (m_j-1)} & \text{if } n = 0; \\ \mathbb{Z}^{2g} & \text{if } n = 1; \\ \mathbb{Z} & \text{if } n = 2; \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows from the collapsed Atiyah-Hirzebruch spectral sequence given in [34, Theorem 5.27] and we obtain $K_0^\Gamma(\underline{E}\Gamma) = H_0^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}}) \oplus H_2^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}})$ and $K_1^\Gamma(\underline{E}\Gamma) = H_1^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}})$. \square

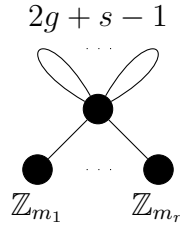


FIGURE 1. A graph of groups for a non-cocompact Fuchsian group.

Proof of (b). Let Γ be a Fuchsian group of signature $[g, s; m_1, \dots, m_r]$ with $s > 0$ and let $\mathcal{F} = \mathcal{FLN}$. In this case we can rearrange the presentation of Γ such that we have a splitting of Γ as an amalgamated free product $\Gamma \cong \mathbb{Z}^{s-1} * \mathbb{Z}_{m_1} * \dots * \mathbb{Z}_{m_r}$. Now, Γ splits as a finite graph of finite groups (Figure 1) and it is easy to see that the Bass-Serre tree of Γ is a model for $\underline{E}\Gamma$.

We will first compute the Bredon homology $H_*^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}})$ with coefficients in the representation ring, then apply the equivariant Atiyah-Hirzebruch spectral sequence. We have a Bredon chain complex

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^r \mathcal{R}_{\mathbb{C}}(\mathbb{Z}_{m_j}) \right) \xleftarrow{\partial} \mathbb{Z}^{2g+s-1} \longleftarrow 0,$$

substituting in $\mathcal{R}_{\mathbb{C}}(\mathbb{Z}_{m_j}) = \mathbb{Z}^{m_j}$ we obtain

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}^{m_j} \right) \xleftarrow{\partial} \mathbb{Z}^{2g+s-1} \longleftarrow 0.$$

Let the first non-zero term have generating set $\langle x_{j,l}, z \mid j = 1, \dots, r, l = 1, \dots, m_j \rangle$ and the second term $\langle a_1, \dots, a_{2g}, c_1, \dots, c_{s-1}, d_1, \dots, d_r \rangle$. It is easy to see the differential ∂ is given by $\partial(a_i) = \partial(b_i) = \partial(c_k) = 0$ and $\partial(d_j) = \sum_{l=1}^{m_j} x_{j,l} - z$. It follows that $H_0^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}}) = \mathbb{Z}^{1+\sum_{j=1}^r (m_j-1)}$, $H_1^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}}) = \mathbb{Z}^{2g+s-1}$ and 0 otherwise. The result now follows from the collapsed Atiyah-Hirzebruch spectral sequence [34, Theorem 5.27]. In particular, we have $K_n^{\Gamma}(\underline{E}\Gamma) = H_n^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}})$ for $n = 0, 1$. \square

5. COMPUTATIONS FOR $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ AND $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$

5.1. Preliminaries. In an abuse of notation, throughout this section we will denote the image $\{\pm A\}$ of a matrix $A \in \mathrm{SL}_2(\mathbb{R})$ in $\mathrm{PSL}_2(\mathbb{R})$ by the matrix A . Recall that for p a prime we have $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) = \mathrm{PSL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{PSL}_2(\mathbb{Z})$, where $\Gamma_0(p)$ is the level p Hecke principle congruence subgroup (see for instance Serre's book "Trees" [47]). The amalgamation is specified by two embeddings of the congruence subgroup $\Gamma_0(p)$ into $\mathrm{PSL}_2(\mathbb{Z})$. The first is given by

$$\Gamma_0(p) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : c \equiv 0 \pmod{p} \right\}$$

and the second via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & pb \\ p^{-1}c & d \end{bmatrix}.$$

In light of this we will collect some facts about each of the groups in the amalgamation. We begin by recording (Table 3) the Fuchsian signatures and the associated Bredon homology for each of the groups $\Gamma_0(p)$ and $\mathrm{PSL}_2(\mathbb{Z})$. Note that when $p \equiv 11 \pmod{12}$ the group $\Gamma_0(p)$ is free.

Lemma 5.1. *The signatures and Bredon homology groups of $\Gamma_0(p)$ and $\mathrm{PSL}_2(\mathbb{Z})$ are given in Table 3.*

p	Signature of $\Gamma_0(p)$	$H_0^{\mathcal{F}}(\underline{E}\Gamma_0(p); R_{\mathbb{C}})$	$H_1^{\mathcal{F}}(\underline{E}\Gamma_0(p); R_{\mathbb{C}})$
2	$[0, 2; 2]$	\mathbb{Z}^2	\mathbb{Z}
3	$[0, 2; 3]$	\mathbb{Z}^3	\mathbb{Z}
$p \equiv 1 \pmod{12}$	$[0, \frac{1}{6}(p-7) + 1; 2, 2, 3, 3]$	\mathbb{Z}^7	$\mathbb{Z}^{\frac{1}{6}(p-7)}$
$p \equiv 5 \pmod{12}$	$[0, \frac{1}{6}(p+1) + 1; 2, 2]$	\mathbb{Z}^3	$\mathbb{Z}^{\frac{1}{6}(p+1)}$
$p \equiv 7 \pmod{12}$	$[0, \frac{1}{6}(p-1) + 1; 3, 3]$	\mathbb{Z}^5	$\mathbb{Z}^{\frac{1}{6}(p-1)}$
$p \equiv 11 \pmod{12}$	$[0, \frac{1}{6}(p+7) + 1;]$	\mathbb{Z}	$\mathbb{Z}^{\frac{1}{6}(p+7)}$
$\mathrm{PSL}_2(\mathbb{Z})$	$[0, 1; 2, 3]$	\mathbb{Z}^4	0

TABLE 3. Fuchsian signatures and Bredon homology groups of $\mathrm{P}\Gamma_0(p)$.

Proof. Let $[0, s; m_1, \dots, m_r]$ be the signature of $\Gamma_0(p)$. We will first compute the ordinary cohomology groups of $\Gamma_0(p)$, then using these we will deduce the signatures, finally the Bredon homology may then be read off of Theorem 4.1(b). Our computation of the cohomology will be near identical to the computation in [1, Section 2]. The key difference is that the modules in [1] are for the lifts of $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathbb{R})$ whereas we are always working with the projectivised groups (see the discussion after [1, Proposition 2.2]). Note that the fact the signature of $\mathrm{PSL}_2(\mathbb{Z})$ is $[0, 1; 2, 3]$ is well known.

Let $G = \mathrm{PSL}_2(p)$ and let Q be the subgroup of equivalence classes of matrices with lower left hand entry equal to zero. Clearly, $Q \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{1}{2}(p-1)}$ (unless $p = 2, 3$ where $Q \cong \mathbb{Z}_p$). Each $\Gamma_0(p)$ fits into a short exact sequence with normal subgroup a congruence subgroup $\Gamma(p)$ isomorphic to a free group and quotient Q .

Now, recall [47, Example 4.2(c)] that $\mathrm{PSL}_2(\mathbb{Z})$ acts on a tree \mathcal{T} with fundamental domain an edge. Moreover, G acts on $\mathcal{T}/\Gamma(p)$. The stabiliser subgroups for both actions are \mathbb{Z}_2 and \mathbb{Z}_3 for the vertices and trivial for the edges. It follows $\Gamma_0(p)$ acts freely on $EQ \times \mathcal{T}$ and so $EQ \times_Q \mathcal{T}/\Gamma_0(p)$ is a model for $B\Gamma_0(p)$.

Let C^* denote the cellular cochains on the B -CW complex $\mathcal{T}/\Gamma(p)$, then by [1, Section 2] we have the following isomorphisms of B -modules

$$C^0 := \mathbb{Z}[G/\mathbb{Z}_2] \mid_B \oplus \mathbb{Z}[G/\mathbb{Z}_3] \mid_B \quad \text{and} \quad C^1 := \mathbb{Z}[G] \mid_B.$$

As in [1, Section 2] we have a long exact sequence

$$\dots \rightarrow H^n(\Gamma_0(p); \mathbb{Z}) \rightarrow H^n(B; C^0) \rightarrow H^n(B; C^1) \rightarrow H^{n+1}(\Gamma_0(p); \mathbb{Z}) \rightarrow \dots$$

Since C^0 is a permutation module, $H^1(B; C^0) = 0$. Calculating ranks yields that $H^1(\Gamma_0(p); \mathbb{Z}) = \mathbb{Z}^{N(p)}$ where

$$N(p) = 1, \quad 1, \quad \frac{1}{6}(p-7), \quad \frac{1}{6}(p+1), \quad \frac{1}{6}(p-1), \quad \frac{1}{6}(p+7),$$

ordered as in Table 3. In [23, Theorem 1.4(b)] it is shown that a Fuchsian group Γ of signature $[0, s; m_1, \dots, m_r]$ has $H^1(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^{s-1}$. In particular, we deduce the signature of $\Gamma_0(p)$ must have the form $[0, N(p) + 1; m_1, \dots, m_r]$.

Now, C^1 is a free B -module and so we have an isomorphism $H^{2n}(\Gamma_0(p); \mathbb{Z}) \cong H^2(B; C^0)$ for all $n \geq 1$. As in [1, Proposition 2.3] we obtain that

$$H^{2n}(\Gamma_0(p); \mathbb{Z}) \cong \mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_6^2, \quad \mathbb{Z}_2^2, \quad \mathbb{Z}_3^2, \quad 0,$$

ordered as in Table 3. The result now follows from the following three facts. Firstly, for a Fuchsian group Γ of signature $[0, s; m_1, \dots, m_r]$, each m_j corresponds to a conjugacy class of maximal finite cyclic subgroups \mathbb{Z}_{m_j} . Secondly, by the proof of [23, Theorem 1.4(b)], each maximal conjugacy

class of finite cyclic subgroups \mathbb{Z}_{m_j} contributes a $\widetilde{H}^*(\mathbb{Z}_{m_j}; \mathbb{Z})$ summand to $\widetilde{H}^*(\Gamma; \mathbb{Z})$. Thirdly, $\mathrm{PSL}_2(\mathbb{Z})$ and hence $\Gamma_0(p)$ has no elements of order 6. \square

Remark 5.2. Let $\widetilde{\Gamma}_0(p)$ denote the lift of $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathbb{R})$. An alternative computation of $H^*(\Gamma_0(p); \mathbb{Z})$ can be achieved by back solving the Lyndon-Hochschild-Serre spectral sequence (see for instance [7, Chapter VII.6]) for the group extension $\mathbb{Z}_2 \twoheadrightarrow \widetilde{\Gamma}_0(p) \twoheadrightarrow \Gamma_0(p)$ which takes the form

$$E_2^{*,*} = H^*(\Gamma_0(p); H^*(\mathbb{Z}_2; \mathbb{Z})) \Rightarrow H^*(\widetilde{\Gamma}_0(p); \mathbb{Z})$$

using the cohomology calculations for $\widetilde{\Gamma}_0(p)$ in [1].

We shall also record the conjugacy classes of finite order elements of $\Gamma_0(p)$ and $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. Note that the only conjugacy classes of finite subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ are one class of groups isomorphic to \mathbb{Z}_2 and one to \mathbb{Z}_3 since $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. The conjugacy classes of finite subgroups of $\Gamma_0(p)$ can be read off of the signature, there is exactly one of order m_j for each $j = 1, \dots, r$.

Lemma 5.3. *The number of conjugacy classes of finite order elements in $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ are those given in Table 4.*

	$p = 2$	$p = 3$	$p \equiv 1 \pmod{12}$	$p \equiv 5 \pmod{12}$	$p \equiv 7 \pmod{12}$	$p \equiv 11 \pmod{12}$
Identity	1	1	1	1	1	1
Order 2	1	2	1	1	2	2
Order 3	4	2	2	4	2	4
Total	6	5	4	6	5	7

TABLE 4. Number of conjugacy classes of finite order elements of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.

Proof. The result follows from the following observation: If there is a conjugacy of elements of order 2 (resp. 3) in $\Gamma_0(p)$, then each of class of elements of order 2 (resp. 3) in $\mathrm{PSL}_2(\mathbb{Z})$ fuses in $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. To see this, consider an element in the first copy of $\mathrm{PSL}_2(\mathbb{Z})$, conjugate it to an element in $\Gamma_0(p)$, and then conjugate it to an element in the other copy of $\mathrm{PSL}_2(\mathbb{Z})$. \square

Lemma 5.4. *Both $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ satisfy the Baum-Connes Conjecture.*

Proof. Since $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) = \mathrm{PSL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{PSL}_2(\mathbb{Z})$, the Bass-Serre tree of the amalgamation is a locally-finite 1-dimensional contractible $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ -CW complex. Moreover, each of the stabilisers Γ_c have $\mathrm{cd}_{\mathbb{Q}}(\Gamma_c) = 1$, being a graph of finite groups. Now, we apply [34, Corollary 5.14] to see that the stabilisers satisfy Baum-Connes and [34, Theorem 5.13] to see that $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ does. The proof is identical for $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. \square

5.2. Computations. There is a long exact Mayer-Vietoris sequence for computing the Bredon homology of an amalgamated free product.

Theorem 5.5. [34, Corollary 3.32] *Let $\Gamma = H *_L K$ and let M be a Bredon module. There is a long exact Mayer-Vietoris sequence:*

$$\begin{array}{ccccc} \dots & \longrightarrow & H_n^{\mathrm{FLN}}(L; M) & \longrightarrow & H_n^{\mathrm{FLN}}(H; M) \oplus H_n^{\mathrm{FLN}}(K; M) \\ & & & & \downarrow \\ \dots & \longleftarrow & H_{n-1}^{\mathrm{FLN}}(L; M) & \longleftarrow & H_n^{\mathrm{FLN}}(\underline{\mathbf{E}}\Gamma; M) \end{array}$$

We are now ready to compute the K -theory of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.

Theorem 5.6 (Theorem 1.5). *Let p be a prime and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, then $K_n^\Gamma(\underline{\mathbb{E}}\Gamma)$ is a free abelian group with rank as given in Table 1. Moreover, since the Baum-Connes Conjecture holds for Γ we have $K_*^\Gamma(\underline{\mathbb{E}}\Gamma) \cong K_*^{\mathrm{top}}(C_r^*(\Gamma))$.*

Proof. There are 6 cases to consider, the two cases when $p = 2, 3$ and the four cases given by $p \equiv 1, 5, 7, 11 \pmod{12}$. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\mathcal{F} = \mathcal{FIN}$. In each case we have the following long exact Mayer-Vietoris sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_2^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) & \longrightarrow & H_1^\mathcal{F}(\Gamma_0(p); \mathcal{R}_\mathbb{C}) & \longrightarrow & (H_1^\mathcal{F}(\mathrm{PSL}_2(\mathbb{Z}); \mathcal{R}_\mathbb{C}))^2 \\
 & & & & & & \downarrow \\
 & & (H_0^\mathcal{F}(\mathrm{PSL}_2(\mathbb{Z}); \mathcal{R}_\mathbb{C}))^2 & \longleftarrow & H_0^\mathcal{F}(\Gamma_0(p); \mathcal{R}_\mathbb{C}) & \longleftarrow & H_1^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) \\
 & & \downarrow & & & & \\
 & & H_0^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) & \longrightarrow & 0 & &
 \end{array}$$

We have computed the Bredon homology groups of $\mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma_0(p)$ in Table 3. Thus, we can separate the above sequence into two sequences. Indeed, $H_1^\mathcal{F}(\mathrm{PSL}_2(\mathbb{Z}); \mathcal{R}_\mathbb{C}) = 0$, so it follows that $H_2^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) \cong H_1^\mathcal{F}(\Gamma_0(p); \mathcal{R}_\mathbb{C})$. The other sequence is then given by the remaining terms.

We will treat the case $p = 2$, the other cases proceed identically. We have $H_2^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) = \mathbb{Z}$ and an exact sequence

$$0 \longrightarrow H_1^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^8 \longrightarrow H_0^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) \longrightarrow 0.$$

We now compute the colimit $H_0^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) = \mathrm{colim}_{\Gamma/H \in \mathrm{Or}_{\mathcal{F}}(\Gamma)} \mathcal{R}_\mathbb{C}(H)$. Since we have a complete description of the conjugacy classes of finite subgroups of Γ and the only inclusions are given by $\{1\} \hookrightarrow \mathbb{Z}_2$ and $\{1\} \hookrightarrow \mathbb{Z}_3$, it follows that $H_0^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) = \mathbb{Z}^6$. Moreover, for the sequence to be exact, it follows the map $\mathbb{Z}^2 \rightarrow \mathbb{Z}^8$ must be an isomorphism onto the kernel of the first map. In particular $H_1^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) = 0$.

The result now follows from the collapsed Atiyah-Hirzebruch spectral sequence given in [34, Theorem 5.27] and we obtain $K_0^\Gamma(\underline{\mathbb{E}}\Gamma) = H_0^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C}) \oplus H_2^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C})$ and $K_1^\Gamma(\underline{\mathbb{E}}\Gamma) = H_1^\mathcal{F}(\Gamma; \mathcal{R}_\mathbb{C})$. We record the Bredon homology groups for the remaining cases in Table 5, the reader can easily verify these. Note that they are always torsion-free and so are completely determined by their \mathbb{Z} -rank. \square

	$p = 2$	$p = 3$	$p \equiv 1 \pmod{12}$	$p \equiv 5 \pmod{12}$	$p \equiv 7 \pmod{12}$	$p \equiv 11 \pmod{12}$
$n = 0$	6	5	4	6	5	7
$n = 1$	0	0	3	1	2	0
$n = 2$	1	1	$\frac{1}{6}(p-7)$	$\frac{1}{6}(p+1)$	$\frac{1}{6}(p-1)$	$\frac{1}{6}(p+7)$

TABLE 5. \mathbb{Z} -rank of the Bredon homology of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.

The computation for $\mathrm{SL}_2(\mathbb{Z}[1/p])$ is almost entirely analogous. We highlight the differences below.

Theorem 5.7 (Theorem 1.6). *Let p be a prime and $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. Then $K_n^\Gamma(\underline{\mathbb{E}}\Gamma)$ is additively isomorphic to the direct sum of two copies of the corresponding equivariant K -homology group of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.*

Proof (Sketch). Let $\mathcal{F} = \mathcal{FIN}$. First, we must compute the Bredon homology of the lifts of $\mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma_0(p)$ to $\mathrm{SL}_2(\mathbb{R})$. For this we use the graph of groups in Figure 1 and note that now every edge group is the same central copy of \mathbb{Z}_2 and the vertex groups change as follows: The vertices with trivial vertex group now have vertex group the same central copy of \mathbb{Z}_2 . The vertex groups isomorphic to \mathbb{Z}_2 are now \mathbb{Z}_4 and the vertex groups isomorphic to \mathbb{Z}_3 are now \mathbb{Z}_6 (each extended by the central \mathbb{Z}_2). Computing the Bredon homology we find that in each case it is isomorphic to the direct sum of two copies of the corresponding Bredon homology group in the projective case.

Now, we apply the long exact Mayer-Vietoris sequence to the amalgamated free product decomposition of $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. Since $H_1^{\mathcal{F}}(\mathrm{SL}_2(\mathbb{Z}); \mathcal{R}_{\mathbb{C}}) = 0$, like in the projective case, the sequence splits into two exact sequences. The computation of $H_2^{\mathcal{F}}(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]))$ is immediate as before and is additively isomorphic to the direct sum of two copies of $H_2^{\mathcal{F}}(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]); \mathcal{R}_{\mathbb{C}})$.

To compute the zeroth and first homology groups we will again use the colimit isomorphism $H_0^{\mathcal{F}}(\Gamma; \mathcal{R}_{\mathbb{C}}) = \mathrm{colim}_{\Gamma/H \in \mathbf{Or}_{\mathcal{F}}(\Gamma)} \mathcal{R}_{\mathbb{C}}(H)$. To use this we obtain a count of the total number of conjugacy classes of finite order elements in $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. To do this use a near identical argument to Lemma 5.3 that takes into account the central \mathbb{Z}_2 subgroup. It follows that the number of conjugacy classes of elements of finite order in $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ is equal to twice the number for the corresponding projective group. It follows that the colimit computation for $H_0^{\mathcal{F}}(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]); \mathcal{R}_{\mathbb{C}})$ is additively isomorphic to the direct sum of two copies of $H_0^{\mathcal{F}}(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]); \mathcal{R}_{\mathbb{C}})$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$.

From here one computes $H_1^{\mathcal{F}}(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]); \mathcal{R}_{\mathbb{C}})$ in an identical manner to the projective case. The resulting groups are isomorphic to the direct sum of two copies of the corresponding Bredon homology groups in the projective case. The result now follows from the collapsed Atiyah-Hirzebruch spectral sequence given in [34, Theorem 5.27] and we obtain $K_0^{\Gamma}(\underline{E}\Gamma) = H_0^{\mathcal{F}}(\Gamma; \mathcal{R}_{\mathbb{C}}) \oplus H_2^{\mathcal{F}}(\Gamma; \mathcal{R}_{\mathbb{C}})$ and $K_1^{\Gamma}(\underline{E}\Gamma) = H_1^{\mathcal{F}}(\Gamma; \mathcal{R}_{\mathbb{C}})$. \square

6. THE UNSTABLE GROMOV-LAWSON-ROSENBERG CONJECTURE

Given a smooth closed n -manifold M a classical question is to ask whether M admits a Riemannian metric of positive scalar curvature. In a vast generalisation of the Atiyah-Singer index theorem, Rosenberg [40] exhibits a class in $KO_n^{\mathrm{top}}(C_r^*(\pi_1(M)))$ which is an obstruction to M admitting a metric of positive scalar curvature.

More precisely, let M be a closed spin n -manifold and $f: M \rightarrow B\Gamma$ be a continuous map for some discrete group Γ . Let $\alpha: \Omega_n^{\mathrm{Spin}}(B\Gamma) \rightarrow KO_n^{\mathrm{top}}(C_r^*(\Gamma))$ be the index of the Dirac operator. If M admits a metric of positive scalar curvature, then $\alpha[M, f] = 0 \in KO_n^{\mathrm{top}}(C_r^*(\Gamma))$

Conjecture 6.1 (The Unstable GLR Conjecture). *Let M be a closed spin n -manifold and $\Gamma = \pi_1(M)$. If $f: M \rightarrow B\Gamma$ is a continuous map which induces the identity on the fundamental groups, then M admits a metric positive scalar curvature if and only if $\alpha[M, f] = 0 \in KO_n^{\mathrm{top}}(C_r^*(\Gamma))$.*

The conjecture has been verified in the case of some finite groups [50, 41, 28, 39], when the group has periodic cohomology, torsion-free groups for which the dimension of $B\Gamma$ is less than 9 [24], and cocompact Fuchsian groups [15]. However, there are counterexamples: The first is due to Schick [48] who disproves the conjecture for the direct product $\mathbb{Z}^4 \times \mathbb{Z}_n$ when n is odd; while other instances have been constructed in [24]. For more information on the Unstable GLR Conjecture the reader should consult [24] and the references therein.

6.1. Proof of Theorem 1.7. We will now prove the conjecture for a large class of groups. Our proof is structurally similar to the proof by Davis-Pearson [15] so we will summarise their method and highlight any differences.

Let ko be the connective cover of KO with covering map p and let D be the ko -orientation of spin bordism. The map α (from above) is obtained by the following composition

$$\Omega_n^{\text{Spin}}(B\Gamma) \xrightarrow{D} ko_n(B\Gamma) \xrightarrow{p} KO_n(B\Gamma) \xrightarrow{\mu_{\mathbb{R}}} KO_n^{\text{top}}(C_r^*(\Gamma))$$

We note that $ko_n(*) = 0$ for $n < 0$ and that p is an isomorphism for $n \geq 0$ on the one point space.

Recall from the introduction that a group Γ satisfies:

(M) If every finite subgroup is contained in a unique maximal finite subgroup.

(NM) If M is a maximal finite subgroup of Γ , then the normaliser $N_{\Gamma}(M)$ of M is equal to M .

(BC) If Γ satisfies the Baum-Connes conjecture.

(PFS) If all maximal finite subgroups of Γ are odd order and have periodic cohomology.

Proposition 6.2. *Let Γ be a group satisfying (BC), (M), and (NM). Let Λ be a set of conjugacy classes of maximal finite subgroups of Γ . There is a commutative diagram with exact rows*

$$\begin{array}{ccccccc} \widetilde{KO}_{n+1}(\underline{B}\Gamma) & \longrightarrow & \bigoplus_{(H) \in \Lambda} \widetilde{KO}_n(BH) & \longrightarrow & \widetilde{KO}_n(B\Gamma) & \longrightarrow & \widetilde{KO}_n(\underline{B}\Gamma) \\ \downarrow \text{id} & & \downarrow \mu_{\mathbb{R}} & & \downarrow \mu_{\mathbb{R}} & & \downarrow \text{id} \\ \widetilde{KO}_{n+1}(\underline{B}\Gamma) & \longrightarrow & \bigoplus_{(H) \in \Lambda} \widetilde{KO}_n^{\text{top}}(C_r^*(H)) & \longrightarrow & \widetilde{KO}_n^{\text{top}}(C_r^*(\Gamma)) & \longrightarrow & \widetilde{KO}_n(\underline{B}\Gamma). \end{array}$$

Proof. First, since Γ satisfies (BC), (M) and (NM) by either [13, Corollary 3.13] or the proof of [13, Theorem 4.1] for any constant functor $\mathbf{E}_c: \mathbf{Or}_{\mathcal{F}}(\Gamma) \rightarrow \mathbf{Spectra}$ by $\Gamma/H \mapsto \mathbf{E}$ there are long exact sequences

$$\dots \longrightarrow \bigoplus_{(H) \in \Lambda} H_n(BH; \mathbf{E}) \longrightarrow \left(\bigoplus_{(H) \in \Lambda} \pi_n(\mathbf{E}) \right) \oplus H_n(B\Gamma; \mathbf{E}) \longrightarrow H_n(\underline{B}\Gamma; \mathbf{E}) \longrightarrow \dots$$

and

$$\dots \longrightarrow \bigoplus_{(H) \in \Lambda} \widetilde{H}_n(BH; \mathbf{E}) \longrightarrow \widetilde{H}_n(B\Gamma; \mathbf{E}) \longrightarrow \widetilde{H}_n(\underline{B}\Gamma; \mathbf{E}) \longrightarrow \dots$$

The result then follows a diagram chase exactly as in [15, Proposition 4], taking $\mathbf{E} = \mathbf{KO}$ and the isomorphism

$$\pi_n \left(\text{hocolim}_{\mathbf{Or}_{\mathcal{F}}(\Gamma)}(\mathbf{E}_c) \right) \cong H_n(\underline{B}\Gamma; \mathbf{E}).$$

□

Theorem 6.3 (Theorem 1.7). *Let Γ be a group satisfying (BC), (M), (NM) and (PFS). If $\underline{B}\Gamma$ is finite and has dimension at most 9, then the Unstable Gromov-Lawson-Rosenberg Conjecture holds for Γ .*

Proof. Let \mathbf{ko} be the spectrum of the connective cover of \mathbf{KO} . Via the cover we obtain a natural transformation $p: \mathbf{ko}_c \rightarrow \mathbf{KO}_c$ of constant $\mathbf{Or}_{\mathcal{F}}(\Gamma)$ - $\mathbf{Spectra}$. From the previous proposition we

obtain a commutative diagram

$$\begin{array}{ccccccc}
\widetilde{ko}_{n+1}(\underline{B}\Gamma) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{ko}_n(BH) & \longrightarrow & \widetilde{ko}_n(B\Gamma) & \longrightarrow & \widetilde{ko}_n(\underline{B}\Gamma) \\
\downarrow p & & \downarrow \mu_{\mathbb{R}} \circ p & & \downarrow \mu_{\mathbb{R}} \circ p & & \downarrow p \\
\widetilde{KO}_{n+1}(\underline{B}\Gamma) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n^{\text{top}}(C_r^*(H)) & \longrightarrow & \widetilde{KO}_n^{\text{top}}(C_r^*(\Gamma)) & \longrightarrow & \widetilde{KO}_n(\underline{B}\Gamma).
\end{array}$$

(†)

By Joachim-Schick [24, Lemma 2.6] p is an isomorphism for $n \geq 6$ and an injection for $n = 5$. Now, suppose that $n \geq 5$ so we are in the setting of the GLR conjecture. Consider an element

$$\beta \in K := \text{Ker}(\mu_{\mathbb{R}} \circ p : ko_n(B\Gamma) \rightarrow KO_n(C_r^*(\Gamma; \mathbb{R})))$$

and note that $K \cong \text{Ker}(\mu_{\mathbb{R}} \circ p : \widetilde{ko}_n(B\Gamma) \rightarrow \widetilde{KO}_n^{\text{top}}(C_r^*(\Gamma)))$. Combining the diagram (†) with the isomorphism $p: \widetilde{ko}_n(\underline{B}\Gamma) \rightarrow \widetilde{KO}_n^{\text{top}}(\underline{B}\Gamma)$ for $n \geq 6$ (injection for $n = 5$), we can deduce that there exists

$$\gamma \in \text{Ker} \left(\bigoplus_{(H)\in\Lambda} ko_n(BH) \rightarrow \bigoplus_{(H)\in\Lambda} KO_n(BH) \right)$$

which maps to β .

For a group L let $ko_n^+(BL)$ be the subgroup of $ko_n(BL)$ given by $D[M, f]$ where M is a positively curved spin manifold and f is a continuous map. In [6] the authors prove for any finite group of odd order with periodic cohomology H , that $ko_n^+(BH) = \text{Ker}(\mu_{\mathbb{R}} \circ p : ko_n(BH) \rightarrow KO_n^{\text{top}}(C_r^*(H)))$. Thus, we have $\gamma \in ko_n^+(BH)$ and $\beta \in ko_n^+(B\Gamma)$. Now, in [51] it is proven that if $D[M, f] \in ko_n^+(BG)$, then M admits a metric of positive scalar curvature. In particular, we are done. \square

Remark 6.4. It is unclear whether the assumption that the finite subgroups having odd order can be dropped. Indeed, it was pointed out to the author by J.F. Davis that the statement of [6, Corollary 2.2] contains a misprint. One should instead (in the notation of [6]) define $\mathcal{Y}_n(B\pi)$ to be the kernel of $A \circ p$ restricted to the subgroup $D(\Omega^n(B\pi)) \subseteq ko_n(B\pi)$. After making this correction, the statements and proofs in the paper are correct. This error caused a mistake in the main theorem of [15] which is only correct if one restricts to Fuchsian groups whose torsion only has odd order.

Corollary 6.5. *Let M be a connected closed spin n -manifold and let $\Gamma = \pi_1(M)$ be a group satisfying (M), (NM) and (PFS). Suppose the assembly map $\mu_{\mathbb{R}}$ is injective and $\underline{B}\Gamma$ is finite of dimension N . If $n \geq \max\{5, N - 4\}$, then M admits a metric with positive scalar curvature if and only if $\alpha[M, f] = 0$.*

6.2. Some examples. In this section we will detail some applications of Theorem 1.7 to various families of groups. These results are new whenever the groups involved are infinite and have torsion.

6.2.1. Graphs of groups. In [42, Theorem 3.1] it is shown that the fundamental groups of graphs of groups with vertex groups satisfying (M) and (NM) and with torsion-free edge groups, satisfy (M) and (NM). It follows that we have the following combination theorem:

Corollary 6.6. *The Γ be a finitely presented fundamental group of a graph of groups such that the vertex groups satisfy (BC), (M) and (NM) and the edge groups are torsion-free and satisfy (BC). If the vertex groups satisfy (PFS) and $\underline{B}\Gamma$ has dimension at most 9, then Γ satisfies the Unstable GLR Conjecture.*

6.2.2. *3-manifold groups.* In [42, Section 3.3] it is shown that 3-manifold groups satisfy (M) and (NM) and there is a well known classification of finite subgroups of orientable connected 3-manifold groups. These are exactly the groups which act freely on the 3-sphere and so have periodic cohomology by [7, Chapter VI.9]. In [33] it is shown 3-manifold groups satisfy (BC). Applying Theorem 1.7 we obtain the following result (which is new whenever Γ is infinite and contains torsion):

Corollary 6.7. *Let M be a closed orientable connected 3-manifold with fundamental group Γ . If Γ has no elements of order 2 then Γ satisfies the Unstable GLR Conjecture.*

6.2.3. *One-relator groups.* In [13, Page 32] it is shown that one-relator groups satisfy (BC), (M) and (NM) and admit a two dimensional model for $\underline{E}\Gamma$. Applying Theorem 1.7 we obtain the following result (which to the authors knowledge is new whenever Γ is infinite and contains torsion):

Corollary 6.8. *Let $\Gamma = \langle X \mid w \rangle$ be a finitely generated one-relator group and suppose w has odd order when interpreted in Γ , then Γ satisfies the Unstable GLR Conjecture.*

6.2.4. *Hilbert modular groups.* Let k be a totally real number field of degree n and \mathcal{O}_k be its ring of integers. The Hilbert modular group of k is defined to be $\mathrm{PSL}_2(\mathcal{O}_k)$ and is a lattice in $\mathrm{PSL}_2(\mathbb{R})^n$. Note that if $k = \mathbb{Q}$ then we recover the classical modular group $\mathrm{PSL}_2(\mathbb{Z})$. Properties (BC), (M) and (NM) are given in [11, Lemma 4.3]. Applying Theorem 1.7 we obtain the following new result:

Corollary 6.9. *Let k be a totally real number field with degree less than or equal to 4. Let $\Gamma \leq \mathrm{PSL}_2(\mathcal{O}_k)$ be finitely presented. If all finite subgroups of Γ are cyclic of odd order then Γ satisfies the Unstable GLR Conjecture.*

Proof. This follows from the fact every finite subgroup of $\mathrm{PSL}_2(\mathbb{R})^n$ is a product of finite cyclic groups. \square

6.2.5. *Subgroups of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for $p \equiv 11 \pmod{12}$.* In this section we will prove the result that many subgroups of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for $p \equiv 11 \pmod{12}$ satisfy the conditions of Theorem 1.7. The result is new whenever the subgroup is infinite, has torsion, and is not isomorphic to a Fuchsian group. We will also compute the KO -theory of $C_r^*(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]); \mathbb{R})$.

Corollary 6.10. *Let $p \equiv 11 \pmod{12}$ and let $\Gamma < \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ be finitely presented. If Γ has no elements of order 2, then Γ satisfies the Unstable GLR Conjecture.*

There are many finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for $p \equiv 11 \pmod{12}$ satisfying the hypothesis of the corollary. Indeed, by the amalgamated free product decomposition, $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ is generated by the two subgroups isomorphic to $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Thus, $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ is a four generated group $\langle a, b, c, d \rangle$ where a and c have order 2, and b and d have order 3. The kernel of the homomorphism $\phi : \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \rightarrow \mathbb{Z}_2$ by $a, c \mapsto 1$ and $b, d \mapsto 0$ is a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ with no 2-torsion.

Proof. The proof follows from applying Theorem 1.7 to the observation that every finite subgroup of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ is cyclic and hence has periodic cohomology and the following lemma. Note that Γ is necessarily a proper subgroup since $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ always contains elements of order 2. \square

Lemma 6.11. *Let p be a prime, then $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ satisfies (M). Moreover, if $p \equiv 11 \pmod{12}$ then $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ satisfies (NM).*

Proof. Since each non-trivial finite subgroup of Γ is of order 2 or 3 it is obvious that Γ satisfies (M). Now, assume $p \equiv 11 \pmod{12}$ and note that $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ satisfies (NM). Recall the amalgamated free product decomposition, $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) = \mathrm{PSL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{PSL}_2(\mathbb{Z})$. The amalgamated subgroup $\Gamma_0(p)$ is torsion-free so we may apply [42, Theorem 3.1]. \square

An alternative direct proof of the calculations of the K -groups of $C_r^*(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]))$ when $p \equiv 11 \pmod{12}$ is as follows. Note that this bypasses the computation of the Bredon homology but does not give us a way to compute either invariant for $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$.

Lemma 6.12. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, then $\underline{B}\Gamma \simeq \bigvee_{b_1(\Gamma_0(p))} S^2$*

Proof. We have

$$\begin{aligned} \underline{B}\Gamma &\simeq \mathrm{hocolim}_{\mathrm{Top}} \left((\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z}) \times_{\mathrm{PSL}_2(\mathbb{Z})} \Gamma) \leftarrow (\underline{\mathrm{E}}\Gamma_0(p) \times_{\Gamma_0(p)} \Gamma) \rightarrow (\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z}) \times_{\mathrm{PSL}_2(\mathbb{Z})} \Gamma) \right) / \Gamma, \\ &\simeq \mathrm{hocolim}_{\mathrm{Top}} \left((\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z}) \times_{\mathrm{PSL}_2(\mathbb{Z})} \Gamma) / \Gamma \leftarrow (\underline{\mathrm{E}}\Gamma_0(p) \times_{\Gamma_0(p)} \Gamma) / \Gamma \rightarrow (\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z}) \times_{\mathrm{PSL}_2(\mathbb{Z})} \Gamma) / \Gamma \right). \end{aligned}$$

Since $\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z}) / \mathrm{PSL}_2(\mathbb{Z})$ is an interval and $\underline{\mathrm{E}}\Gamma_0(p) / \Gamma_0(p)$ is a finite graph, we have

$$\underline{B}\Gamma \simeq \mathrm{hocolim}_{\mathrm{Top}} \left(I \leftarrow \bigvee_{b_1(\Gamma_0(p))} S^1 \rightarrow I \right),$$

but I is contractible, so up to homotopy this becomes a suspension of a wedge of circles. In particular, $\underline{B}\Gamma \simeq \bigvee_{b_1(\Gamma_0(p))} S^2$. \square

Theorem 6.13. *Let $p \equiv 11 \pmod{12}$ and let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, then*

$$K_0^{\mathrm{top}}(C_r^*(\Gamma)) = \mathbb{Z}^{7 + \frac{1}{6}(p+7)} \quad \text{and} \quad K_1^{\mathrm{top}}(C_r^*(\Gamma)) = 0.$$

Proof. Let Λ be a set of representatives of finite subgroups of Γ . By [13, Theorem 4.1(a)] we have a short exact sequence

$$0 \rightarrow \bigoplus_{(H) \in \Lambda} \widetilde{K}_n^{\mathrm{top}}(C_r^*(H)) \rightarrow K_n^{\mathrm{top}}(C_r^*(\Gamma)) \rightarrow K_n(\underline{B}\Gamma) \rightarrow 0.$$

The only nontrivial part now is computing $K_n(\underline{B}\Gamma)$, but we have already shown that $\underline{B}\Gamma$ is homotopy equivalent to a wedge of $\bigvee_{b_1(\Gamma_0(p))} S^2$, i.e. a wedge of 2-spheres. Thus, we can simply apply the homological Atiyah-Hirzebruch spectral sequence (which collapses trivially) to obtain that $K_0(\underline{B}\Gamma) = \mathbb{Z}^{\frac{1}{6}(p+7)+1}$ and $K_1(\underline{B}\Gamma) = 0$. \square

A near identical argument can be used to compute the KO -groups of $C_r^*(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]))$ when $p \equiv 11 \pmod{12}$.

Theorem 6.14. *Let $p \equiv 11 \pmod{12}$ be a prime and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. Except for an extension problem in dimensions congruent to 1, 3 and 4 modulo 8, we have for $n = 0, \dots, 7$ that*

$$KO_n^{\mathrm{top}}(C_r^*) = \mathbb{Z}^5, \quad \mathbb{Z}_2^3, \quad \mathbb{Z}^{2 + \frac{1}{6}(p+7)} \oplus \mathbb{Z}_2^3, \quad \mathbb{Z}_2^{\frac{1}{6}(p+7)}, \quad \mathbb{Z}^5 \oplus \mathbb{Z}_2^{\frac{1}{6}(p+7)}, \quad 0, \quad \mathbb{Z}^{2 + \frac{1}{6}(p+7)}, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Proof. Let Λ be a set of representatives of finite subgroups of Γ . As before, by [13, Theorem 4.1(a)] we have a short exact sequence

$$0 \rightarrow \bigoplus_{(H) \in \Lambda} \widetilde{KO}_n^{\mathrm{top}}(C_r^*(H)) \rightarrow KO_n^{\mathrm{top}}(C_r^*(\Gamma)) \rightarrow KO_n(\underline{B}\Gamma) \rightarrow 0.$$

Now, $KO_n(\underline{B}\Gamma) \cong KO_n(*) \oplus KO_{n-2}(*)^{\frac{1}{6}(p+7)}$ and the groups $KO_n^{\mathbb{Z}_m}(*) \cong KO_n^{\text{top}}(C_r^*(\mathbb{Z}_m))$ are given in [17, Section 2.1]. \square

Our methods leave open the following.

Question 6.15. Let p be a prime, then does $\text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ satisfy the Unstable GLR Conjecture? What about a lattice in $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{Q}_p)$?

REFERENCES

- [1] A. Adem and N. Naffah, *On the Cohomology of $\text{SL}_2(\mathbb{Z}[1/p])$* , in *Geometry and Cohomology in Group Theory* (London Mathematical Society Lecture Note Series no. 252), 1–9, Cambridge University Press. doi: 10.1017/CB09780511666131.002
- [2] M.P.G. Aparicio, P. Julg and A. Valette *The Baum–Connes conjecture: an extended survey*. In: Chamseddine A., Consani C., Higson N., Khalkhali M., Moscovici H., Yu G. (eds) *Advances in Noncommutative Geometry* (20202. Springer, Cham. doi: 10.1007/978-3-030-29597-4_3
- [3] P. Baum, A. Connes, and N. Higson, *Classifying spaces for proper actions and K -theory of group C^* -algebras*, *Contemporary Mathematics* **167** (1994), 241–291.
- [4] P. Baum and M. Karoubi, *On the Baum–Connes conjecture in the real case*, *Quarterly Journal of Mathematics*, **55** (2004), no. 3, 231–235. doi: 10.1093/qmath/hag051
- [5] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan’s Property (T) (New Mathematical Monographs)*. Cambridge: Cambridge University Press 2008. doi: 10.1017/CB09780511542749
- [6] B. Botvinnik, P.B. Gilkey and S. Stolz, *The Gromov–Lawson–Rosenberg conjecture for groups with periodic cohomology*, *Journal of Differential Geometry*, **46** (1997), 374–405. doi: 10.4310/jdg/1214459973
- [7] K.S. Brown, *Cohomology of Groups*. Graduate Texts in Mathematics 87. New York: Springer-Verlag 1982. doi: 10.1007/978-1-4684-9327-6
- [8] R.R. Bruner and J.P.C. Greenlees, *Connective Real K -Theory of Finite Groups, Mathematical Surveys and Monographs* **169**. American Mathematical Society, Providence, RI 2010.
- [9] A.T. Bui and G. Ellis, *The homology of $\text{SL}_2(\mathbb{Z}[1/m])$ for small m* , *Journal of Algebra*, **408** (2014), 102–108. doi: 10.1016/j.jalgebra.2013.11.002.
- [10] A.T. Bui, A.D. Rahm, and M. Wendt, *The Farrell–Tate and Bredon homology for $\text{PSL}_4(\mathbb{Z})$ via cell subdivisions*, *Journal of Pure and Applied Algebra*, **223** (2019), no. 7, 2872–2888. doi 10.1016/j.jpaa.2018.10.002
- [11] M. Bustamante and L.J.S. Saldaña, *On the algebraic K -theory of the Hilbert modular group*, *Algebraic and Geometric Topology*, **16** (2016), no. 4, 2107–2125. doi: 10.2140/agt.2016.16.2107
- [12] J. Davis and W. Lück, *Spaces over a category and assembly maps in isomorphism conjectures in K - and L -theory*, *K -Theory*, **15** (1998), 201–252. doi: 10.1023/A:1007784106877
- [13] J. Davis and W. Lück, *The p -chain spectral sequence*, *K -Theory*, **30** (2003), no. 1, 71–104. doi: 10.1023/B:KTHE.0000015339.41953.04
- [14] J. Davis and W. Lück, *The topological K -theory of certain crystallographic groups*, *Journal of Noncommutative Geometry*, **7** (2013), 373–431. doi: 10.4171/JNCG/121
- [15] J. Davis and K. Pearson, *The Gromov–Lawson–Rosenberg conjecture for cocompact Fuchsian groups*, *Proceedings of the American Mathematical Society*, **131** (2003), 3571–3578. doi:10.1090/S0002-9939-03-06905-3
- [16] D. Degrijse and I.J. Leary, *Equivariant vector bundles over classifying spaces for proper actions*, *Algebraic & Geometric Topology*, **17** (2017), no. 1, 131–156. doi: 10.2140/agt.2017.17.131
- [17] M. Fuentes-Rumi, *The equivariant K - and KO -theory of certain classifying spaces via an equivariant Atiyah–Hirzebruch spectral sequence*, preprint (2020). arXiv: 1905.02972
- [18] R. Flores, S. Pooya and A. Valette, *K -theory and K -homology for the lamplighter groups of finite groups*, *Proceedings of the London Mathematical Society*, **115** (2017), no. 3, 1207–1226. doi: 10.1112/plms.12061
- [19] W. Fulton and J. Harris, *Representation theory. A first course*. Graduate Texts in Mathematics 129. New York: Springer-Verlag 1991. doi: 10.1007/978-1-4612-0979-9
- [20] H.-W. Henn, *The cohomology of $\text{SL}(3, \mathbb{Z}[1/2])$* , *$K$ -Theory*, **16** (1999), no. 4, 299–359.
- [21] N. Higson and G. Kasparov, *Operator K -theory for groups which act properly and isometrically on Hilbert space*, *Electronic Research Announcements of the American Mathematical Society*, **3** (1997), 131–142. doi: 10.1090/S1079-6762-97-00038-3
- [22] N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum–Connes Conjecture*, *Geometric and Functional Analysis*, **12** (2002), no. 2, 330–354. doi: 10.1007/s00039-002-8249-5

- [23] S. Hughes, *Cohomology of Fuchsian Groups and Non-Euclidean Crystallographic Groups*, preprint (2019). arXiv: 1910.00519
- [24] M. Joachim and T. Schick, *Positive and negative results concerning the Gromov-Lawson-Rosenberg conjecture*, Geometry and topology: Aarhus (1998), Contemporary Mathematics, vol. 258, American Mathematical Society, Providence, RI, 2000, pp. 213–226. doi:10.1090/conm/258/04066
- [25] G.G. Kasparov, *Operator K-theory and its applications*, Journal of Soviet Mathematics, **37** (1987), no. 6, 1373–1396. doi: 10.1007/BF01103851
- [26] G.G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Inventiones Mathematicae, **91** (1988), no. 1, 147–201. doi: 10.1007/BF01404917
- [27] S. Katok, *Fuchsian Groups*, The University of Chicago Press, Chicago, 1992.
- [28] S. Kwasik and R. Schultz, *Positive scalar curvature and periodic fundamental groups*, Commentarii Mathematici Helvetici, **65** (1990), 271–286. doi: 10.1007/BF02566607
- [29] J.-F. Lafont, I.J. Ortiz, A.D. Rahm and R. Sánchez-García, *Equivariant K-homology for hyperbolic reflection groups*, The Quarterly Journal of Mathematics, **69** (2018), no. 4, 1475–1505. doi: 10.1093/qmath/hay030
- [30] X. Li, *K-theory for generalized Lamplighter groups*, Proceedings of the American Mathematics Society, **147** (2019), 4371–4378. doi: 10.1090/proc/14619
- [31] W. Lück and R. Stamm, *Computations of K- and L-theory of cocompact planar groups*, K-Theory, **21** (2000), no. 3, 249–292. doi:10.1023/A:1026539221644
- [32] A.M. Macbeath, *The classification of non-euclidean plane crystallographic groups*, Canadian Journal of Mathematics. Journal Canadien de Mathématiques, **19** (1967), 1192–1205. doi: 10.4153/CJM-1967-108-5
- [33] M. Matthey, H. Oyono-Oyono and W. Pitsch, *Homotopy invariance of higher signatures and 3-manifold groups*, Bulletin de la Société Mathématique de France, **136** (2008), no. 1, 1–25. doi: 10.24033/bsmf.2547
- [34] G. Mislin and A. Valette, *Proper group actions and the Baum-Connes conjecture*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, 2003.
- [35] S. Pooya, *K-theory and K-homology of the wreath products of finite with free groups*, Illinois Journal of Mathematics, **63** (2019) no. 2, 317–334. doi: 10.1215/00192082-7768735
- [36] S. Pooya and A. Valette, *K-theory for the C*-algebras of solvable Baumslag-Solitar groups*, Glasgow Mathematical Journal, **60** (2018) no. 2, 481–486. doi:10.1017/S0017089517000210
- [37] A.D. Rahm, *Homology and K-theory of the Bianchi groups*, Comptes Rendus Mathématique, **349** (2011), no. 11–12, 615–619. doi: 10.1016/j.crma.2011.05.014.
- [38] A.D. Rahm, *On the equivariant K-homology of PSL_2 of the imaginary quadratic integers*, Annales de l’Institut Fourier, **66** (2016), no. 4, 1667–1689. doi: 10.5802/aif.3047
- [39] J. Rosenberg, *C*-algebras, positive scalar curvature, and the Novikov conjecture II*, in Geometric Methods in Operator Algebras (Kyoto 1983), H. Araki and E. G. Effros, eds., Pitman Research Notes in Math, no. 123, Longman/Wiley 1986, 341–374.
- [40] J. Rosenberg, *C*-algebras, positive scalar curvature, and the Novikov conjecture, III*, Topology, **25** (1986), 319–336.
- [41] J. Rosenberg and S. Stolz, *Manifolds of positive scalar curvature*, in Algebraic Topology and its Applications, MSRI Publications 27, Springer 1994, 241–267.
- [42] L.J.S. Saldaña, (2020). *On the dimension of groups that satisfy certain conditions on their finite subgroup*, Glasgow Mathematical Journal, ahead of press. doi:10.1017/S0017089520000531
- [43] L.J.S. Saldaña and M. Velásquez, *The algebraic and topological K-theory of the Hilbert modular group*, Homology, Homotopy and Applications, **20** (2018) no. 2, 377–402. doi: 10.4310/HHA.2018.v20.n2.a19
- [44] R. Sánchez-García, *Bredon homology and equivariant K-homology of $SL(3, \mathbb{Z})$* , Journal of Pure and Applied Algebra, **212** (2008), no. 5, 1046–1059. doi: 10.1016/j.jpaa.2007.07.019
- [45] R. Sánchez-García, *Equivariant K-homology for some Coxeter groups*, Journal of the London Mathematical Society, **75** (2007), no. 3, 773–790. doi: 10.1112/jlms/jdm035
- [46] R. Sánchez-García, *Equivariant K-homology of the classifying space for proper actions*, Ph.D. thesis, University of Southampton, 2005.
- [47] J.-P. Serre, *Trees*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. doi: 10.1007/978-3-642-61856-7
- [48] T. Schick, *A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture*, Topology, **37** (1998), 1165–1168. doi: 10.1016/S0040-9383(97)00082-7
- [49] C. Soulé, *The cohomology of $SL_3(\mathbb{Z})$* , Topology **17** (1978), 1–22. doi: 10.1016/0040-9383(78)90009-5

- [50] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Annals of Mathematics, **136** (1992), 511-540. doi: 10.2307/2946598
- [51] S. Stolz, *Positive scalar curvature metrics - existence and classification questions*, Proceedings of the International Congress of Mathematicians (Zurich 1994), Birkhäuser, 1995, 625-636. doi: 10.1007/978-3-0348-9078-6_56