GRAPHS AND COMPLEXES OF LATTICES

SAM HUGHES

ABSTRACT. We study lattices acting on CAT(0) spaces via their commensurated subgroups. To do this we introduce the notions of a graph of lattices and a complex of lattices giving graph and complex of group splittings of CAT(0) lattices. Using this framework we characterise irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices by C^*-simplicity and give a necessary condition for lattices in products with a Euclidean factor to be biautomatic. We also construct non-residually finite uniform lattices acting on arbitrary products of right-angled buildings and non-biautomatic lattices acting on the product of $\mathbb{E}^n$ and a right-angled building.

1. INTRODUCTION

Let H be a locally compact group with Haar measure μ. A discrete subgroup $\Gamma \leqslant H$ is a *lattice* if the covolume $\mu(H/\Gamma)$ is finite. We say the lattice *uniform* is H/Γ is cocompact and *non-uniform otherwise*. We say a lattice Γ in a product $H_1 \times H_2$ is *algebraically irreducible* if no finite index subgroup of Γ splits as a direct product of two infinite groups, otherwise we say Γ is *reducible*. We will give a topological interpretation of irreducibility for CAT(0) lattices in Section 2.3. Given a pair of locally compact groups H_1 and H_2 there are a number of basic questions one can ask:

(Q1) Does $H_1 \times H_2$ contain algebraically irreducible lattices?
(Q2) What are the generic properties of an algebraically irreducible lattice?

In the classical setting of lattices in semisimple Lie groups and linear algebraic groups over local fields these questions are well studied. Indeed, there are deep theorems such as the Margulis normal subgroup theorem, super-rigidity theorem, and the arithmeticity theorem [Mar91].

The non-classical setting is more complicated and was initiated by studying lattices in the full automorphism group of a locally-finite polyhedral complex. A striking example of the non-classical setting is given by the work of Burger and Mozes [BM97; BM00a; BM00b]. The authors constructed torsion-free simple groups which could be realised as cocompact irreducible lattices in a product of automorphism groups of locally-finite trees.

MATHEMATICAL INSTITUTE, ANDREW WILES BUILDING, OBSERVATORY QUATER, UNIVERSITY OF OXFORD, OXFORD OX2 6GG, UK

E-mail address: sam.hughes@maths.ox.ac.uk.

More generally, we can consider lattices acting on CAT(0) spaces which encompasses symmetric spaces, non-positively curved manifolds, Euclidean and hyperbolic buildings, and more. The reader is referred to [BH99] for a comprehensive introduction to the theory.

Assumption 1.1. *Throughout this paper, all spaces are finite dimensional and all actions of groups on graphs or polyhedral complexes are assumed to be* admissible. *That is, each element of a group fixes pointwise each cell it stabilises.*

A systematic study of the full isometry groups of CAT(0) spaces and their lattices was undertaken by Caprace and Monod [CM09b; CM09a; CM19]. The authors showed in [CM09b, Theorem 1.6], that under mild hypotheses on a CAT(0) space X, there is finite index subgroup of $H \leqslant \mathrm{Isom}(X)$ which splits as

$$(1) \qquad H \cong \mathrm{Isom}(\mathbb{E}^n) \times S_1 \times \cdots \times S_p \times D_1 \times \cdots \times D_q,$$

for some $n, p, q \geqslant 0$, where each S_i is an almost connected simple Lie group with trivial centre and each D_j is a totally disconnected irreducible group with trivial amenable radical. Moreover by [CM09b, Addendum 1.8], X itself splits as

$$(2) \qquad X = \mathbb{E}^n \times X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q$$

where each X_i is an irreducible symmetric space of non-compact type and each Y_j is an irreducible minimal CAT(0)-space.

Taking these decompositions as a starting point motivates a new approach towards CAT(0) groups, that is, understanding the lattices in each of the factors individually and then how the factors interact. The later question is the central goal of this paper: To provide a combinatorial framework for studying lattices in products of irreducible CAT(0) spaces and deduce properties of the algebraically irreducible lattices. To this end we introduce the notion of a *graph of lattices* (Definition 3.2) with fixed locally-finite Bass-Serre $\mathcal{T}$ (we will also assume that the tree is unimodular and its automorphism group is non-discrete, these are essentially non-degeneracy conditions so that there are tree lattices). Note that in the case of a product of two trees a similar construction was considered by Benakli and Glasner [BG02].

Roughly a graph of lattices is a graph of groups such that all local groups are commensurable finite-by-{H-lattices} equipped with a morphism to H. We use this to study lattices in the product of $T := \mathrm{Aut}(\mathcal{T})$ and closed subgroups H of the isometry group of a fairly generic CAT(0) space. We prove a structure theorem for $(H \times T)$-lattices. That is, we show every $(H \times T)$-lattice gives rise to a graph of H-lattices and conversely, we give necessary and sufficient conditions for a graph of H-lattices to be an $(H \times T)$-lattice.

Theorem A (Theorem 3.3). *Let X be a proper* CAT(0) *space and let $H = \mathrm{Isom}(X)$ contain a uniform lattice. Let $(A, \mathcal{A}, \psi)$ be a graph of H-lattices with locally-finite unimodular*

non-discrete Bass-Serre tree $\mathcal{T}$, *and fundamental group* Γ. *Suppose* $T = \text{Aut}(\mathcal{T})$ *admits a uniform lattice.*

(1) *Assume A is finite. If for each local group A_σ the kernel $\text{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$, then Γ is a uniform $(H \times T)$-lattice and hence a $\text{CAT}(0)$ group. Conversely, if Λ is a uniform $(H \times T)$-lattice, then Λ splits as a finite graph of uniform H-lattices with Bass-Serre tree $\mathcal{T}$.*

(2) *Under the same hypotheses as* (1), Γ *is quasi-isometric to $X \times \mathcal{T}$.*

(3) *Assume X is a $\text{CAT}(0)$ polyhedral complex. Let μ be the normalised Haar measure on H. If for each local group A_σ the kernel $K_\sigma = \text{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$ and the sum $\sum_{\sigma \in VA} \mu(A_\sigma)/|K_\sigma|$ converges, then Γ is a $(H \times T)$-lattice. Conversely, if Λ is a $(H \times T)$-lattice, then Λ splits as a graph of H-lattices with Bass-Serre tree $\mathcal{T}$.*

We also introduce an analogous construction we call a *complex of lattices* (Definition 5.1) by replacing the tree with a $\text{CAT}(0)$ polyhedral complex. In this setting we prove a completely analogous structure theorem (Theorem 5.2). In the process we deduce some consequences about commensurated subgroups of $\text{CAT}(0)$ groups (see also Corollary 5.4) which generalises splittings of lattices in products of trees and twin buildings.

Corollary B (Corollary 5.3). *Let $X = X_1 \times X_2$ be a proper cocompact minimal $\text{CAT}(0)$ space and $H = \text{Isom}(X_1) \times \text{Isom}(X_2)$. Suppose X_1 is a $\text{CAT}(0)$ polyhedral complex. Then, for any uniform H-lattice Γ, the cell stabilisers of X_1 in Γ are commensurated, commensurable, and isomorphic to finite-by-$\{\text{Isom}(X_2)\text{-lattices}\}$.*

Let Γ be a discrete group. The *reduced C^*-algebra* of Γ, denoted $C_r^*(\Gamma)$, is the norm closure of the algebra of bounded operators on $\ell^2(\Gamma)$ by the left regular representation of Γ. We say Γ is *C^*-simple* if $C_r^*(\Gamma)$ has exactly two norm-closed two-sided ideals 0 and $C_r^*(\Gamma)$ itself. We refer the reader to [Har07] for a general survey and [Bre+17] for a number of recent developments. In the setting of $\text{CAT}(0)$ groups there is a characterisation of C^*-simple $\text{CAT}(0)$ cubical groups [KS16] and of linear groups [Bre+17, Theorem 1.6]. In Section 4 we study C^*-simplicity of a lattice in a product space with a tree factor (Theorem 4.2). We obtain the following characterisation of irreducible $(\text{Isom}(\mathbb{E}^n) \times T)$-lattices.

Theorem C (Theorem 4.3). *Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \text{Aut}(\mathcal{T})$. Let $n \geqslant 1$ and $\Gamma < \text{Isom}(\mathbb{E}^n) \times T$ be a uniform lattice. Then, Γ is C^*-simple if and only if Γ is algebraically irreducible.*

In Section 6 we extend the work of Leary and Minasyan [LM21] to obtain a necessary condition for biautomaticity for lattices in a product with non-trivial Euclidean de Rham factor (Theorem 6.1). We will give the relevant definitions in Section 6.

In Section 7 we adapt a construction of Horbez and Huang [HH20] to extend actions from a regular tree to the universal cover of a Salvetti complex $\widetilde{S}_L$ with defining graph L. In particular, from a graph of lattices, one obtains a complex of lattices. With a mild hypothesis on the graph L, we use this construction to obtain algebraically irreducible non-biautomatic uniform lattices acting on $\widetilde{S}_L \times \mathbb{E}^n$ for $n \geqslant 2$ (Example 5). We also deduce a consequence about quasi-isometric rigidity of right angled Artin groups with centre containing $\mathbb{Z}^2$.

Corollary D (Example 5 and Corollary 7.5)**.** *Let $m \geqslant 3$. Let K be a pointed flag complex on $[m]$, and let $L = \bigvee_{i=1}^{5} K$. If A_L is irreducible, then there exists an algebraically irreducible uniform lattice in $\mathrm{Aut}(\widetilde{S}_L) \times \mathrm{Isom}(\mathbb{E}^2)$ which is not virtually biautomatic nor residually finite. In particular, $A_L \times \mathbb{Z}^2$ is not quasi-isometrically rigid even amongst* CAT(0) *groups.*

In [Tho06], Thomas constructs a functor from graphs of groups covered by a fixed biregular tree $\mathcal{T}$ to complexes of groups covered by a fixed "sufficiently symmetric" right-angled building X with parameters determined by the valences of $\mathcal{T}$. We will give the relevant definitions in Section 8.1. In Theorem 8.4 we show that Thomas' functor theorem takes a graph of lattices to a complex of lattices and in particular $(H \times T)$-lattices to $(H \times A)$-lattices, where $T = \mathrm{Aut}(\mathcal{T})$, $A = \mathrm{Aut}(X)$, and H is a closed subgroup of the isometry group of a CAT(0) space (under mild hypothesis). As consequences we construct more CAT(0) groups which are not virtually biautomatic (Corollary 8.5) and both uniform and non-uniform irreducible lattices in products of fairly arbitrary hyperbolic and Euclidean buildings (Corollary 8.7). We highlight one special case here:

Corollary E (Special case of Corollary 8.5)**.** *Let X be the right-angled building of a regular m-gon of uniform thickness $10n$ and let $A = \mathrm{Aut}(X)$. For each $n \geqslant 2$ there exists a weakly irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice which is not virtually biautomatic nor residually finite. In particular, if Y is irreducible, then the direct product of a uniform A-lattice with $\mathbb{Z}^2$ is not quasi-isometrically rigid even amongst* CAT(0) *groups.*

Bibiliographical note. This paper was part of a longer paper contained in the author's PhD thesis in a chapter of the same name (see [Hug21a, Paper 5]). This has been split at the request of a referee. A sequel to this paper *"Irreducible lattices fibring over the circle"* [Hug21b] will contain many of the remaining results from [Hug21a, Paper 5] and some new results. A number of group presentations, and results regarding residual finiteness, ℓ^2-Betti numbers, and autostackability will only exist in the thesis version. Finally, we remark the results here have been used in [Hug21c] to construct a lattice (and in fact a hierarchically hyperbolic group) in a product of trees which is not virtually torsion-free.

2. Preliminaries

2.1. **Lattices and covolumes.** Let H be a locally compact topological group with right invariant Haar measure μ. A discrete subgroup $\Gamma \leqslant H$ is a *lattice* if the covolume $\mu(H/\Gamma)$ is finite. A lattice is *uniform* if H/Γ is compact and *non-uniform* otherwise. Let S be a right H-set such that for all $s \in S$, the stabilisers H_s are compact and open, then if $\Gamma \leqslant H$ is discrete the stabilisers are finite.

Let X be a locally finite, connected, simply connected simplicial complex. The group $H = \text{Aut}(X)$ of simplicial automorphisms of X naturally has the structure of a locally compact topological group, where the topology is given by uniform convergence on compacta.

Theorem 2.1 (Serre's covolume formula [Ser71]). *Let X be a locally finite simply-connected simplicial complex. Let $\Gamma \leqslant H$ be a lattice with fundamental domain Δ, then there is a normalisation of the Harr measure μ on H, depending only on X, such that for each discrete subgroup $\Gamma < H$ we have*

$$\mu(H/\Gamma) = \text{Vol}(X/\Gamma) := \sum_{v \in \Delta^{(0)}} \frac{1}{|\Gamma_v|}. \qquad \square$$

Note that T the automorphism group of a locally finite tree $\mathcal{T}$ admits lattices if and only if the group T is unimodular (that is the left and right Haar measures coincide). In this case we say $\mathcal{T}$ is *unimodular*.

2.2. **Non-positive curvature.** We will be primarily interested in lattices in the isometry groups of CAT(0) spaces, we will call these groups CAT(0) *lattices* (note that a uniform CAT(0) lattice is a CAT(0) group). We begin by recording several facts about the structure and isometry groups of general CAT(0) spaces. The definitions and results here are largely due to Caprace and Monod [CM09b; CM09a; CM19].

An isometric action of a group H on a CAT(0) space X is *minimal* if there is no non-empty H-invariant closed convex subset $X' \subset X$, the space X is *minimal* if $\text{Isom}(X)$ acts minimally on X. Note that by [CM09b, Proposition 1.5], if X is cocompact and geodesically complete, then it is minimal. The *amenable radical* of a locally compact group

H is the largest amenable normal subgroup. We can now state Caprace and Monod's group and space decomposition theorems mentioned in the introduction.

Theorem 2.2. [CM09b, Theorem 1.6] *Let X be a proper $\mathrm{CAT}(0)$ space with finite dimensional Tits' boundary and assume $\mathrm{Isom}(X)$ has no global fixed point in ∂X. There is a canonical closed, convex, $\mathrm{Isom}(X)$-stable subset $X' \subseteq X$ such that $G = \mathrm{Isom}(X')$ has a finite index, open, characteristic subgroup $H \trianglelefteq G$ that admits a canonical decomposition*

$$H \cong \mathrm{Isom}(\mathbb{E}^n) \times S_1 \times \cdots \times S_p \times D_1 \times \cdots \times D_q,$$

for some $n, p, q \geqslant 0$, where each S_i is an almost connected simple Lie group with trivial centre and each D_j is a totally disconnected irreducible group with trivial amenable radical. $\square$

Theorem 2.3. [CM09b, Addendum 1.8] *Let X' and H be as above, then*

$$X' \cong \mathbb{E}^n \times X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q$$

where each X_i is an irreducible symmetric space and each Y_j is an irreducible minimal $\mathrm{CAT}(0)$-space. $\square$

2.3. **Irreducibility.** Two notions of irreducibility will feature in this paper; for uniform $\mathrm{CAT}(0)$ lattices they are equivalent due to a theorem of Caprace–Monod.

Let $X = \mathbb{E}^n \times X_1 \times \cdots \times X_m$ be a product of irreducible proper $\mathrm{CAT}(0)$ spaces with each X_i not quasi-isometric to $\mathbb{E}^1$ and let Γ be a lattice in $H = H_0 \times H_1 \times \cdots \times H_m :=$ $\mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_m)$, with for each $i \geqslant 1$ the group H_i is non-discrete, cocompact, and acting minimally on X_i. Suppose $n = 0$, then we say Γ is *weakly irreducible* if the projection of Γ to each proper subproduct $H_I := \prod_{i \in I} H_i$ for $I \subset \{1, \ldots, m\}$ is non-discrete.

Now, suppose Γ is a uniform lattice. If $n = 1$, then Γ is always reducible by [CM19, Theorem 2]. If $n \geqslant 2$, then we observe that Γ is contained in $\prod_{j=1}^{\ell} \mathrm{Isom}(\mathbb{E}^{k_j}) \times \prod_{i=1}^{m} H_i$ where $\ell \geqslant 1$, $\sum_{j=1}^{\ell} k_j = n$, and each k_j is minimal (so ℓ is maximal amongst all choices of orthonormal bases for $\mathbb{R}^n$). Denote each $\mathrm{Isom}(\mathbb{E}^{k_j})$ by E_j and the corresponding orthogonal group by O_j. Then for Γ to be *weakly irreducible* we require that each $k_j \geqslant 2$, and that the projection $\pi_{I,J}$ of Γ to each proper subproduct, $G_{I,J} := \prod_{j \in J} O_j \times \prod_{i \in I} H_i$ for $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, \ell\}$, of H is non-discrete (here at least one of I or J is a proper subset).

We say Γ is *algebraically irreducible* if Γ has no finite index subgroup splitting as the direct product of two infinite groups.

Remark 2.4. We record the following remarks concerning these definitions.

(1) The complexity of the definition in the presence of the Euclidean factor is an artefact of the examples found by Leary–Minasyan [LM21]. Here the authors constructed weakly irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices (where T is the automorphism

group of some locally-finite tree), we will denote one of these by $\mathrm{LM}(A)$. The necessity of the of the decomposition $\prod_j E_j$ is because $\Gamma = \mathrm{LM}(A) \times \mathrm{LM}(A)$ has non-discrete projections to any subproduct of $\mathrm{Isom}(\mathbb{E}^{2n}) \times T \times T$ but is clearly algebraically reducible. However, $\Gamma < \mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Isom}(\mathbb{E}^n) \times T \times T$ and here Γ has discrete projections to proper subproducts.

(2) The reason we consider projections to subproducts of H_i and $\mathrm{O}(k_j)$ instead of the whole of $\mathrm{Isom}(\mathbb{E}^{k_j})$ is as follows: If Γ projects non-discretely to some $\mathrm{Isom}(\mathbb{E}^{k_j})$ but has a discrete projection into the corresponding $\mathrm{O}(k_j)$, then Γ virtually splits as $\Gamma' \times \mathbb{Z}$ with Γ' infinite. Indeed, in this case, the image of Γ in $\mathrm{O}(k_j)$ is finite, so a finite index subgroup of Γ acts on some $\mathbb{E}^{k_j}$ by translations. In particular, this finite index subgroup fixes a point in ∂X, and so by the discussion after [CM19, Theorem 2], Γ splits as $\Gamma' \times \mathbb{Z}$.

(3) The necessity that $k_j \geqslant 2$ is because if some $k_j = 1$, then $\pi_{\mathrm{O}(n)}(\Gamma) \leqslant \mathrm{O}(1) \cong \mathbb{Z}_2$ and so by the previous remark Γ virtually splits as $\Gamma' \times \mathbb{Z}$.

With these definitions in hand we are able to use the following result of Caprace–Monod.

Theorem 2.5. [CM09a, Theorem 4.2] *Let X be a proper $\mathrm{CAT}(0)$ space, $H < \mathrm{Isom}(X)$ a closed subgroup acting cocompactly on X, and $\Gamma < H$ a uniform lattice.*

(1) If Γ is irreducible as an abstract group, then for for finite index subgroup $\Gamma_0 < \Gamma$ and any Γ_0-equivariant splitting $X = X_1 \times X_2$ with X_1 and X_2 non-compact, the projection of Γ_0 to both $\mathrm{Isom}(X_1)$ and $\mathrm{Isom}(X_2)$ is non-discrete.

(2) If in addition the H-action is minimal, then the converse holds. $\square$

Finally, we restate a result of Caprace–Monod which we can use as criterion to determine non-residual finiteness of lattices in products.

Theorem 2.6. [CM09a, Theorem 4.10] *Let X be a proper $\mathrm{CAT}(0)$ space such that $G = \mathrm{Isom}(X)$ acts cocompactly and minimally. Let $\Gamma < \mathrm{Isom}(X)$ be a finitely generated algebraically irreducible lattice. Let $\Gamma' = \Gamma \cap H$, where H is given in Theorem 2.2. If the projection of Γ' to an irreducible factor of X has non-trivial kernel, then Γ is not residually finite.* $\square$

3. Graphs of lattices

In this section we will review Bass-Serre theory and tree lattices. These tools will be fundamental to us in the following chapters. We will then define a graph of lattices and prove the structure theorem for $(H \times T)$-lattices.

3.1. Graphs of groups. We shall state some of the definitions and results of Bass-Serre theory. In particular, the action will be on the right. We follow the treatment of Bass [Bas93]. Throughout a graph $A = (VA, EA, \iota, \tau)$ should be understood as it is defined

by Serre [Ser03], with edges in oriented pairs indicated by e and $\bar{e}$, and maps $\iota(e)$ and $\tau(e)$ from each edge to its initial and terminal vertices. We will, however, often talk about the geometric realisation of a graph as a metric space. In this case the graph should be assumed to be simplicial (possibly after subdividing) and should have exactly one undirected edge e for each pair $(e, \bar{e})$. We will often not distinguish between the combinatorial and metric notions.

A *graph of groups* $(A, \mathcal{A})$ consists of a graph A together with some extra data $\mathcal{A} = (V\mathcal{A}, E\mathcal{A}, \Phi\mathcal{A})$. This data consists of *vertex groups* $A_v \in V\mathcal{A}$ for each vertex v, *edge groups* $A_e = A_{\bar{e}} \in E\mathcal{A}$ for each (oriented) edge e, and monomorphisms $(\alpha_e : A_e \to A_{\iota(e)}) \in \Phi$ for every oriented edge in A. We will often refer to the vertex and edge groups as *local groups* and the monomorphisms as *structure maps*.

The *path group* $\pi(\mathcal{A})$ has generators the vertex groups A_v and elements t_e for each edge $e \in E\mathcal{A}$ along with the relations:

$$\left\{ \begin{array}{c} \text{The relations in the groups } A_v, \\ t_{\bar{e}} = t_e^{-1}, \\ t_e \alpha_{\bar{e}}(g) t_e^{-1} = \alpha_e(g) \text{ for all } e \in E\mathcal{A} \text{ and } g \in A_e = A_{\bar{e}}. \end{array} \right\}$$

We will often abuse notation and write $\mathcal{A}$ for a graph of groups. The *fundamental group of a graph of groups* can be defined in two ways. Firstly, considering reduced loops based at a vertex v in the graph of groups, in this case the fundamental group is denoted $\pi_1(\mathcal{A}, v)$ (see [Bas93, Definition 1.15]). Secondly, with respect to a maximal or spanning tree of the graph. Let X be a spanning tree for A, we define $\pi_1(\mathcal{A}, X)$ to be the group generated by the vertex groups A_v and elements t_e for each edge $e \in E\mathcal{A}$ with the relations:

$$\left\{ \begin{array}{c} \text{The relations in the groups } A_v, \\ t_{\bar{e}} = t_e^{-1} \text{ for each (oriented) edge } e, \\ t_e \alpha_{\bar{e}}(g) t_e^{-1} = \alpha_e(g) \text{ for all } g \in A_e, \\ t_e = 1 \text{ if } e \text{ is an edge in } X. \end{array} \right\}$$

Note that the definitions are independent of the choice of basepoint v and spanning tree X and both definitions yield isomorphic groups so we can talk about *the fundamental group* of $\mathcal{A}$, denoted $\pi_1(\mathcal{A})$.

Let G be the fundamental group corresponding to the spanning tree X. For every vertex v and edge e, A_v and A_e can be identified with their images in G. We define a tree with vertices the disjoint union of all coset spaces G/A_v and edges the disjoint union of all coset spaces G/A_e respectively. We call this graph the *Bass-Serre tree* of $\mathcal{A}$ and note that the action of G admits X as a fundamental domain.

Given a group G acting on a tree $\mathcal{T}$, there is a *quotient graph of groups* formed by taking the quotient graph from the action and assigning edge and vertex groups as the stabilisers of a representative of each orbit. Edge monomorphisms are then the inclusions, after conjugating appropriately if incompatible representatives were chosen.

Theorem 3.1. [Bas93] *Up to isomorphism of the structures concerned, the processes of constructing the quotient graph of groups, and of constructing the fundamental group and Bass-Serre tree are mutually inverse.* □

Let $(A, \mathcal{A})$ and $(B, \mathcal{B})$ be graphs of groups. A *morphism of graphs of groups* $\phi : (A, \mathcal{A}) \to (B, \mathcal{B})$ consists of:

(1) A graph morphism $f : A \to B$.

(2) Homomorphisms of local groups $\phi_v : A_v \to B_{f(v)}$ and $\phi_e = \phi_{\bar{e}} : A_e \to B_{f(e)}$.

(3) Elements $\gamma_v \in \pi_1(\mathcal{B}, f(v))$ for each $v \in VA$ and $\gamma_e \in \pi(\mathcal{B})$ for each $e \in EA$ such that if $v = i(e)$ then
- $\delta_e := \gamma_v^{-1} \gamma_e \in \mathcal{B}_{f(v)}$;
- $\phi_a \circ \alpha_e = \mathrm{Ad}(\delta_e) \circ \alpha_{f(e)} \circ \phi_e$.

3.2. **A Structure theorem.** In this section we will define a graph of lattices and prove the structure theorem for $(H \times T)$-lattices.

Definition 3.2 (Graph of lattices). Let H be a locally compact group with Haar measure μ. A *graph of H-lattices* $(A, \mathcal{A}, \psi)$ is a graph of groups $(A, \mathcal{A})$ equipped with a morphism $\psi : \mathcal{A} \to H$ such that:

(1) Each local group $A_\sigma \in \mathcal{A}$ is covirtually an H-lattice and the image $\psi(A_\sigma)$ is an H-lattice;

(2) The local groups are commensurable in $\Gamma = \pi_1(\mathcal{A})$ and their images are commensurable in H;

(3) For each $e \in EA$ the element t_e of the path group $\pi(\mathcal{A})$ is mapped under ψ to an element of $\mathrm{Comm}_H(\psi_e(A_e))$.

Theorem 3.3 (The Structure Theorem - Theorem A). *Let X be a finite dimensional proper* $\mathrm{CAT}(0)$ *space and let $H = \mathrm{Isom}(X)$ contain a uniform lattice. Let $(A, \mathcal{A}, \psi)$ be a graph of H-lattices with locally-finite unimodular non-discrete Bass-Serre tree $\mathcal{T}$, and fundamental group Γ. Suppose $T = \mathrm{Aut}(\mathcal{T})$ admits a uniform lattice.*

(1) *Assume A is finite. If for each local group A_σ the kernel $\mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$, then Γ is a uniform $(H \times T)$-lattice and hence a $\mathrm{CAT}(0)$ group. Conversely, if Λ is a uniform $(H \times T)$-lattice, then Λ splits as a finite graph of uniform H-lattices with Bass-Serre tree $\mathcal{T}$.*

(2) *Under the same hypotheses as* (1)*, Γ is quasi-isometric to $X \times \mathcal{T}$.*

(3) *Assume X is a $\mathrm{CAT}(0)$ polyhedral complex. Let μ be the normalised Haar measure on H. If for each local group A_σ the kernel $K_\sigma = \mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$ and the sum $\sum_{\sigma \in VA} \mu(A_\sigma)/|K_\sigma|$ converges, then Γ is a $(H \times T)$-lattice. Conversely, if Λ is a $(H \times T)$-lattice, then Λ splits as a graph of H-lattices with Bass-Serre tree $\mathcal{T}$.*

We will divert the majority of the proof to the proof of Theorem 5.2 due to the similarity of the theorem statement and arguments involved in the proof. The difference arises from the fact that the category of graphs of groups is not equivalent to the category of 1-complexes of groups (see [Tho06, Proposition 2.1]) due to the difference in morphisms. We highlight the key differences below.

Proof. We first prove (1). The "if direction" is the same as Theorem 5.2(1). For the converse note that an $(H \times T)$-lattice Γ splits as a graph of groups $(A, \mathcal{A})$ by the fundamental theorem of Bass-Serre theory and the projection to H induces a morphism $\pi_H : \mathcal{A} \to H$. The same argument as Theorem 5.2(1) implies that the local groups are commensurable covirtually commensurable H-lattices. In particular, the images of the elements $t_e \in \pi(\mathcal{A})$ for $e \in EA$ are contained in $\mathrm{Comm}_H(\pi_H(A_\sigma))$ for every local group A_σ. ◆

We now prove (2). By (1) Γ acts properly discontinuously cocompactly on $X \times \mathcal{T}$. The result follows from the Švarc-Milnor Lemma [BH99, p. I.8.19]. ◆

The proof of (3) is almost identical to (1) we will highlight the differences. Since X is a CAT(0) polyhedral complex, it follows that $X \times \mathcal{T}$ is. Now, we may apply Serre's Covolume Formula to $\Gamma = \pi_1(\mathcal{A})$. Let Δ be a fundamental domain for Γ acting on $X \times \mathcal{T}$, then the covolume of Γ may be computed as

$$\sum_{\sigma \in \Delta^{(0)}} \frac{1}{|\Gamma_\sigma|} = \sum_{\sigma \in \pi_{\mathcal{T}}(\Delta^{(0)})} \sum_{\tau \in \pi_{\mathcal{T}}^{-1}(\sigma)} \frac{1}{|\Gamma_\tau|} = \sum_{\sigma \in \pi_{\mathcal{T}}(\Delta^{(0)})} \frac{1}{|K_\sigma|} \sum_{\tau \in \pi_{\mathcal{T}}^{-1}(\sigma)} \frac{|K_\sigma|}{|\Gamma_\tau|} = \sum_{\sigma \in \pi_{\mathcal{T}}(\Delta^{(0)})} \frac{\mu(\pi_H(\Gamma_\sigma))}{|K_\sigma|}.$$

Since $\pi_{\mathcal{T}}(\Delta^{(0)})$ can be identified with VA and the later sum converges by assumption, it follows as before that Γ acts faithfully properly discontinuously and isometrically with finite covolume on $X \times Y$. For the converse we proceed as in Theorem 5.2(3). ◆   □

3.3. Reducible lattices.

Let X be a proper minimal CAT(0) space and $H = \mathrm{Isom}(X)$. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \mathrm{Aut}(\mathcal{T})$. We will now characterise reducible uniform $(H \times T)$-lattices by both their projections to H and T, and by the separability of the vertex stabilisers in the projection to T. Moreover, if H is linear, we will show that all such lattices are linear, and thus, residually finite. We say that a subgroup $\Lambda \leqslant \Gamma$ is *separable* if it is the intersection of finite-index subgroups of Γ, *virtually normal* if Λ contains a finite index subgroup N such that $N \trianglelefteq \Gamma$, and *weakly separable* if it is the intersection of virtually normal subgroups of Γ.

Proposition 3.4. *Let X be a proper minimal irreducible* CAT(0) *space or $\mathbb{E}^2$ and let $H = \mathrm{Isom}(X)$. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(H \times T)$-lattice equipped with projections π_H and π_T to H and T respectively, then the following are equivalent:*

(1) $\pi_H(\Gamma)$ is an H-lattice;
(2) $\pi_T(\Gamma)$ is a T-lattice;

(3) *For every vertex $v \in \mathcal{T}$, the projection of the vertex stabiliser $\pi_T(\Gamma_v)$ is separable in $\pi_T(\Gamma)$;*

(4) *There is a vertex $v \in \mathcal{T}$ such that the projection of the vertex stabiliser $\pi_T(\Gamma_v)$ is weakly separable in $\pi_T(\Gamma)$;*

(5) Γ *is a reducible $(H \times T)$-lattice.*

Proof. First, we will show that (1) implies (2), our proof for this case largely follows [BM00b, Proposition 1.2]. Assume $\pi_H(\Gamma)$ is an H-lattice, then $\Gamma \cdot T$ is closed and so $\Gamma \cap T$ is a uniform T-lattice. Now, $\pi_T(\Gamma)$ normalises $\Gamma \cap T$ and hence by [BM00a, p. 1.3.6] is discrete. Thus, $\pi_T(\Gamma)$ is discrete and so is a lattice in T.

Next, we will show that (2) implies (1). Assume $\pi_T(\Gamma)$ is a lattice in T and consider the kernel K of the action of Γ on $\mathcal{T}$. We will show that K is a finite index subgroup of $\pi_H(\Gamma)$. Assume that K has infinite index, then $\pi_H(\Gamma)/K \leqslant \pi_T(\Gamma)$ is an infinite subgroup of the vertex stabiliser, a profinite group, and so cannot be discrete. Thus, K has finite index in $\pi_H(\Gamma)$. Since K acts trivially on $\mathcal{T}$ we see that $K = \Gamma \cap H$. Since $\Gamma \cdot H$ is closed it follows K is an H-lattice. Thus, $\pi_H(\Gamma)$ is virtually a lattice in H and therefore an H-lattice.

Clearly, (5) implies (1) and (2). We will now prove that (1) and (2) imply (5). By the previous paragraph we have $K \leqslant \pi_H(\Gamma)$ finite index. Let $\Gamma_T = \{\gamma \mid (e, \gamma) \in \Gamma\}$, we want to show that Γ_T is a uniform T-lattice. Since all uniform T-lattices are commensurable Γ_T will be a finite index subgroup of $\pi_T(\Gamma)$. By the first paragraph we see Γ_T is a uniform lattice. Thus, $K \times \Gamma_T$ is a finite index subgroup of Γ and so Γ is reducible.

Now, evidently (3) implies (4). To see that (4) implies (5) we apply [Cap+19, Corollary 30] to $\pi_T(\Gamma)$, noting that a cocompact action on a leafless tree does not preserve any subtree, in particular, $\pi_T(\Gamma)$ is discrete. Finally, we show that (5) implies (3). Observe that $\pi_T(\Gamma)$ is a virtually free T-lattice which splits as a finite graph of finite groups. Since $\pi_T(\Gamma)$ is a finite graph of finite groups, the vertex stabilisers are separable subgroups. $\square$

One immediate consequence of the theorem is that we can determine whether a lattice is irreducible simply by considering the projections to either H or T. Also, note that if H is the automorphism group of a unimodular leafless tree then we recover [BM00b, Proposition 1.2] and and [Cap+19, Corollary 32].

We also have the following observations about the linearity and residual finiteness of reducible lattices.

Proposition 3.5. *With the same notation as before, assume H is linear (or lattices in H are residually finite). If Γ is a uniform reducible $(H \times T)$-lattice, then Γ is linear (resp. residually finite).*

Proof. If Γ is reducible, then Γ is virtually a direct product of a linear (resp. residually finite) group with a virtually free group. In particular, Γ is virtually a direct product of linear (resp. residually finite) groups and therefore linear (resp. residually finite). $\qquad\square$

Corollary 3.6. *With H and T as before, assume H is linear. If Γ is a finitely generated uniform $(H \times T)$-lattice, then exactly one of the following holds:*

(1) Γ is reducible and therefore linear (hence residually finite);

(2) Γ is irreducible and linear (hence residually finite);

(3) Γ is irreducible and non-residually finite.

Moreover, if H is a connected centre-free simple linear algebraic group without compact factors and Γ is irreducible and linear, then Γ is arithmetic and just-infinite.

Proof. The first case follows from the previous proposition. Now, assume Γ is irreducible and $\pi_H(\Gamma)$ is injective, then π_H is a faithful linear representation of Γ and we are in the second case. Since Γ is linear, π_T must be injective otherwise Γ would contradict Theorem 2.6. Now, if either of π_T or π_H are not injective, then by Theorem 2.6 we see that Γ is not residually finite. Note that π_T not being injective necessarily implies that π_H is not injective because otherwise Γ would admit a faithful linear representation, contradicting non-residual finiteness. To prove the 'moreover' note that Γ is just-infinite follows from the Bader-Shalom Normal Subgroup Theorem [BS06] applied to the closure of Γ in $H \times T$. The arithmeticity of Γ follows from [BFS19]. $\qquad\square$

Let $vb_p(\Gamma)$ denote the *pth virtual Betti number* of Γ which is defined to be the maximum of the pth Betti number over all finite index subgroups of Γ, or ∞ if the set is unbounded.

Proposition 3.7. *With H and T as before, assume H is a connected centre-free semisimple linear algebraic group without compact factors. Let Γ be a finitely generated uniform irreducible $(H \times T)$-lattice. If $vb_1(\Gamma) > 0$, then Γ is not residually finite. In particular, if $b_1(\mathcal{T}/\Gamma) > 0$, then Γ is not residually finite.*

Proof. Since Γ is irreducible, by the previous corollary, either Γ is linear and just-infinite, or Γ is not residually finite. Now, if the virtual Betti number of Γ is greater than zero, then a finite index subgroup Γ' of Γ admits $\mathbb{Z}$ as a quotient and so cannot be just infinite. Hence, Γ' is not residually finite and so neither is Γ.

The quotient space $\mathcal{T}/\Gamma$ gives rise to a graph of groups splitting of Γ with Bass-Serre tree $\mathcal{T}$. An easy application of the Γ-equivariant Mayer-Vietoris sequence (see [Bro94, Chapter IV.9] for the details) applied to $\mathcal{T}$ shows that $b_1(\Gamma) \geqslant b_1(\mathcal{T}/\Gamma)$. Alternatively note that $\Gamma \twoheadrightarrow \pi_1(\mathcal{A}) \cong F_k$ where $\mathcal{A}$ is the geometric realisation of the graph $\mathcal{T}/\Gamma$ and so homotopy equivalent to a wedge of k copies of S^1. $\qquad\square$

3.4. **Examples.**

Example 1 (Change of tree). Given an edge transitive but not vertex transitive irreducible $(H \times T_{k,\ell})$-lattice Γ one may construct a non-residually finite irreducible $(H \times T_{mk,n\ell})$-lattice for all $m, n \geqslant 2$ as follows:

Firstly, note Γ splits as a graph of H-lattices. Indeed, $\Gamma = A *_C B$ where A, B and C are covirtually H-lattices. We may assume that A stabilises a vertex of valence k and B stabilises a vertex of valence ℓ. Let N_A and N_B be finite groups of order m and n respectively and pick split extensions $\widetilde{A} = N_A \rtimes A$ and $\widetilde{B} = N_B \rtimes B$. We may construct a graph of lattices by considering the graph of groups corresponding to $\widetilde{A} *_C \widetilde{B}$. The representations of $\widetilde{A}$ and $\widetilde{B}$ are the given by the composites $\widetilde{A} \twoheadrightarrow A \to H$ and $\widetilde{B} \twoheadrightarrow B \to H$. The resulting fundamental group $\widetilde{\Gamma}$ acts on the $(mk, n\ell)$-regular tree. Moreover, the lattice is irreducible by Proposition 3.4 and the non-discreteness of the projection to H.

We will compare the notion of a graph of lattices with the "universal covering trick" of Burger–Mozes [BM00a, Section 1.8] and generalised by Caprace–Monod [CM09a, Section 6.C]. In particular, we will show how in many cases one can obtain a graph of lattices from the universal covering trick. We take the opportunity to point out that many of the groups constructed in the previous sections cannot be obtained from universal covering trick.

Example 2 (The universal covering trick). Let A be the geometric realisation of a locally finite graph (not reduced to a single point) and let $Q < \mathrm{Isom}(A)$ be a vertex transitive closed subgroup. Let C be an infinite profinite group acting level transitively on a locally finite rooted tree $\mathcal{T}_0$. Let B be the 1-skeleton of the square complex $A \times \mathcal{T}_0$ and let $\mathcal{T}$ be the universal cover. Define D to be the extension $1 \rightarrowtail \pi_1(B) \rightarrowtail D \twoheadrightarrow C \times Q \twoheadrightarrow 1$. By [CM09a, Proposition 6.8], there exists a CAT(0) space Y such that $D \rightarrowtail \mathrm{Isom}(Y)$ is a closed subgroup, and D acts cocompactly and minimally without fixed point at infinity.

The classical situation where this is applied is as follows: Let Q be a product of p-adic Lie groups, H be a product of real Lie groups and $\Gamma < H \times Q$ to be an S-arithmetic irreducible lattice. Let A be the 1-skeleton of the Bruhat-Tit's building for X, let $\mathcal{T}$ be the universal cover of A and let $T = \mathrm{Aut}(\mathcal{T})$. Now, Γ lifts to a weakly irreducible lattice $\widetilde{\Gamma} < H \times Q \times T$ and the corresponding graph of lattices is obtained by considering the graph A/Γ equipped with local groups given by the stabilisers of the action of Γ on A.

As an brief application we will construct (virtually) torsion-free irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T_{10n})$-lattices. Note that for n odd these constitute the first explicit examples of such irreducible lattices.

Example 3. Recall from [LM21, Example 9.2] the Leary-Minasyan group $\mathrm{LM}(A)$ where A is the matrix corresponding to the Pythagorean triple $(3, 4, 5)$ which acts on $\mathbb{E}^2 \times \mathcal{T}_{10}$. (Note that these groups were classified up to isomorphism by Valiunas [Val20].) By [LM21,

Example 9.4], this has presentation

$$\mathrm{LM}(A) = \langle a, b, t \mid [a, b],\ ta^2 b^{-1} t^{-1} = a^2 b,\ tab^2 t^{-1} = a^{-1} b^2 \rangle.$$

Using this group we will construct a virtually torsion-free irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice where T is the automorphism group of the $10n$-regular tree for all $n \geqslant 3$.

Let $\mathbb{Z}^n = \langle a_0, \dots, a_{n-1} \rangle$ and let $F = \langle f \rangle$ be a cyclic group of order n acting on L by cyclically permuting the a_i. Let $L = \mathbb{Z}^n \rtimes F$, this is a crystallographic group and so embeds into $\mathrm{Isom}(\mathbb{E}^n)$. Now, consider the $(n \times n)$-matrix B given by

$$B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & I_{n-2} \end{bmatrix}.$$

We define Γ_n to be the HNN extension of L by the matrix B, the Bass-Serre tree of this HNN extension will be regular of valence $10n$. This has generators $a_0, \dots, a_{n-1}, f, t$ and relations

$$f^n = 1,\ [a_i, a_j] = 1,\ f a_i f^{-1} = a_{i+1} \pmod{n},\ [a_2, t] = 1,\ \dots,\ [a_{n-1}, t] = 1,$$

$$ta_0^2 a_1^{-1} t^{-1} = a_0^2 a_1,\ ta_0 a_1^2 t^{-1} = a_0^{-1} a_1^2,$$

where $i, j \in \{0, \dots, n-1\}$. Here the first three sets of relation come from L, the relations $[a_i, t] = 1$ for $i \geqslant 2$ come from the fact B fixes $\{a_2, \dots, a_{n-1}\}$ point-wise, and the last two relations arise from the action of B on $\langle a_0, a_1 \rangle$. Now, let $a := a_0$, then we may write Γ_n as

$$\Gamma_n = \langle a, f, t \mid f^n = 1,\ ta^2 a^{-f} t^{-1} = a^2 a^f,\ ta(a^2)^f t^{-1} = a^{-1}(a^2)^f,\ [a^{f^i}, a^{f^j}] = 1 \rangle$$

for $i, j \in \{0, \dots, n-1\}$. Thus, Γ_n is a 3 generator, $\frac{1}{2}n(n-1) + 3$ relator group.

To see Γ_n is irreducible note that $\pi_{\mathrm{O}(n)}(\Gamma)$ is not virtually contained in some $\prod_j \mathrm{O}(k_j) < \mathrm{O}(n)$. Indeed, consider the subgroup generated by the $\pi_{\mathrm{O}(n)}(f)$-orbit of $\pi_{\mathrm{O}(n)}(t)$. To show Γ_n is virtually torsion-free note that every torsion element of Γ_n has non-trivial image in $\pi_{\mathrm{O}(n)}(\Gamma_n)$. This is generated by the images of f and t and so is a finitely generated linear group and hence has a finite index torsion-free subgroup P_n. The preimage of P_n in Γ_n is torsion-free.

4. AN APPLICATION TO C^*-SIMPLICITY

In this section we will consider the C^*-simplicity of $(H \times T)$-lattices. Let Γ be a discrete group. Recall that the *reduced C^*-algebra* of Γ, denoted $C_r^*(\Gamma)$, is the norm closure of the algebra of bounded operators on $\ell^2(\Gamma)$ by the left regular representation of Γ. We say Γ is *C^*-simple* if $C_r^*(\Gamma)$ has exactly two norm-closed two-sided ideals 0 and $C_r^*(\Gamma)$ itself.

The C^*-simplicity of graphs of groups has been considered before [HP11], however, the methods developed there are not applicable to $(H \times T)$-lattices because the vertex and edge groups are all commensurable. Instead, we will apply the machinery developed in

[Bre+17] to prove the C^*-simplicity of $(H \times T)$-lattices via properties of either H or the action on $\mathcal{T}$.

Let Γ be a group. We say a subgroup H is *normalish* if for every $n \geqslant 1$ and $t_1 \ldots, t_n$ the intersection $\bigcap_{i=1}^n H^{t_i}$ is infinite.

Proposition 4.1. *Let Γ be the fundamental group of a (possibly infinite) graph of finite groups with leafless Bass-Serre tree $\mathcal{T}$ not quasi-isometric to $\mathbb{R}$. If Γ is infinite, not virtually cyclic and acts faithfully on $\mathcal{T}$, then Γ is C^*-simple.*

Proof. As Γ is not finite or virtually cyclic Γ has a positive (possibly infinite) first L^2-Betti number. Indeed, the chain complex of the Bass-Serre tree $C_*(\mathcal{T}; \mathcal{U}\Gamma)$, which is concentrated in dimension 0 and 1, may be used to compute the L^2-homology. As Γ is infinite the boundary map is surjective and so the L^2-homology is concentrated in degree 1. We may pair each orbit of 0-cells v with an orbit of 0-cells e contained in the boundary of e, in each case the dimension of the $\mathcal{U}\Gamma$-module is $1/|\Gamma_v|$ or $1/|\Gamma_e|$, and $1/|\Gamma_e| - 1/|\Gamma_v| \geqslant 0$. Since Γ is non-trivial and not virtually cyclic some of these inequalities must be strict. In particular, we conclude Γ has a (possibly infinite) non-trivial first L^2-Betti number. Since Γ has a trivial amenable radical and a non-trivial L^2-Betti number we may apply [Bre+17, Theorem 6.5] to deduce that Γ is C^*-simple.

Alternatively, we first note that any normalish subgroup of Γ contains a free subgroup since Γ is a faithful graph of finite groups and is not virtually cyclic. Now, we apply [Bre+17, Theorem 6.2] to deduce that Γ is C^*-simple. $\square$

The following theorem and corollary give a partial answer to two questions of de la Harpe [Har07] and consider the more general case of an arbitrary graph of groups. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \mathrm{Aut}(\mathcal{T})$. The theorem implies the following lattices are C^*-simple:

- H is a semisimple Lie group with trivial centre and Γ is a graph of S-arithmetic lattices. This new whenever Γ is not residually finite. To see this, apply (1) and (2a);
- Γ is a lattice in a product of trees. To see this, apply (2c);
- Γ is the fundamental group of a graph of lattices where each vertex and edge group acts on the universal cover of a Salvetti complex corresponding to a right-angled Artin group with trivial centre. To see this, apply (1) and (2a) to [Bre+17, Theorem 1.6];
- H is the automorphism group of an affine building with no irreducible factor isometric to $\mathbb{E}^n$ and Γ is an irreducible $(H \times T)$-lattice. To see this, apply (2a);
- H is the automorphism group of a hyperbolic building and Γ is an irreducible $(H \times T)$-lattice. To see this, apply (2a);
- H is a product of the above and Γ is an irreducible $(H \times T)$-lattice. To see this, apply (2a);

- $\mathrm{Isom}(\mathbb{E}^n)$ and Γ is an irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. Note this characterises irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices and will follow from (2b) (see Corollary 4.3).

The results in this list are new whenever the $(H \times T)$-lattices in question are not cubical or linear groups.

Theorem 4.2. *Let $X = X_1 \times \cdots \times X_k$ be a product of proper minimal cocompact $\mathrm{CAT}(0)$-spaces each not isometric to $\mathbb{R}$ and let $H = \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_k)$ act without fixed point at infinity. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \mathrm{Aut}(\mathcal{T})$. Let $n \geqslant 0$ and $\Gamma < \mathrm{Isom}(\mathbb{E}^n) \times H \times T$ be a finitely generated lattice.*

(1) Assume Γ is reducible and $n = 0$. Then Γ is C^*-simple if and only if $\Gamma \cap H$ is C^*-simple, and Γ has the icc property.

(2) Assume Γ is weakly irreducible. If one of the following holds:

    (a) H-lattices have no normalish amenable subgroups;

    (b) $\mathrm{Ker}(\pi_T)$ is trivial and $\mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n) \times H})$ is infinite;

    (c) H-lattices have a non-zero L^2-Betti number and trivial amenable radical;

then Γ is C^*-simple.

Proof. In the reducible case Γ virtually splits as $F_n \times \Gamma_H$. The result follows from the following three observations [Har07, Proposition 19 (i, iii, iv)], a direct product of two C^*-simple groups is C^*-simple, finite index subgroups of C^*-simple groups are simple, and a virtually C^*-simple group is C^*-simple if and only if it satisfies the icc property.

Now, assume Γ is irreducible. We will show that (2a) implies C^*-simplicity. Since Γ is finitely generated $\mathcal{G} = \mathcal{T}/\Gamma$ is finite. We first show that any amenable normalish subgroup N of Γ must fix a vertex of $\mathcal{T}$. Let $g \in \Gamma$ act as a hyperbolic element on $\mathcal{T}$, choose any other element $h \in \Gamma$ acting hyperbolically on $\mathcal{T}$ with an axis not equal to g, then any normalish subgroup N containing g contains the free group $\langle g, g^h \rangle$ and so cannot be amenable. Thus, N fixes a vertex of $\mathcal{T}$. Now, by Theorem 3.3 every vertex and edge stabiliser of Γ is a finite-by-{H-lattice} group. Since by assumption H-lattices do not contain any normalish amenable subgroups, neither does Γ. It remains to verify that Γ has no finite normal subgroups, but Γ has trivial amenable radical by [CM09a, Corollary 2.7]. In particular the result now follows from [Bre+17, Theorem 6.2].

We next prove (2b) implies C^*-simplicity. Let $K = \mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n) \times H})$, we have that Γ is an extension of K by $\pi_{\mathrm{Isom}(\mathbb{E}^n) \times H}(\Gamma)$. Now, K is a (possibly infinite) graph of finite groups acting faithfully on $\mathcal{T}$. Indeed, restricting $\pi := \pi_{\mathrm{Isom}(\mathbb{E}^n) \times H}$ to a vertex stabiliser $\Gamma_v < \Gamma$ of the action on $\mathcal{T}$, by Theorem 3.3 we see $\mathrm{Ker}(\pi|_{\Gamma_v})$ is finite. Every finite subgroup of Γ, and hence K, is conjugate to a finite subgroup of some vertex stabiliser. Thus, the graph of groups decomposition is given by $\mathcal{T}/K$.

We claim K is not virtually infinite cyclic. Indeed, if K was virtually cyclic, then there exists a commensurated infinite cyclic subgroup $Z < K < \Gamma$. By [CM19, Theorem 2(i)] Z acts properly on $\mathbb{E}^n$ in the decomposition of X. But $Z < K$, a contradiction.

It follows the group K is C^*-simple by Proposition 4.1. Because $\mathrm{Ker}(\pi_T)$ is trivial, every element acts non-trivially on $\mathcal{T}$ and so the centraliser $C_\Gamma(K)$ is trivial. Now, we apply [Bre+17, Theorem 1.4] to prove the result.

Finally, we will prove (2c) implies C^*-simplicity. We apply the cohomology Γ-equivariant Mayer-Vietoris sequence with $\mathcal{U}\Gamma$ coefficients arising from filtering $E\Gamma$ by the Bass-Serre tree [Bro94, Chapter VII.9]. Since $\mathcal{T}$ is not a quasi-line there is a vertex v connected to an edge e such that the stabilisers satisfy $|\Gamma_v : \Gamma_e| \geqslant 3$, thus the L^2-Betti numbers of Γ_e are at least 3 times the L^2-Betti number of Γ_v. Now, additivity of von Neumann dimension over exact sequences and a simple counting argument implies every $(H \times T)$-lattice must have a non-trivial L^2-Betti number. Alternatively, we note that every $(H \times T)$-lattice is measure equivalent to $L \times F_r$ where L is an H-lattice and F_r is a free group. Now, an application of the Kunneth formula yields that $L \times F_r$ has a non-trivial L^2-Betti number and so by Gaboriau's theorem [Gab02, Theorem 6.3] so does every $(H \times T)$-lattice. By [CM09a, Corollary 2.7] every $(H \times T)$-lattice has trivial amenable radical, the result follows from [Bre+17, Theorem 6.5]. □

Theorem 4.3 (Theorem C). *Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \mathrm{Aut}(\mathcal{T})$. Let $n \geqslant 1$ and $\Gamma < \mathrm{Isom}(\mathbb{E}^n) \times T$ be a uniform lattice. Then, Γ is C^*-simple if and only if Γ is irreducible.*

Proof. Suppose Γ is reducible, then Γ has a finite index subgroup Λ splitting as $\mathbb{Z} \times \Lambda'$ which has non-trivial amenable radical and so cannot be C^*-simple. Suppose Γ is irreducible, then by Proposition 4.4 Γ acts on $\mathcal{T}$ faithfully. Now, Γ is non-residually finite by Theorem 2.6 so $\mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)})$ is non-trivial (since otherwise Γ would be a finitely generated linear group). Moreover, $\mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)})$ is infinite (if it was finite it would fix $\mathcal{T}$). Now, by Theorem 4.2(2b) we deduce that Γ is C^*-simple □

The following propositions give criteria for irreducibility in terms of the action of Γ on $\mathcal{T}$.

Proposition 4.4. *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. Then Γ is weakly and algebraically irreducible if and only if Γ acts on $\mathcal{T}$ faithfully.*

We will also state and prove the analogous result for lattices in the product of the automorphism group of a CAT(0) polyhedral complex (with mild hypothesis) and $\mathrm{Isom}(\mathbb{E}^n)$. We will prove both results simultaneously.

Proposition 4.5. *Let X be an irreducible locally finite CAT(0) polyhedral complex and let $A = \mathrm{Aut}(X)$ act cocompactly and minimally. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice, then Γ is weakly and algebraically irreducible if and only if Γ acts on X faithfully.*

Proof of Proposition 4.4 and 4.5. Assume Γ is irreducible. By [CM19, Corollary 3], Γ has finite amenable radical B. Such a non-trivial element $g \in B$ stabilises a vertex of the Bass-Serre tree $\mathcal{T}$ (resp. complex X). Now, either g has infinitely many conjugates which contradicts the finiteness of B, or g stabilises the whole of $\mathcal{T}$ (resp. X) and so is contained in $\Gamma \cap \mathrm{Isom}(\mathbb{E}^n)$.

Suppose we are in the later case, then B, by the graph (resp. complex) of lattices description of Γ, acts faithfully on every vertex stabiliser of Γ. Let t be an infinite order element of $\pi_{\mathrm{O}(n)}(\Gamma)$ such that the intersection of the complements of the fixed point subspaces of t and B in $\mathbb{E}^n$ is positive dimensional (and so at least dimension 2) - such an element exists by the definition of irreducibility. Consider a vertex stabiliser Γ_v in the action on $\mathcal{T}$ (resp. X) and let E^+ denote a subgroup of Γ_v corresponding to a 2-dimensional subspace fixed by neither t nor B.

Claim: $L := \bigcap_{n \in \mathbb{Z}} \Gamma_v^{t^n} \cap E^+$ *is trivial.*

The claim implies that every element of E^+ acts non-trivially on $\mathcal{T}$ (resp. X) and so since B acts non-trivially on E^+ must also act non-trivially on $\mathcal{T}$ (resp. X). A contradiction. Thus Γ acts faithfully on $\mathcal{T}$ (resp. X).

Proof of claim: Suppose the intersection L is non-trivial, then L is normalised by $\langle t \rangle$. It follows that $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(L)$ is normalised by $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\langle t \rangle)$ which is clearly nonsensical unless $L = \{1\}$ because t is an infinite order rotation of the flat $\pi_{\mathbb{E}^n}(E^+ \otimes \mathbb{E})$. ◆

Suppose, for the converse, that Γ acts on $\mathcal{T}$ (resp. X) faithfully. Decompose minimally the closure of $\pi_{\mathrm{O}(n)}(\Gamma)$ as $\prod_{i=1}^{\ell} O_i$ where $O_i = \mathrm{O}(k_i)$ and $\sum_{i=1}^{\ell} k_i = n$. To show Γ is irreducible we need to show that each $k_i \geqslant 2$, and that π_T (resp. π_A), π_{O_I} and $\pi_{O_I \times T}$ (resp. $\pi_{O_I \times A}$) are non-discrete for each proper subset $I \subset \{1 \dots, \ell\}$.

First, suppose some $k_i = 1$, then by [CM19] Γ fixes a points in $\partial X \times \mathbb{E}^n$ and so virtually splits as $\Gamma' \times \mathbb{Z}$. It follows from [CM19, Theorem 2(ii)] that the direct factor $\mathbb{Z}$ acts on X trivially, contradicting faithfulness of the action. Thus, each $k_i \geqslant 2$.

Now, by [CM19, Theorem 2] there is a commensurated free abelian subgroup L of Γ of rank n which fixes a vertex of $\mathcal{T}$ (resp. X). Since Γ acts on $\mathcal{T}$ (resp. X) faithfully, the projection of L to T (resp. A) fixes a vertex and so is an infinite subgroup of a profinite group. In particular, the projection of L and hence, π_T (resp. π_A) are non-discrete.

Next, suppose some π_{O_I} is discrete, then by an identical argument to the previous paragraph Γ virtually splits as direct product $\Gamma' \times \mathbb{Z}$ and the direct factor $\mathbb{Z}$ acts on X trivially contradicting faithfulness of the action. Thus, the projections π_{O_I} are non-discrete for each proper subset $I \subset \{1 \dots, \ell\}$.

Finally, suppose some $\pi_{O_I} \times T$ (resp. $\pi_{O_I} \times A$) is discrete. Note that by [CM19, Theorem 2] the largest rank of a commensurated abelian subgroup of Γ is exactly n. Since Γ acts on $\mathcal{T}$ (resp. X) faithfully, the projection is injective. Let $m = \sum_{i \in I} k_i$ and note that this is strictly less than n It follows that the projection of Γ is an $(\mathrm{Isom}(\mathbb{E}^m) \times T)$-lattice (resp. $(\mathrm{Isom}(\mathbb{E}^m) \times A)$-lattice). In particular, the largest rank of a commensurated

abelian subgroup of Γ is exactly m, but $m < n$, a contradiction. Thus, the projections $\pi_{O_I} \times T$ (resp. $\pi_{O_I} \times A$) are non-discrete for each proper subset $I \subset \{1 \ldots, \ell\}$. $\qquad\square$

5. Complexes of lattices

In this section we will introduce the notion of a complex of H-lattices. We will then prove a structure theorem analogous to Theorem 3.3 for these complexes of H-lattices.

5.1. **Complexes of groups.** The definitions in this section are adapted from [Tho06, Section 1.4] and [Hae91; Hae92]. Throughout this section if X is a polyhedral complex then X' is its first barycentric subdivision. This is a simplicial complex with vertices VX' and edges EX'. Each $e \in EX'$ corresponds to cells $\tau \subset \sigma$ of X and so we may orient them from σ to τ. We will write $i(e) = \sigma$ and $t(e) = \tau$. We say two edges e and f of X' are *composable* if $i(e) = t(f)$, in which case there exists an edge $g = ef$ of X' such that $i(c) = i(e)$ and $t(c) = t(f)$, and e, f and g form the boundary of a 2-simplex in X. We denote the set of composable edges by E^2X'.

A *complex of groups* $G(X) = (G_\sigma, \psi_e, g_{e,f})$ over a polyhedral complex X is given by the following data:

(1) For each vertex σ of VX', a group G_σ called the *local group at σ*.
(2) For each edge e of EX', a monomorphism $\psi_e : G_{i(e)} \to G_{t(e)}$ called the *structure map*.
(3) For each pair of composable edges e and f, an element $g_{e,f} \in G_{t(e)}$ called the *twisting element*. We require these elements to satisfy the following conditions:
   (a) For $(e, f) \in EX'$, we have $\mathrm{Ad}(g_{e,f})\psi_{ef} = \psi_e\psi_f$.
   (b) For each triple of composable edges a, b and c we have a *cocycle condition*
   $$\psi_a(g_{b,a}) = g_{c,b}g_{cb,a}.$$

We say $G(X)$ is *simple* if each of the twisting elements $g_{e,f}$ are the identity.

Some complexes of groups arise from actions on polyhedral complexes. Let G be a group acting without inversions on a polyhedral complex Y. Let $X = Y/G$ with natural projection $p : Y \to X$. For each $\sigma \in VX'$, choose a lift $\tilde{\sigma} \in VY'$ such that $p\tilde{\sigma} = \sigma$. The local group G_σ is the stabiliser of $\tilde{\sigma}$ in G, and the structure maps and twisting elements are given by further choices. The resulting complex of groups $G(X)$ is unique up to isomorphism. A complex of groups isomorphic to a complex of groups arising from a group action is called *developable*.

Let $G(X)$ be a complex of groups over a polyhedral complex X. Let T be a maximal tree in the 1-skeleton of X' and fix a basepoint σ in T. The *fundamental group* of $G(X)$, denoted $\pi_1(G(X), \sigma_0)$, is generated by the set

$$\coprod_{\sigma \in VX'} G_\sigma \coprod \{e^+, e^- : e \in EX'\}$$

subject to the relations

$$\begin{cases} \text{the relations in the groups } G_\sigma, \\ (e^+)^{-1} = e^- \text{ and } (e^-)^{-1} = e^+, \\ e^+ f^+ = g_{e,f}(ef)^+, \ \forall (e,f) \in E^2 X', \\ \psi_e(g) = e^+ g e^-, \ \forall g \in G_{i(e)}, \\ e^+ = 1, \ \forall e \in T. \end{cases}$$

If $G(X)$ is developable, then it has a *universal cover* $\widetilde{G(X)}$. This is a simply connected polyhedral complex, equipped with an action of $G = \pi_1(G(X), \sigma_0)$ such that the complex of groups given by $\widetilde{G(X)}/G$ is isomorphic to $G(X)$.

Let $G(X) = (G_\sigma, \psi_e)$ and $H(Y) = (H_\tau, \psi_f)$ be complexes of groups over polyhedral complexes X and Y. Let $f : X' \to Y'$ be a simplicial map sending vertices to vertices and edges to edges. A *morphism* $\Phi : G(X) \to H(Y)$ over f consists of:

(1) A homomorphism $\phi_\sigma : G_\sigma \to H_{f(\sigma)}$ for each $\sigma \in VX'$.
(2) For each $e \in EX'$ an element $g_e \in H_{t(f(e))}$ such that
   (a) $\mathrm{Ad}(g_e)\psi_{f(e)}\phi_{i(e)} = \phi_{t(e)}\psi_e$;
   (b) For all $(a,b) \in E^2 X'$ we have $\phi_{t(a)}(g_{a,b})g_{ab} = g_e\psi_{f(a)}(g_b)g_{f(a),f(b)}$.

5.2. Complexes of lattices.

In this section we introduce complexes of lattices in analogy with the graphs of lattices we defined previously.

Definition 5.1 (Complex of lattices)**.** Let H be a locally compact group with Haar measure μ. A *complex of H-lattices* $(G(X), \psi)$ is a developable complex of groups equipped with a morphism ψ to H such that:

(1) For each $\sigma \in VX'$, the local group G_σ is covirtually an H-lattice and the image $\psi(G_\sigma)$ is an H-lattice;
(2) The local groups are commensurable in $\Gamma = \pi_1(G(X), \sigma)$ and their images are commensurable in H.
(3) For each $e \in EX'$, the elements e^+ and e^- in Γ are mapped to elements of $\mathrm{Comm}_H(\psi(G_\sigma))$.

The analogous structure theorem is given as follows.

Theorem 5.2. *Let X be a finite dimensional proper $\mathrm{CAT}(0)$ space and let $H = \mathrm{Isom}(X)$ contain a uniform lattice. Let $(G(Z), \psi)$ be a complex of H-lattices over a polyhedral complex Z, with universal cover Y, and fundamental group Γ. Suppose $A = \mathrm{Aut}(Y)$ admits a uniform lattice.*

(1) Assume Z is finite and Y is a $\mathrm{CAT}(0)$ space. If for each local group G_σ the kernel $\mathrm{Ker}(\psi|_{G_\sigma})$ acts faithfully on Y, then Γ is a uniform $(H \times A)$-lattice and hence a $\mathrm{CAT}(0)$ group. Conversely, if Λ is a uniform $(H \times A)$-lattice, then Λ splits as a finite complex of uniform H-lattices with universal cover Y.

(2) *Under the same hypotheses as* (1), Γ *is quasi-isometric to* $X \times Y$.

(3) *Assume* X *is a* $\mathrm{CAT}(0)$ *polyhedral complex and* Y *is a* $\mathrm{CAT}(0)$ *space. Let* μ *be the normalised Haar measure on* H. *If for each local group* G_σ *the kernel* $K_\sigma = \mathrm{Ker}(\psi|_{G_\sigma})$ *acts faithfully on* Y *and the sum* $\sum_{\sigma \in VZ} \mu(G_\sigma)/|K_\sigma|$ *converges, then* Γ *is a* $(H \times A)$-*lattice. Conversely, if* Λ *is a* $(H \times A)$-*lattice, then* Λ *splits as a finite complex of* H-*lattices with universal cover* Y.

Note that by definition we are assuming all complexes of lattices are developable complexes of groups.

Proof. We first prove (1). The fundamental group Γ of $G(Z)$ acts on the universal cover Y and on X via the homomorphism $\psi : \Gamma \to H$. The action on the product space $X \times Y$ is properly discontinuous cocompact and by isometries. The kernel of the action is contained in the intersection $\bigcap_{\sigma \in Z'} \mathrm{Ker}(\psi|_{G_\sigma})$. But this acts faithfully on Y, thus, the action is faithful. It follows Γ is an $(H \times A)$-lattice.

We now prove the converse. Assume Γ is an $(H \times A)$-lattice, and note that the action of Γ on Y yields a developable complex of groups $G(Z) = (\Gamma_\sigma, \psi_a, g_{a,b})$ with spanning tree T and equipped with a homomorphism $\pi_H : \Gamma \to H$. It suffices to show the local groups corresponding to the vertices of Z are covirtually H-lattices. Indeed, for an edge $e \in EZ'$, if the index $|\Gamma_{t(e)} : \psi_e(\Gamma_{i(e)})|$ is infinite, then the universal cover of $G(Z)$ would not be locally finite. It follows that all of the local groups are commensurable and hence, commensurable in H. Consequently, the elements e^+ and e^- for all $e \in E^2 Z'/T$ in Γ must commensurate the local groups.

Let $\sigma \in Y$ be a vertex and consider the stabiliser $\Gamma_\sigma < \Gamma$ for the action on $X \times Y$. Suppose Γ_σ does not act cocompactly on $X \times \sigma$, then there is no compact set whose Γ_σ translates cover $X \times \sigma$. Let D be a non-compact set whose Γ_σ-translates cover $X \times \sigma$, but there is a compact set C whose Γ translates cover $X \times Y$. We may arrange our subsets such that $C' = C \cap (X \times \sigma) \subseteq D$. In particular, there are elements $g_i \in \Gamma/\Gamma_\sigma$ whose translates of C' cover D. But some of these elements must fix $X \times \sigma$ yielding a contradiction. Hence, Γ_σ is cocompact.

It is clear that $\mathrm{Ker}(\Gamma_\sigma \to H)$ is finite. Otherwise Γ would act with infinite point stabilisers on $X \times Y$ contradicting the discreteness of Γ. It remains to show that the projection $\overline{\Gamma}_\sigma$ of Γ_σ to H is discrete. Assume that $\overline{\Gamma}_\sigma$ is not discrete, then there does not exists a neighbourhood N of $1 \in H$ such that $N \cap \overline{\Gamma}_\sigma = \{1\}$. But this implies there does not exist a neighbourbood N' of $1 \in H \times A$ such that $N' \cap \Gamma = \{1\}$ which contradicts the discreteness of Γ. It follows Γ_σ is covirtually an H-lattice.

The final step is to show the elements e^+ and e^- for each $e \in EX'$ are mapped to elements of $\mathrm{Comm}_H(\pi_H(\Gamma_\sigma))$. But this is immediate since the local groups map to H with finite kernel, the elements e^+ and e^- commensurate the local groups, and so must still preserve the appropriate conjugation relations in the map to H. ◆

We now prove (2). By (1), Γ acts properly discontinuously cocompactly on $X \times Y$. The result follows from the Švarc-Milnor Lemma [BH99, p. I.8.19]. ◆

The proof of (3) is almost identical to (1) we will highlight the differences. Since X is a CAT(0) polyhedral complex, it follows that $X \times Y$ is. Now, we may apply Serre's Covolume Formula to Γ. Let Δ be a fundamental domain for Γ acting on $X \times Y$, then the covolume of Γ may be computed as

$$\sum_{\sigma \in \Delta^{(0)}} \frac{1}{|\Gamma_\sigma|} = \sum_{\sigma \in \pi_Y(\Delta^{(0)})} \sum_{\tau \in \pi_Y^{-1}(\sigma)} \frac{1}{|\Gamma_\tau|} = \sum_{\sigma \in \pi_Y(\Delta^{(0)})} \frac{1}{|K_\sigma|} \sum_{\tau \in \pi_Y^{-1}(\sigma)} \frac{|K_\sigma|}{|\Gamma_\tau|} = \sum_{\sigma \in \pi_Y(\Delta^{(0)})} \frac{\mu(\pi_X(\Gamma_\sigma))}{|K_\sigma|}.$$

Since $\pi_Y(\Delta^{(0)})$ can be identified with Z and the later sum converges by assumption, it follows as before that Γ acts faithfully properly discontinuously and isometrically with finite covolume on $X \times Y$. For the converse the only adjustment required is that the compact sets C and C' in the proof of (1) should be replaced with ones of finite covolume. The remainder of the proof is identical. ◆ □

The following corollaries are immediate.

Corollary 5.3 (Corollary B). *Let $X = X_1 \times X_2$ be a proper cocompact minimal CAT(0) space and $H = \mathrm{Isom}(X_1) \times \mathrm{Isom}(X_2)$. Suppose X_1 is a CAT(0) polyhedral complex. Then, for any uniform H-lattice Γ, the cell stabilisers of X_1 in Γ are commensurated and isomorphic to finite-by-$\{\mathrm{Isom}(X_2)$-lattices$\}$.*

Corollary 5.4. *Let $X = \mathbb{E}^n \times M \times \prod_{i=1}^m X_i$, where $n \geqslant 0$, M is a symmetric space with no compact factors and corresponding semisimple Lie group H with trivial centre, and each X_i is a cocompact minimal proper CAT(0) polyhedral complex. Let $G = \mathrm{Isom}(\mathbb{E}^n) \times H \times \prod_{i=1}^m \mathrm{Isom}(X_i)$. Then, every uniform G-lattice has a commensurated subgroup, isomorphic to a finite-by-$\{\mathbb{Z}^n \times L\}$ group where L is a uniform H-lattice, and which fixes a vertex v in $\prod_{i=1}^m X_i$ and acts properly cocompactly on $\mathbb{E}^n \times M \times \{v\}$.*

6. Biautomaticity in the presence of a Euclidean factor

In this section we give a condition to determine the failure of biautomaticity for a CAT(0) group in the presence of a non-trivial Euclidean de Rham factor.

For the rest of this section we fix the following notation and terminology, the treatment roughly follows [LM21, Section 2] and [Eps+92, Section 2.3, 2.5]. Let $\mathcal{A}$ be a finite set and let Γ be a group with a map $\mu : \mathcal{A} \to \Gamma$. We say that Γ is *generated by* $\mathcal{A}$ if the unique extension of μ to the homomorphism from the free monoid $\mathcal{A}^*$ to Γ is surjective. We will call elements of $\mathcal{A}^*$ *words* and for any $w \in \mathcal{A}^*$, if $\mu(w) = g$ for some $g \in \Gamma$, we will say w *represents* g. We will always assume $\mathcal{A}$ is closed under inversion, that is, there is an involution $i : \mathcal{A} \to \mathcal{A}$ such that $\mu(i(a)) = \mu(a)^{-1}$, in this case we will denote $i(a)$ as a^{-1}. Any *subset* $\mathcal{L} \subseteq \mathcal{A}^*$ will be called a *language over* $\mathcal{A}$.

An *automatic structure* for a group Γ is a pair $(\mathcal{A}, \mathcal{L})$, where $\mathcal{A}$ is a finite generating set of Γ equipped with a map $\mu : \mathcal{A} \to \Gamma$ and closed under inversion, and $\mathcal{A} \subseteq \mathcal{A}^*$ is a language satisfying three conditions. Firstly, $\mu(\mathcal{L}) = \Gamma$, secondly $\mathcal{L}$ is a *regular language*, that is, it is accepted by some finite state automaton, and thirdly, it satisfies a fellow traveller property (which we will not make precise here). We say $(\mathcal{A}, \mathcal{L})$ is *biautomatic structure* if both $(\mathcal{A}, \mathcal{L})$ and $(\mathcal{A}, \mathcal{L}^{-1})$ are automatic structures. A group Γ is said to be *automatic (resp. biautomatic)* if it admits an automatic (resp. biautomatic) structure.

A (bi)automatic structure is *finite-to-one* if $|\mu^{-1}(g) \cap \mathcal{A}| < \infty$ for all $g \in \Gamma$. As noted in [LM21, Page 8] by [Eps+92, Theorem 2.5.1] it may be assumed that all (bi)automatic structures are finite-to-one. So without loss of generality we will make this assumption and we will also suppose that all the automata in this paper have no dead states.

A subgroup $H < \Gamma$ is $\mathcal{L}$-*quasiconvex* if there exists $\kappa \geqslant 0$ such that for any path p in the Cayley graph of Γ with respect to $\mathcal{A}$, starting at 1_Γ, ending at some $h \in H$, and labelled by a word $w \in \mathcal{L}$, then every vertex of p lies in the κ-neighbourhood of H. The main examples of $\mathcal{L}$-quasiconvex subgroups are centralisers of finite subsets as proved in [GS91, Proposition 4.3] and [Eps+92, Theorem 8.3.1 and Corollary 8.3.5].

Theorem 6.1. *Let $X = \prod_{i=1}^m X_i$ be a product of proper irreducible $\mathrm{CAT}(0)$ spaces each not isometric to $\mathbb{E}$ and $H < \mathrm{Isom}(X)$ be a closed subgroup acting minimally and co-compactly on X. Let $n \geqslant 2$ and let Γ be an $(\mathrm{Isom}(\mathbb{E}^n) \times H)$-lattice. If the projection $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is not discrete, then Γ is not virtually biautomatic.*

Proof. Assume $(\mathcal{B}, \mathcal{L})$ is a biautomatic structure on Γ. By [CM19, Theorem 2(i)] there exists a commensurated free abelian subgroup $A \leqslant \Gamma$ acting properly on $\mathbb{E}^n$ of rank n.

Claim: *There is a finite index subgroup of A that is $\mathcal{L}$-quasiconvex.*

By the Flat Torus Theorem the rank of a maximal abelian subgroup of Γ is bounded by the rank of a maximal flat in $X \times \mathbb{E}^n$. Let F be such a flat acted on by A. Fix a set of generators S_A for A and a set of generators S containing S_A for the maximal abelian subgroup containing A stabilising F.

We may split X into a product $Y_1 \times Y_2$ where A acts trivially on Y_1 and non-trivially on Y_2. For $j = 1, 2$ let $K_j = \mathrm{Isom}(Y_j) \cap H$. Since, A acts trivially on Y_1 it follows A and $\Gamma \cap K_1$ commute. Now, Γ splits as a complex of $(\mathrm{Isom}(\mathbb{E}^n) \times K_1)$-lattices. Note that any symmetric space factor of X is contained in Y_1 since A projects trivially by [CM19, Lemma 6]. In particular, A is a subgroup of a vertex group Γ_v, which is covirtually virtually isomorphic to $A \times K_v$, where K_v is a lattice in K_1. Define S_K to be a set of generators for K_v and for each $s \in S_K$ let $s' \in K_v$ be some element which does no commute with s. Define a set $S_K' = \{s, s' : s \in S_K\}$ and note that it is finite.

Let $N = \mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)})$. For each irreducible factor Z_j for $j = 1, \ldots, \ell$ of Y_2 choose some element $g_j \in N < \Gamma$ which acts non-trivially on Z_j. Note the kernel N is non-empty since otherwise Γ would be a finitely generated linear group and hence residually finite,

contradicting [CM09b, Theorem 2(iv)]. Now, we can choose such an element so that it centralises a finite index subgroup of A. Indeed, we may choose $g_j \in N$. Since A is commensurated and N is a normal subgroup, g_j centralises a finite index subgroup of A. For each g_j pick another element g_j' which centralises a finite index subgroup of A and does not commute with g_j. Let $S_{Y_2} = \{g_j, g_j' : j = 1 \ldots, \ell\}$ and note that it is finite. Let $A' = \left(\bigcap_{g \in S_{Y_2}} A^g \right) \cap A$, since this is the intersection of finitely many commensurable subgroups A' is a finite index subgroup of A. By construction A' is the centraliser of the finite set $S_K' \cup S_{Y_2} \cup S_A$. Thus, by [GS91, Proposition 4.3], A' is $\mathcal{L}$-quasiconvex. ◆

Now, there exists an element in $\bar{t} \in \pi_{O(n)}(\Gamma)$ with infinite order, let t denote a preimage of $\bar{t}$ in Γ. By [LM21, Corollary 5.4], there is a finite index subgroup $\Gamma^0 \trianglelefteq \Gamma$ such that every finitely generated subgroup of Γ^0 centralises a finite index subgroup of A. After passing to a suitable power we may assume $t^k \in \Gamma^0$. But $\langle t^k \rangle$ does not centralise a finite index subgroup of A, a contradiction. Hence, there is no biautomatic structure on Γ. Since the hypotheses on Γ pass to finite index subgroups, it follows Γ is not virtually biautomatic. □

Example 4. The group Γ_n for each $n \geqslant 2$ constructed in Example 3 is an irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T_{10n})$-lattice that is not virtually biautomatic.

Remark 6.2. In light of M. Valiunas' result [Val21, Theorem 1.2] Theorem 6.1 can be strengthened to state that Γ does not embed into any biautomatic group. It may also be possible to simplify the proof using their result.

7. Products with Salvetti complexes

In this section we will adapt a construction of Horbez and Huang [HH20, Proposition 4.5] to extend actions from trees to Salvetti complexes. Horbez–Huang constructed an example of a non-uniform lattice acting on the universal cover of the Salvetti complex $\widetilde{S}_L$ provided L is not a complete graph. We generalise this to construct a tower of uniform lattices in $\mathrm{Aut}(\widetilde{S}_L)$ and with an additional hypothesis on L non-biautomatic lattices in $\mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Aut}(\widetilde{S}_L)$.

7.1. **Graph and polyhedral products.** Let K be a simplicial complex on the vertex set $[m] := \{1, \ldots, m\}$. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i) \mid i \in [m]\}$ be a collection of CW-pairs. The *polyhedral product* of $(\underline{X}, \underline{A})$ and K, is the space

$$(\underline{X}, \underline{A})^K := \bigcup_{\sigma \in K} \prod_{i=1}^m Y_i^\sigma \subseteq \prod_{i=1}^m X_i \quad \text{where} \quad Y_i^\sigma = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Let K be a simplicial complex on $[m]$ vertices. Let $\underline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_m\}$ be a set of discrete groups. The *graph product* of $\underline{\Gamma}$ and K, denoted $\underline{\Gamma}^K$ is quotient of the free product $*_{i \in [m]} \Gamma_i$ by the relations $[\gamma_i, \gamma_j] = 1$ for all $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$ if i and j are connected by an

edge in K. Let $B\underline{\Gamma} = \{B\Gamma_1, \ldots, B\Gamma_n\}$. The graph product $\underline{\Gamma}^K$ is the fundamental group of the polyhedral product $X = (B\underline{\Gamma}, *)^K$. Moreover, if K is a *flag complex*, i.e. every nonempty set of vertices which are pairwise connected by edges spans a simplex, then X is a $K(\underline{\Gamma}^K, 1)$ [Sta15, Theorem 1.1].

If every vertex group in a graph product $\underline{\Gamma}^L$ is $\mathbb{Z}$ then we call the group a *right-angled Artin group (RAAG)* and denote $\underline{\Gamma}^L$ by A_L. In this case we will identify the generating set of A_L with the vertex set VL of L. The polyhedral product $(S^1, *)^L$ is a classifying space for A_L, is referred to as the *Salvetti complex* for A_L and denoted by S_L. We denote the universal cover by $\widetilde{S}_L$.

7.2. Extending actions over the Salvetti complex.

We will now adapt the construction of Horbez and Huang [HH20, Proposition 4.5] to extend actions from trees to Salvetti complexes and present some applications.

Construction 7.1. *Let L be a finite simplicial graph on vertices $\{v_1, \ldots, v_m\}$ and suppose $\langle v_1, \ldots, v_k \rangle = F_k < A_L$ is a free subgroup. Let Γ be a group acting on $\mathcal{T}_{2k}$ by isometries such that the action is label-preserving, then the action of Γ on $\mathcal{T}$ extends to an action of $\widetilde{\Gamma}$ on $\widetilde{S}_L$ by isometries. Moreover, if Γ is a T_{2k}-lattice then $\widetilde{\Gamma}$ is an $\mathrm{Aut}(\widetilde{S}_L)$-lattice.*

Proof. Define $\phi : A_L \twoheadrightarrow F_k$ by $v \mapsto 1$ unless $v \in \mathcal{V}$ and let $\pi : \widetilde{S}_L \to X$ be the covering space corresponding to $\mathrm{Ker}(\phi)$. Let Γ be a group acting on $\mathcal{T}_{2k}$ preserving the labelling, we want to extend the action of Γ on $\mathcal{T}_{2k}$ to an action on X.

We may identify the vertex set of $\mathcal{T}_{2k}$ with the vertex set of X via the embedding of $\mathcal{T}_{2k} \rightarrowtail X$. We orient each edge of $\widetilde{S}_L$ and endow X with the induced labelling and orientation. The 1-skeleton $X^{(1)}$ of X is obtained from $\mathcal{T}_{2k}$ by attaching to each vertex of $\mathcal{T}_{2k}$ a circle for each $v \in VL \backslash \mathcal{V}$.

Since Γ acts by isometries on $\mathcal{T}_{2k}$ label preservingly, it follows Γ acts by isometries on $X^{(1)}$ label preservingly and preserves the orientation of edges in $VL \backslash \mathcal{V}$. It follows the action extends to X. Let $\widetilde{\Gamma}$ be the group of lifts of all automorphisms in Γ, we have a short exact sequence

$$1 \longrightarrow \mathrm{Aut}(\pi) \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1.$$

We have $\widetilde{S}_L / \widetilde{\Gamma} = X/\Gamma$ so there is a bijection between the $\widetilde{\Gamma}$-orbits of $\widetilde{S}_L^{(0)}$ and the Γ-orbits of $\mathcal{T}_{2k}^{(0)}$. For a vertex $v \in X$, each lift of $g \in \mathrm{Stab}_\Gamma(v)$ fixes a unique vertex $\tilde{v} \in \widetilde{S}_L$. In particular, the cardinality of the vertex stabilisers is preserved. It follows from Serre's covolume formula that if Γ was a T_{2k}-lattice, then $\widetilde{\Gamma}$ is an $\mathrm{Aut}(\widetilde{S}_L)$-lattice. $\square$

Proposition 7.2. *There is an ascending tower of lattices in $T_4 = \mathrm{Aut}(\mathcal{T}_4)$ with label preserving action.*

Proof. The groups will be index two subgroups of the HNN extensions constructed in [BK90, Example 7.4]. We describe them here for the convenience of the reader. Let

$V_r = \{f : \mathbb{Z}_r \to \mathbb{Z}_2 \ : \ f \text{ a function}\} \cong \mathbb{Z}_2^r$ and $\alpha_r \in \mathrm{Aut}(V_r)$ by $\alpha_r(f)(i) = f(i+1)$. Let $W_r = \{f \in V_r \ : \ f(0) = 1\} \cong \mathbb{Z}_2^{r-1}$ and define Γ_r to be the HNN extension

$$\langle V_r, t \mid f^t = \alpha_r(f) \ \forall f \in W_r \rangle.$$

By [BK90, Proposition 7.6] the group Γ_r acts faithfully on $\mathcal{T}_4$ with quotient a loop (one vertex and one edge) and covolume $1/m^r$. Moreover, if $r|r'$ then $\Gamma_r \leqslant \Gamma_{r'}$ with index $m^{r'-r}$ and so for $r \geqslant 2$, the sequence $(\Gamma_{r^s})_{s \geqslant 1}$ is an infinite ascending chain in $\mathrm{Lat}_u(\mathcal{T}_4)$.

Now, define $\phi : \Gamma_r \to \mathbb{Z}_2$ by $\phi(V_r) = 0$ and $\phi(t) = 1$. The kernel Λ_r is an index two subgroup which satisfies the same properties as Γ_r except now the quotient has fundamental domain the first barycentric subdivision of a loop (two vertices and two edges) and covolume $2/m^r$. $\qquad\square$

Corollary 7.3. *Let L be a finite flag complex which is not a full simplex, then the automorphism group $\mathrm{Aut}(\widetilde{S}_L)$ of the universal cover of the Salvetti complex contains a tower of uniform lattices.*

Proof. Fix $r \geqslant 2$. We apply Construction 7.1 to the lattices Λ_{r^s} for $s \geqslant 1$ in the preceding proposition and obtain a sequence of lattices $\widetilde{\Lambda}_{r^s}$ in $\mathrm{Aut}(\widetilde{S}_L)$. The group $\widetilde{\Lambda}_{r^s}$ has two orbits of vertices, each stabilised by a group of order m^{r^s}, it follows from Serre's Covolume Formula that $\widetilde{\Lambda}_{r^s}$ has covolume equal to $2/m^{r^s}$. It remains to show that the inclusions $\Lambda_{r^s} \rightarrowtail \Lambda_{r^{s'}}$ induce inclusions $\widetilde{\Lambda}_{r^s} \rightarrowtail \widetilde{\Lambda}_{r^{s'}}$ for $s' < s$. Consider the covering space $\pi : \widetilde{S}_L \to X$ where X is as in Construction 7.1. Note that X and hence $\mathrm{Aut}(\pi)$ does not depend on r or s since each group acts with the same fundamental domain. In particular, as $\Lambda_{r^s} < \Lambda_{r^{s'}}$ we have $\widetilde{\Lambda}_{r^s} < \widetilde{\Lambda}_{r^{s'}}$ for $s < s'$. $\qquad\square$

Let J be a simplicial complex on $[m]$ and let $V \subseteq [m]$. The *double of J over V*, denoted $\mathcal{D}(J; V)$ is the simplicial complex with vertices $[m]\backslash V \cup \{v^+, v^- : v \in V\}$ and simplices described as follows: $[w_1^{\epsilon_1}, \ldots, w_n^{\epsilon_n}]$, where $\epsilon_i \in \{+, -, \ \}$, spans an n-simplex in $\mathcal{D}(J, V)$ if and only if $[w_1, \ldots, w_n]$ spans an n-simplex in J.

Theorem 7.4. *Let $m \geqslant 3$ and $k \geqslant 2$. Let K be a pointed flag complex on $[m]$, and let $L = \bigvee_{i=1}^{k} K$. Let X be a proper $\mathrm{CAT}(0)$ space and assume $H < \mathrm{Isom}(X)$ acts cocompactly and minimally. The following conclusions hold:*

(1) *Let Γ be a group acting on $\mathcal{T}_{2k}$ by isometries. Then the action of Γ on $\mathcal{T}$ extends to an action of $\widetilde{\Gamma}$ on $\widetilde{S}_L$ by isometries.*

(2) *If Γ is a uniform lattice in $H \times T_{2k}$, then $\widetilde{\Gamma}$ is a uniform lattice in $H \times \mathrm{Aut}(\widetilde{S}_L)$.*

(3) *If in addition X is a $\mathrm{CAT}(0)$ polyhedral complex and Γ is an $(H \times T_{2k})$-lattice, then $\widetilde{\Gamma}$ is an $(H \times \mathrm{Aut}(\widetilde{S}_L))$-lattice.*

(4) *If the projection of Γ to H (resp. T_{2k}) is non-discrete, then so is the projection of $\widetilde{\Gamma}$ to H (resp. $\mathrm{Aut}(\widetilde{S}_L)$).*

Proof. The proof of (1) is identical to Construction 7.1 except now we do not require the action to be label preserving on $\mathcal{T}_{2k}$. Indeed, the assumption that $L = \bigvee_{i=1}^{k} W$ implies that $T_{2k} \hookrightarrow \operatorname{Aut}(\widetilde{S}_L)$. Indeed, marking a non-basepoint vertex in K we obtain a set $\mathcal{V}$ of k vertices in L which generate a free subgroup $A_{\mathcal{V}}$ of A_L. The surjection $\phi \colon A_L \twoheadrightarrow A_{\mathcal{V}}$ by $v \mapsto 1$ unless $v \in \mathcal{V}$ induces a covering map $\pi \colon \widetilde{S}_L \to X$ with deck transformation group $\operatorname{Ker}(\phi)$. Now, we may identify the vertices of $\mathcal{T}_{2k}$ with X and build $X^{(1)}$ from $\mathcal{T}_{2k}$ by attaching to each vertex a circle for each $V \in VL \backslash \mathcal{V}$. The remainder of X can then be built inductively by noting that the link of a vertex in X is $\mathcal{D}(L; \mathcal{V})$. Now, any isometry of $\mathcal{T}_{2k}$ clearly extends over X by permuting sheets of lifts of S_K in X. In particular, we can extend any action on $\mathcal{T}_{2k}$ to $\widetilde{S}_L$.

Now, let Γ be the group as in the theorem statement. The desired group $\widetilde{\Gamma}$ can now be described as the group of lifts of all automorphisms in Γ. In particular we obtain an extension $\operatorname{Aut}(\pi) \to \widetilde{\Gamma} \to \Gamma$. ◆

The proof of (2) follows from taking the diagonal embedding $\widetilde{\Gamma} \rightarrowtail H \times \operatorname{Aut}(\widetilde{S}_L)$ and then noting that the quotient $(\widetilde{S}_L \times X)/\widetilde{\Gamma}$ is compact and that cardinality of each of the vertex stabilisers is finite. ◆

We prove (3) in the same manner, noting the covolume on the product space is finite by Serre's Covolume Formula. ◆

The images of the projections of Γ and $\widetilde{\Gamma}$ to H coincide. Since any element of Γ which acts non-trivially on $\mathcal{T}_{2k}$ lifts to an element acting non-trivially on $\widetilde{S}_L$, the non-discreteness of $\pi_{T_{2k}}(\Gamma)$ implies the non-discreteness of $\pi_{\operatorname{Aut}(\widetilde{S}_L)}(\widetilde{\Gamma})$. This proves (4). ◆ $\qquad\square$

Example 5. Applying the previous theorem to the Leary-Minasyan group $\operatorname{LM}(A)$ which acts irreducibly on the product of a 10-regular tree and $\mathbb{E}^2$ we obtain a lattice Γ in $\operatorname{Isom}(\mathbb{E}^2) \times \operatorname{Aut}(\widetilde{S}_L)$. Moreover, the projection to either factor is non-discrete. Thus, if $\widetilde{S}_L$ is irreducible, then Γ is algebraically irreducible by Theorem 2.5 and non-residually finite by Theorem 2.6. By Theorem 6.1 the group Γ is not virtually biautomatic.

Recall that a group Γ is *quasi-isometrically rigid* if every group quasi-isometric to Γ is virtually isomorphic to Γ. The quasi-isometric rigidity of right angled Artin groups has received a lot of attention recently (see for instance [Hua18] and the references therein). The following corollary is immediate and appears to be new if L has no induced 4-cycle [Hua18] and A_L is not a free group [LM21].

Corollary 7.5 (Corollary D). *Let $m \geqslant 3$. Let K be a pointed flag complex on $[m]$, and let $L = \bigvee_{i=1}^{5} K$. If A_L is irreducible, then there exists a weakly irreducible uniform lattice in $\operatorname{Aut}(\widetilde{S}_L) \times \operatorname{Isom}(\mathbb{E}^n)$ which is not virtually biautomatic nor residually finite. In particular, $A_L \times \mathbb{Z}^2$ is not quasi-isometrically rigid even amongst* CAT(0) *groups.*

Proof. The group Γ constructed in Example 5 is algebraically irreducible, non-residually finite, and quasi-isometric to $A_L \times \mathbb{Z}^2$. Algebraic irreducibility is a virtual isomorphism

invariant but $A_L \times \mathbb{Z}^2$ is algebraically reducible. In particular, $A_L \times \mathbb{Z}^2$ is quasi-isometric to Γ but not virtually isomorphic to Γ and so cannot be quasi-isometrically rigid. $\qquad\square$

Remark 7.6. It seems likely that one could take a polyhedral product of locally CAT(0) cube complexes over a flag complex and then repeat the above constructions to obtain towers of lattices in the automorphism group of the universal cover and more weakly irreducible lattices in mixed products.

8. A functor theorem

In this section will show that the functors introduced by A. Thomas in [Tho06] take graphs of H-lattices with a fixed Bass-Serre tree to complexes of H-lattices whose development is a "sufficiently symmetric" right-angled building (we will make this precise later). Finally, we will combine these tools to construct a number of examples. In particular, non-residually finite (Isom($\mathbb{E}^n$) $\times A$)-lattices where A is the automorphism group of a sufficiently symmetric right-angled building, and non-residually finite algebraically irreducible lattices in products of arbitrarily many isometric and non-isometric sufficiently symmetric right-angled buildings.

8.1. **Right angled buildings.** Let (W, I) be a right-angled Coxeter system. Let N be the finite nerve of (W, I) and P' be the simplicial cone on N' with vertex x_0. A *right-angled building* of type (W, I) is a polyhedral complex X equipped with a maximal family of subcomplexes called *apartments*. Such an apartment is isometric to the Davis complex for (W, I) and the copies of P' in X are called *chambers*. Moreover, the apartments and chambers satisfy the axioms for a Bruhat–Tits building.

Let $\mathcal{S}$ denote the set of $J \subseteq I$ such that $W_J \leqslant W$ is finite. Note that $W_\varnothing = \{1\}$ so $\varnothing \in \mathcal{S}$. For each $i \in I$, the vertex P' of *type* $\{i\}$ will be called an *i-vertex*, and the union of the simplices of P' which contains the i-vertex but not x_0 will be called the *i-face* There is a one-to-one correspondence between the vertices of P' and the types $J \in \mathcal{S}$.

Let X be a right-angled building. A vertex of X has a type $J \in \mathcal{S}$ induced by the types of P'. For $i \in I$ an *$\{i\}$-residue* of X is the connected subcomplex consisting of all chambers which meet in a given i-face. The cardinality of the $\{i\}$-residue is the number of copies of P' in it.

Theorem 8.1 ([HP03])**.** *Let (W, I) be a right-angled Coxeter system and $\{q_i : i \in I\}$ a set of integers such that $q_i \geqslant 2$. Then up to isometry there exists a unique building X of type (W, I) such that for each $i \in I$ the $\{i\}$-residue of X has cardinality q_i.*

If (W, I) is generated by reflections in an n-dimensional right-angled hyperbolic polygon P, then P' is the barycentric subdivision of P. Moreover, the apartments of X are

isometric to $\mathbb{R}\mathbf{H}^n$. In this case we call X a *hyperbolic building*. We remark that a right-angled building can be expressed as the universal cover of a polyhedral product, however, we will not use this observation elsewhere.

Remark 8.2. Let (W, I) be a right-angled Coxeter system with parameters $\{q_i\}$ and nerve N. Let E_i be a set of size q_i and let CE_i denote the simplicial cone on E_i, denote the collections of these by $\underline{E}$ and $C\underline{E}$ respectively. The right-angled building of type (W, I) with parameters $\{q_i\}$ is the universal cover of the polyhedral product $(C\underline{E}, \underline{E})^N$.

8.2. **A functor theorem.** In this section we will recap a functorial construction of A. Thomas which takes graphs of groups with a given universal covering tree to complexes of groups with development a right-angled building. We will then show that this functor takes graphs of lattices to complexes of lattices and deduce some consequences.

Let X be a right-angled building of type (W, I) and parameters $\{q_i\}$ with chamber P'. Suppose $m_{i_1, i_2} = \infty$ and define the following two symmetry conditions due to Thomas [Tho06]:

(T1) There exists a bijection g on I such that $m_{i,j} = m_{g(i),g(j)}$ for all $i, j \in I$, and $g(i_1) = i_2$.

(T2) There exists a bijection $h : \{i \in I : m_{i_1,i} < \infty\} \to \{i \in I : m_{i_2,i} < \infty\}$ such that $m_{i,j} = m_{h(i),h(j)}$ for all i, j in the domain, $h(i_1) = i_2$, and for all i in the domain $q_i = q_{h(i)}$.

We include the construction adapted from [Tho06] for completeness and for utility in the proofs of the new results which will follow. An example of the construction for a graph of groups consisting of a single edge is given in Figure 1
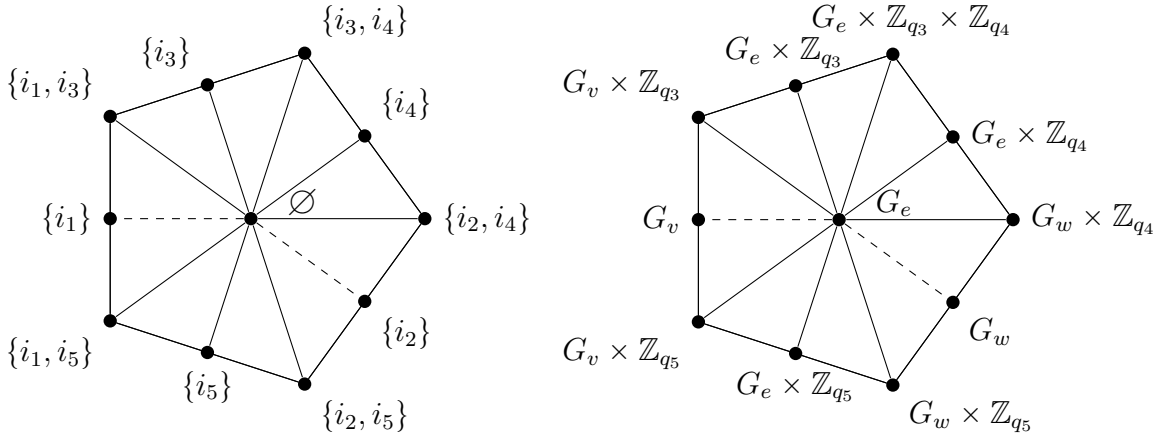

FIGURE 1. The left pentagon shows a labelling of the types $J \in \mathcal{S}$. The right pentagon shows the local groups after applying Thomas' functor to a graph of groups with a single edge. In both pentagons the dashed line shows the embedding of the graph. If the graph of groups has a single vertex, then $G_v = G_w$, $q_1 = q_2$, $q_3 = q_4$, the edge $(\{i_1, i_5\}, \{i_1\})$ is glued to $(\{i_2, i_5\}, \{i_2\})$, and the edge $(\{i_1, i_3\}, \{i_1\})$ is glued to $(\{i_2, i_4\}, \{i_2\})$.

Construction 8.3 (Thomas' Functor [Tho06])**.** *Let X be a right-angled building of type (W, I) and parameters $\{q_i\}$. For each $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$ let $\mathcal{T}$ be the (q_{i_1}, q_{i_2})-biregular tree. Suppose (T1) holds and if $q_{i_1} = q_{i_2}$ then (T2) holds with g an extension of h. Then there is functor $F : \mathcal{G}(\mathcal{T}) \to \mathcal{C}(X)$ preserving faithfulness and coverings.*

We will construct F as a composite $F_2 \circ F_1$. We first define $F_1 : \mathcal{G} \to \mathcal{C}_1$. Let $(A, \mathcal{A})$ be a graph of groups and $|A|$ the geometric realisation of A. We will construct a complex of groups $F_1(A)$ over $|A|$. For the objects we have:

- The local groups at the vertices of $|A|$ are the vertex groups of $\mathcal{A}$.
- For all $e \in EA$ let $\sigma_e = \sigma_{\bar{e}}$ be the vertex of the barycentric subdivision $|A|'$ at the midpoint of e.
- The local group at σ_e in $F_1(A)$ is $A_e = A_{\bar{e}}$.
- A monomorphism $\alpha_e : A_e \to A_{i(e)}$ in A induces the same monomorphism in $F_1(A)$.

Let $\phi : A \to B$ be a morphism of graphs of groups over a map of graphs f, note that by [Tho06, Proposition 2.1] F_1 is not injective on morphisms. We define $F_1(\phi)$ as follows:

- The map f induces a polyhedral map $f' : |A|' \to |B|'$ so we will define $F_1(\phi) : F_1(A) \to F_1(B)$ over f.
- Now take the morphisms on the local groups to be the same as for ϕ.

Let $\mathcal{C}(\mathcal{T}) = \mathrm{Im}(F_1(\mathcal{G}(\mathcal{T})))$ and $G(Y) \in \mathcal{C}(\mathcal{T})$. Now, we will define $F_2 : \mathcal{C}(\mathcal{T}) \to \mathcal{C}(X)$ as follows:

- We first embed Y' into a canonically constructed polyhedral complex $F_2(Y)$. For each $e \in EY$ let P'_e be a copy of P' and identify the midpoint of e with the cone vertex x_0 of P'_e.
- If Y is 2-colourable with colours i_1 and i_2 (from the valences of the Bass-Serre tree if $q_{i_1} \neq q_{i_2}$), then we identify the vertex of e of type i_j with the i_j-vertex of P'_e.
- Suppose Y is not 2-colourable. If $e \in EY$ is not a loop in Y then identify one vertex of e with the i_1-vertex of P'_e and the other with the i_2-vertex. If e forms a loop then we attach P'_e/h (where h is the isometry from the assumption) and identify the vertex of e to the image of the i_1- and i_2-vertices of in P'_e/h.
- Glue together, either by preserving type on the i_1- and i_2-faces or by the isometry h, the faces of the the P'_e and P'_e/h whose centres correspond to the same vertex of Y. Let $F_2(Y)$ denote the resulting polyhedral complex.
- Note that $Y' \rightarrowtail F_2(Y)$ and that each vertex of $F_2(Y)$ has a unique type $J \in \mathcal{S}$ or two types J and $h(J)$ where $i_1 \in J \in \mathcal{S}$ and h is the isometry from the assumption.
- Fix the local groups and structure maps induced by the embedding of Y' in $F(Y)$. For each $i \in I$ let $G_i = \mathbb{Z}_{q_i}$ and for $J \subseteq I$ let $G_J = \prod_{j \in J} G_j$. For each $e \in EY$ let G_e be the local group at the midpoint of e.

- Let $J \in \mathcal{S}$ such that neither i_1 or i_2 are in J. The local group at a vertex of type J is $G_e \times G_J$. The structure maps between such local groups are the natural inclusions.
- Let $J \in \mathcal{S}$ and suppose $i_k \in J$ for one of $k = 1$ or $k = 2$. Since $m_{i_1, i_2} = \infty$ both i_1 and i_2 cannot be in J. Let F_e be the i_k-face of P'_e or the glued face of P'_e/h. The vertex of type J in P'_e or P'_e/h is contained in F_e. Let v be the vertex of Y identified with the centre of F_e and let G_v be the local group at v in $G(Y)$
- The local group at the vertex of type J is $G_v \times G_{J \setminus \{i_k\}}$. For each $J' \subset J$ with $i_k \in J'$ the structure map $G_v \times G_{J' \setminus \{i_k\}} \rightarrowtail G_v \times G_{J \setminus \{i_k\}}$ is the natural inclusion. For each $J' \subset J$ with $i_k \notin J'$ the structure map $G_e \times G_{J'} \rightarrowtail G_v \times G_{J \setminus \{i_k\}}$ is the product of the structure map $G_e \rightarrowtail G_v$ in $G(Y)$ and the natural inclusion.

Now, let $\phi : G(Y) \to H(Z)$ be a morphism in $\mathcal{C}(\mathcal{T})$ over a non-degenerate polyhedral map $f : Y \to Z$. We will define $F_2(\phi)$ as follows:

- If Y and Z are two colourable f extends to a polyhedral map $F_2(f) : F_2(Y) \to F_2(Z)$. Otherwise we use (T1) to construct $F_2(f)$.
- If $\tau \in VF(Y)$ then $G_\tau = G_\sigma \times G_J$ where σ is a vertex of Y'. The homomorphism of local groups $G_\sigma \times G_J \to H_{f(\sigma)} \times G_J$ is ϕ_σ on the first factor and the identity on the other factors.
- Let $a \in EF(Y)$. If ψ_a, the structure map along $a \in F_2(G(Y))$, has a structure map ψ_b from $G(Y)$ as its first factor, put $F_2(\phi)(b) = \phi(a)$. Otherwise set $F_2(\phi)(b) = 1$.

We will now show the functor takes graphs of lattices to complexes of lattices and deduce a number of consequences. For locally compact group H let $\mathrm{Lat}(H)$ denote the set of H-lattices and let $\mathrm{Lat_u}(H)$ denote the set of uniform H-lattices.

Theorem 8.4. *Let Y be a right-angled building of type (W, I) and parameters $\{q_i\}$ and let $A = \mathrm{Aut}(Y)$. For each $i_1, i_2 \in I$ such that $m_{i_1, i_2} = \infty$ let $\mathcal{T}$ be the (q_{i_1}, q_{i_2})-biregular tree and let $T = \mathrm{Aut}(\mathcal{T})$. Suppose (T1) holds and if $q_{i_1} = q_{i_2}$ then (T2) holds with g an extension of h, and let $F : \mathcal{G}(\mathcal{T}) \to \mathcal{C}(Y)$ be Thomas' functor. Let X be a finite dimensional proper $\mathrm{CAT}(0)$ space and assume $H = \mathrm{Isom}(X)$ contains a cocompact lattice. The following conclusions hold:*

(1) If $G(\mathcal{T})$ is a graph of H-lattices, then $F(G(\mathcal{T}))$ is a complex of H-lattices.

(2) F induces an inclusion of sets $\mathrm{Lat_u}(H \times T) \rightarrowtail \mathrm{Lat_u}(H \times A)$.

(3) If Y is a $\mathrm{CAT}(0)$ polyhedral complex then F induces an inclusion of sets $\mathrm{Lat}(H \times T) \rightarrowtail \mathrm{Lat}(H \times A)$.

Let Γ be a uniform $(H \times T)$-lattice and let $F\Gamma$ be the corresponding $(H \times A)$-lattice.

(4) $\pi_T(\Gamma)$ is discrete if and only if $\pi_A(F\Gamma)$ is discrete. Moreover, $\pi_H(\Gamma) = \pi_H(F\Gamma)$.

(5) If Γ satisfies any of $\{$algebraically irreducible, non-residually finite, not virtually torsion free$\}$, then so does $F\Gamma$.

Proof. We first prove (1). We will first verify the conditions on the local groups and then construct a morphism to H. Let $(B, \mathcal{B}, \psi)$ be a graph of H-lattices and consider the image $L(Z)$ of $\mathcal{B}$ under F. Here $Z = F(B)$. Each local group in $L(Z)$ is of the form $G_\sigma \times G_J$ where G_σ is a local group in $\mathcal{B}$ and G_J is a finite product of finite cyclic groups. We have a morphism $\psi : \mathcal{B} \to H$ such that the image of each local group G_σ is an H-lattice and the restriction to G_σ has finite kernel. Thus, by construction the local groups in $L(Z)$ are commensurable in $\pi_1(L(Z))$. We define $F(\psi_\sigma)$ to be the composite $\psi|_{G_\sigma} \circ \pi_\sigma : G_\sigma \times G_J \twoheadrightarrow G_\sigma \to \psi(G_\sigma)$, thus commensurability of the images in H is immediate.

We will now deal with the edges. Note the twisting elements in $L(Z)$ are all trivial and the complex of groups H has all structure maps the identity. Let the structure maps in $L(Z)$ be denoted by λ_a for $a \in EZ'$ and the structure maps in $\mathcal{B}$ by α_e for $e \in EB$. The family of elements $(t_e)_{e \in EB}$ in the path group $\pi(\mathcal{B})$ are mapped under ψ to elements of $\text{Comm}_H(\psi(G_\sigma))$ where G_σ is some local group. Now, let $a \in EZ'$, then by construction a either corresponds to a subdivision of an edge a in EB in which case we define $(F\psi)(a) = \psi(a)$. Or, a corresponds to a inclusion of local groups $G_\sigma \times G_{J'} \to G_\sigma \times G_J$, in which case we define $(F\psi)(a) = 1_H$.

It remains to verify the two edge axioms for a morphism. For each $a \in EZ'$ corresponding to the subdivision of an edge a in EB we have

$$\text{Ad}((F\psi)(a)) \circ F(\psi_{i(a)}) = \text{Ad}(\psi(a)) \circ \psi_{i(a)} \circ \pi_a = \psi_{t(a)} \circ \alpha_a \circ \pi_a = F(\psi_{t(a)}) \circ F(\alpha_a),$$

where π_a is the surjection $G_a \times G_J \twoheadrightarrow G_a$. For any other edge $a \in EZ'$ we have

$$\text{Ad}((F\psi)(a)) \circ F(\psi_{i(a)}) = F(\psi_{i(a)}) \text{ and } F(\psi_{t(a)}) \circ \lambda_a = F(\psi_{i(a)}).$$

Finally, the other condition that $(F\psi)(ab) = (F\psi)(a)(F\psi)(b)$ for $(a, b) \in E^2Z'$ is verified trivially. Thus, $F(\mathcal{B}) = L(Z)$ is a complex of H-lattices. ◆

We will next prove (2). Let Γ be an $(H \times T)$-lattice. By Theorem 3.3, Γ splits as graph of H-lattices $\mathcal{B}$. Thus, by (1) we obtain a complex of H-lattices $F(\mathcal{B})$ with fundamental group Λ. By Theorem 5.2(1) it suffices to show that for each local group G_σ in $F(\mathcal{B})$ the kernel $K_\sigma = \text{Ker}(\pi_H|_{FG_\sigma})$ acts faithfully on X. Now, K_σ is a direct product of $L_\sigma = \text{Ker}(\pi_H|_{G_\sigma})$ with a direct product of cyclic groups G_J, where G_σ is a local group in $\mathcal{B}$. By construction G_J acts faithfully on X and by Theorem 3.3, K_σ acts faithfully on $\mathcal{T}$ which embeds into Y. In particular, K_σ acts faithfully on X. ◆

We will next prove (3). We construct a complex of lattices as in the previous case. The proof for (3) is now identical once we have verified that covolume condition in Theorem 5.2(3). Let c denote the covolume of an $(H \times T)$-lattice Γ with associated graph of lattices $(B, \mathcal{B})$, this is given by the formula $c = \sum_{\sigma \in VA} \mu(\Gamma_\sigma) < \infty$. Now, every vertex of the complex $Z = F(B)$ has local group isomorphic to a finite extension of some Γ_σ. In

particular we may bound $\sum_{\sigma \in Z} \mu(\Gamma_\sigma)$ by $\ell \times c$ where ℓ is the number of vertices in the finite Coxeter nerve of X. ◆

The proof of (4) follows from the proof of (1). ◆

The proof of (5) follows from either applying Theorem 2.5 to (4) (algebraically irreducible) or the fact $\Gamma \rightarrowtail F\Gamma$ and the properties of residual finiteness and virtual torsion-freeness are subgroup closed. ◆   □

8.3. **Examples and applications.** In this section we will detail some sample examples and applications of the functor theorem.

We can obtain a number of examples by applying Thomas' functor to any irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. This will give a non-biautomatic group acting properly discontinuously cocompactly on $\mathbb{E}^n \times X$ where X is a sufficiently symmetric right-angled building. More precisely, we have the following corollary:

Corollary 8.5 (General version of Corollary E)**.** *Let Y be a right-angled building of type (W, I) and parameters $\{q_i\}$ and let $A = \mathrm{Aut}(Y)$. For each $i_1, i_2 \in I$ such that $m_{i_1, i_2} = \infty$ let $\mathcal{T}$ be the (q_{i_1}, q_{i_2})-biregular tree and let $T = \mathrm{Aut}(\mathcal{T})$. Suppose (T1) holds and if $q_{i_1} = q_{i_2}$ then (T2) holds with g an extension of h and let $F : \mathcal{G}(\mathcal{T}) \to \mathcal{C}(Y)$ be Thomas' functor. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice and suppose $\pi_{\mathrm{O}(n)}(\Gamma)$ is infinite. Then, $F\Gamma$ is a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice which is not virtually biautomatic nor residually finite. In particular, if Y is irreducible, then the direct product of a uniform A-lattice with $\mathbb{Z}^2$ is not quasi-isometrically rigid.*

Proof. By Theorem 8.4 $F\Gamma$ is a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice with a non-discrete projection to $\mathrm{O}(n)$. That $F\Gamma$ is not virtually biautomatic then follows from Theorem 6.1. The failure of quasi-isometric rigidity follows from the fact that the direct product of a uniform A lattice with $\mathbb{Z}^2$ is reducible, whereas, the weakly irreducible lattice is algebraically irreducible by Theorem 2.5 and so does not virtually split as a direct product of two infinite groups. In particular, the groups cannot by virtually isomorphic.   □

Example 6. Let $\Gamma = \mathrm{LM}(A)$ where A is the matrix corresponding to the Pythagorean triple $(3, 4, 5)$. Recall the group acts on $\mathbb{E}^2 \times \mathcal{T}_{10}$. Let X be the right angled building whose Coxeter nerve is the regular pentagon and whose parameters are given by $q_1 = q_2 = 10$, $q_3 = q_4 = k$, and $q_5 = \ell$. Let A be the automorphism group of X and consider $F\Gamma$ the image of Γ under Thomas' functor F as in Figure 1. By Theorem 8.4, the group $F\Gamma$ is a non-residually finite $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice with non-discrete projections to both factors and is irreducible as an abstract group. Moreover, by the previous corollary, $F\Gamma$ is not virtually biautomatic.

We will now construct a presentation for $\Lambda_{k,\ell} := F\Gamma$. The group has generators a, b, x_3, x_4, x_5, t and relations

$$x_3^k = x_4^k = x_5^\ell = 1,\ [a, b],\ [a, x_3],\ [a, x_4],\ [a, x_5],\ [b, x_3],\ [b, x_4],\ [b, x_5],\ [x_3, x_4],$$

$$ta^2b^{-1}t^{-1} = a^2b, \ tab^2t^{-1} = a^{-1}b^2, \ tx_3t^{-1} = x_4, [t, x_5].$$

Proposition 8.6. *The group $\Lambda_{2,2}$ in Example 6 is virtually torsionfree. This is witnessed by the index 16 subgroup*

$$\Delta := \langle a, \ b, \ x_3tx_4t^{-1}, \ x_3x_4t^{-2}, \ (x_5x_3)^2, \ (x_5x_4)^2, \ t^{-1}x_3x_4t^{-1}, \ (tx_5x_4t^{-1})^2 \rangle.$$

Proof. The quotient $\Lambda_{2,2}/\Delta$ is isomorphic to $D_4 \times \mathbb{Z}_2$ which has order 16. By construction every torsion element of $\Lambda_{2,2}$ is conjugate to some power of x_3, x_4, x_5 or x_3x_4. Indeed, every torsion element is contained in a vertex or edge stabiliser of the action on the pentagonal building and acts trivially on $\mathbb{E}^2$. Each of these elements is mapped to a non-trivial element of $D_4 \times \mathbb{Z}_2$. In particular, the kernel Δ is torsion-free. $\qquad\square$

Example 7. Let $n \geqslant 2$ and let Γ_n be the irreducible lattice constructed in Example 3 acting on $\mathbb{E}^n \times \mathcal{T}_{10n}$. Let X be a right angled building satisfying (T1) and (T2) with automorphism group A and parameters $\{q_i\}$ all equal to $10n$. Applying Thomas' functor and Theorem 8.4 to Γ_n we obtain a non-residually finite $(\text{Isom}(\mathbb{E}^n) \times A)$-lattice with non-discrete projections to both factors. Moreover by Corollary 8.5, Γ_n is not virtually biautomatic.

We will now show the existence of non-residually finite lattices in arbitrary products of sufficiently symmetric isometric and non-isometric right-angled buildings. We note that Bourdon's "hyperbolization of Euclidean buildings" [Bou00, Section 1.5.2] can be used to construct weakly irreducible uniform lattices in products of hyperbolic buildings. We will provide a number of examples to show that the groups we construct here are distinct.

Corollary 8.7. *Let Γ be a weakly irreducible lattice in product of trees $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n$ such that $\mathcal{T}_k$ is (t_{k_1}, t_{k_2})-biregular. Let $X_1 \times \cdots \times X_n$ be a product of irreducible right angled buildings satisfying (T1) and (T2). Suppose X_k is of type (W_k, I_k), has parameters $\{t_{k_1}, t_{k_2}, q_{k_3}, \ldots, q_{k_{n_k}}\}$ where $m_{k_{i_1}, k_{i_2}} = \infty$ and $A_k = \text{Aut}(X_k)$. Then, the lattice $\Lambda = F^n\Gamma$ obtained by applying Thomas' functor n times (once for each tree $\mathcal{T}_k$ corresponding to the building X_k) is a lattice in $A_1 \times \cdots \times A_n$, is weakly and algebraically irreducible, and is non-residually finite.*

Proof. Let $T_k = \text{Aut}(\mathcal{T}_k)$. The result follows from applying Theorem 8.4 n times as follows. Consider Γ as a graph of $(T_2 \times \cdots \times T_n)$-lattices and apply F to obtain a $(A_1 \times T_2 \times \cdots \times T_n)$-lattice with the desired properties (non-residual finiteness follows from the fact that the projection to $T_2 \times \cdots \times T_n$ has a non-trivial kernel). Now, consider $F\Gamma$ as a graph of $(A_1 \times T_3 \times \ldots T_n)$-lattices and proceed by induction on the index k. $\qquad\square$

Example 8. In [BM00b; BM97] Burger and Mozes construct for each pair of sufficiently large even integers (m, n) a finitely presented simple group as a uniform lattices in a product of trees $\mathcal{T}_m \times \mathcal{T}_n$ (for more examples see [Rat07b; Rat07a; Rad20]). Applying

Theorem 8.7, we obtain uniform non-residually finite algebraically and weakly irreducible lattices acting on a product of buildings $X_1 \times X_2$ each satisfying (T1) and (T2) with X_1 having some parameters equal to m and X_2 having some parameters equal to n.

Example 9. In [Hug21c] the author constructed a non-virtually torsion-free irreducible uniform lattice in a product of two locally finite trees. Applying Theorem 8.7, we obtain uniform non-residually finite non-virtually torsion-free algebraically and weakly irreducible lattices acting on a product of buildings.

9. SOME QUESTIONS

In every example of an $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice known to the author, the lattice is virtually torsion-free.

Question 9.1. *Are there non-virtually torsion-free $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices?*

Since it is possible to characterise $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice in terms of C^*-simplicity, it would be interesting to recover the characterisation for complexes of $\mathrm{Isom}(\mathbb{E}^n)$-lattices.

Question 9.2. *Are the algebraically irreducible non-biautomatic groups constructed in Section 7 and Section 8 C^*-simple?*

More generally we ask:

Question 9.3. *When is a $\mathrm{CAT}(0)$ lattice C^*-simple?*

The characterisation of algebraically irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices in terms of C^*-simplicity suggests the following question:

Question 9.4. *Can C^*-simplicity of a Leary-Minasyan group $\mathrm{LM}(A)$ be determined by properties of the matrix A?*

REFERENCES

[BFS19]   Uri Bader, Alex Furman, and Roman Sauer. "An adelic arithmeticity theorem for lattices in products". In: *Math. Z.* 293.3-4 (2019), pp. 1181–1199. DOI: 10.1007/s00209-019-02241-9.

[BS06]   Uri Bader and Yehuda Shalom. "Factor and normal subgroup theorems for lattices in products of groups". In: *Invent. Math.* 163.2 (2006), pp. 415–454. DOI: 10.1007/s00222-005-0469-5.

[Bas93]   Hyman Bass. "Covering theory for graphs of groups". In: *J. Pure Appl. Algebra* 89.1-2 (1993), pp. 3–47. DOI: 10.1016/0022-4049(93)90085-8.

[BK90]   Hyman Bass and Ravi Kulkarni. "Uniform tree lattices". In: *J. Amer. Math. Soc.* 3.4 (1990), pp. 843–902. DOI: 10.2307/1990905.

[BG02]   Nadia Benakli and Yair Glasner. "Automorphism groups of trees acting locally with affine permutations". In: *Geom. Dedicata* 89 (2002), pp. 1–24. DOI: 10.1023/A:1014290316936.

[Bou00]   Marc Bourdon. "Sur les immeubles fuchsiens et leur type de quasi-isométrie". In: *Ergodic Theory Dynam. Systems* 20.2 (2000), pp. 343–364. DOI: 10.1017/S014338570000016X.

[Bre+17]   Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa. "*C**-simplicity and the unique trace property for discrete groups". In: *Publ. Math. Inst. Hautes Études Sci.* 126 (2017), pp. 35–71. DOI: 10.1007/s10240-017-0091-2.

[BH99]   Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature.* Vol. 319. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999, pp. xxii+643. ISBN: 3-540-64324-9. DOI: 10.1007/978-3-662-12494-9.

[Bro94]   Kenneth S. Brown. *Cohomology of groups.* Vol. 87. Graduate Texts in Mathematics. Corrected reprint of the 1982 original. Springer-Verlag, New York, 1994, pp. x+306. ISBN: 0-387-90688-6.

[BM97]   Marc Burger and Shahar Mozes. "Finitely presented simple groups and products of trees". In: *C. R. Acad. Sci. Paris Sér. I Math.* 324.7 (1997), pp. 747–752. DOI: 10.1016/S0764-4442(97)86938-8.

[BM00a]   Marc Burger and Shahar Mozes. "Groups acting on trees: from local to global structure". In: *Inst. Hautes Études Sci. Publ. Math.* 92 (2000), 113–150 (2001).

[BM00b]   Marc Burger and Shahar Mozes. "Lattices in product of trees". In: *Inst. Hautes Études Sci. Publ. Math.* 92 (2000), 151–194 (2001).

[Cap+19]   Pierre-Emmanuel Caprace, Peter H. Kropholler, Colin D. Reid, and Phillip Wesolek. "On the residual and profinite closures of commensurated subgroups". In: *Mathematical Proceedings of the Cambridge Philosophical Society* (2019), pp. 1–22. DOI: 10.1017/S0305004119000264.

[CM09a]   Pierre-Emmanuel Caprace and Nicolas Monod. "Isometry groups of non-positively curved spaces: discrete subgroups". In: *J. Topol.* 2.4 (2009), pp. 701–746. DOI: 10.1112/jtopol/jtp027.

[CM09b]   Pierre-Emmanuel Caprace and Nicolas Monod. "Isometry groups of non-positively curved spaces: structure theory". In: *J. Topol.* 2.4 (2009), pp. 661–700. DOI: 10.1112/jtopol/jtp026.

[CM19]   Pierre-Emmanuel Caprace and Nicolas Monod. *Erratum and addenda to "Isometry groups of non-positively curved spaces: discrete subgroups".* 2019. arXiv: 1908.10216 [math.GR].

[Eps+92]   David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. *Word processing in groups.* Jones and Bartlett Publishers, Boston, MA, 1992, pp. xii+330. ISBN: 0-86720-244-0.

[Gab02]   Damien Gaboriau. "Invariants l^2 de relations d'équivalence et de groupes". In: *Publ. Math. Inst. Hautes Études Sci.* 95 (2002), pp. 93–150. DOI: 10.1007/s102400200002.

[GS91]   S. M. Gersten and H. B. Short. "Rational subgroups of biautomatic groups". In: *Ann. of Math. (2)* 134.1 (1991), pp. 125–158. DOI: 10.2307/2944334.

[Hae91]   André Haefliger. "Complexes of groups and orbihedra". In: *Group theory from a geometrical viewpoint (Trieste, 1990).* World Sci. Publ., River Edge, NJ, 1991, pp. 504–540.

[Hae92]   André Haefliger. "Extension of complexes of groups". In: *Ann. Inst. Fourier (Grenoble)* 42.1-2 (1992), pp. 275–311.

[HP03]   Frédéric Haglund and Frédéric Paulin. "Constructions arborescentes d'immeubles". In: *Math. Ann.* 325.1 (2003), pp. 137–164. DOI: 10.1007/s00208-002-0373-x.

[Har07]   Pierre de la Harpe. "On simplicity of reduced *C**-algebras of groups". In: *Bull. Lond. Math. Soc.* 39.1 (2007), pp. 1–26. DOI: 10.1112/blms/bdl014.

[HP11]   Pierre de la Harpe and Jean-Philippe Préaux. "*C**-simple groups: amalgamated free products, HNN extensions, and fundamental groups of 3-manifolds". In: *J. Topol. Anal.* 3.4 (2011), pp. 451–489. DOI: 10.1142/S1793525311000659.

[HH20]   Camille Horbez and Jingyin Huang. *Measure equivalence classification of transvection-free right-angled Artin groups.* 2020. arXiv: 2010.03613 [math.GR].

[Hua18]   Jingyin Huang. "Commensurability of groups quasi-isometric to RAAGs". In: *Invent. Math.* 213.3 (2018), pp. 1179–1247. DOI: 10.1007/s00222-018-0803-3.

[Hug21a]   Sam Hughes. "Equivariant cohomology, lattices, and trees". PhD thesis. School of Mathematical Sciences, University of Southampton, 2021.

[Hug21b]   Sam Hughes. *Irreducible lattices fibring over the circle.* In preparation. 2021.

[Hug21c]   Sam Hughes. "Lattices in a product of trees, hierarchically hyperbolic groups, and virtual torsion-freeness". To appear in: *Bull. London Math. Soc.* (2021). arXiv: 2105.02847 [math.GR].

[KS16]   Aditi Kar and Michah Sageev. "Ping pong on CAT(0) cube complexes". In: *Comment. Math. Helv.* 91.3 (2016), pp. 543–561. DOI: 10.4171/CMH/395.

[LM21]   Ian J. Leary and Ashot Minasyan. "Commensurating HNN extensions: nonpositive curvature and biautomaticity". In: *Geom. Topol.* 25.4 (2021), pp. 1819–1860. DOI: 10.2140/gt.2021.25.1819.

[Mar91]   G. A. Margulis. *Discrete subgroups of semisimple Lie groups.* Vol. 17. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1991, pp. x+388. ISBN: 3-540-12179-X. DOI: 10.1007/978-3-642-51445-6.

[Rad20]   Nicolas Radu. "New simple lattices in products of trees and their projections". In: *Canad. J. Math.* 72.6 (2020). With an appendix by Pierre-Emmanuel Caprace, pp. 1624–1690. DOI: 10.4153/s0008414x19000506.

[Rat07a]   Diego Rattaggi. "A finitely presented torsion-free simple group". In: *J. Group Theory* 10.3 (2007), pp. 363–371. DOI: 10.1515/JGT.2007.028.

[Rat07b]   Diego Rattaggi. "Three amalgams with remarkable normal subgroup structures". In: *J. Pure Appl. Algebra* 210.2 (2007), pp. 537–541. DOI: 10.1016/j.jpaa.2006.10.008.

[Ser71]   Jean-Pierre Serre. "Cohomologie des groupes discrets". In: *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970).* 1971, 77–169. Ann. of Math. Studies, No. 70.

[Ser03]   Jean-Pierre Serre. *Trees.* Springer Monographs in Mathematics. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. Springer-Verlag, Berlin, 2003, pp. x+142. ISBN: 3-540-44237-5.

[Sta15]   Mentor Stafa. "On the fundamental group of certain polyhedral products". In: *J. Pure Appl. Algebra* 219.6 (2015), pp. 2279–2299. DOI: 10.1016/j.jpaa.2014.09.001.

[Tho06]   Anne Thomas. "Lattices acting on right-angled buildings". In: *Algebr. Geom. Topol.* 6 (2006), pp. 1215–1238. DOI: 10.2140/agt.2006.6.1215.

[Val20]   Motiejus Valiunas. *Isomorphism classification of Leary-Minasyan groups.* 2020. arXiv: 2011.08143 [math.GR].

[Val21]   Motiejus Valiunas. *Leary-Minasyan subgroups of biautomatic groups.* 2021. arXiv: 2104.13688 [math.GR].