PROFINITE RIGIDITY OF FIBRING

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ABSTRACT. We introduce the classes of TAP groups, in which various types of algebraic fibring are detected by the non-vanishing of twisted Alexander polynomials. We show that finitely presented LERF groups lie in the class $\mathsf{TAP}_1(R)$ for every integral domain R, and deduce that algebraic fibring is a profinite property for such groups. We offer stronger results for algebraic fibring of products of limit groups, as well as applications to profinite rigidity of Poincaré duality groups in dimension 3 and RFRS groups.

1. INTRODUCTION

Our understanding of profinite properties of fundamental groups of compact 3-manifolds has seen a lot of recent progress. One particularly noteworthy statement is the theorem of Bridson–McReynolds–Reid–Spitler [BMRS20] saying that the fundamental groups of some hyperbolic 3-manifolds (including the Weeks manifold) are profinitely rigid in the absolute sense, that is, each is distinguished from every other finitely generated residually finite group by its set of finite quotients.

Restricting attention solely to 3-manifold groups, we have two remarkable results: First, Jaikin-Zapirain [JZ20] showed that if the profinite completion of the fundamental group of a compact orientable aspherical 3-manifold is isomorphic to that of $\pi_1(\Sigma) \rtimes \mathbb{Z}$ with Σ a compact orientable surface, then the manifold fibres over the circle. Second, Liu [Liu20] proved that there are at most finitely many finite-volume hyperbolic 3-manifolds with isomorphic profinite completions of their fundamental groups. Both theorems rely in a crucial way on the following result of Friedl–Vidussi.

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Theorem 1.1 ([FV11b, Theorem 1.1])**.** *Let R be a Noetherian unique factorisation domain (UFD). Let M be a compact, orientable, connected 3-manifold with empty or toroidal boundary. An epimorphism $\varphi\colon \pi_1(M) \to \mathbb{Z}$ is induced by a fibration $M \to \mathbb{S}^1$ if and only if for every epimorphism $\alpha\colon \pi_1(M) \twoheadrightarrow Q$ with finite image the associated first twisted Alexander polynomial $\Delta_{1,R}^{\varphi,\alpha}$ over R is non-zero.*

The result relies in a key way on a special case proved by Friedl–Vidussi in an earlier work [FV08], where the group $\pi_1(M)$ is additionally assumed to be locally extended residually finite (LERF, or subgroup separable). Once this is established, the above result follows by a series of arguments based on the work of Wilton–Zalesskii [WZ10] and Wise [Wis12].

The interest in fibring has surpassed its roots in manifold topology finding numerous applications within the realm of geometric group theory, for example in the construction of subgroups of hyperbolic groups with exotic finiteness properties [JNW21, IMM20, IMM21, IMP21, Fis22, IP22], exotic higher rank phenomena [Kro18, Hug22], the existence of uncountably many groups of type FP [Lea18b, Lea18a, KLS20, BL20], a connection between fibring of RFRS groups and ℓ^2-Betti numbers [Kie20b, Fis21], and the construction of analogues of the Thurston polytope for various classes of groups [FL17, FT20, Kie20a].

The version of Theorem 1.1 for LERF groups $\pi_1(M)$ is the starting point for our investigations. First, we introduce the notion of *TAP groups* (standing for Twisted Alexander Polynomial), that is groups in which the twisted Alexander polynomials control algebraic fibring, see Definition 3.1. We then show that in fact all finitely presented LERF groups are TAP – see Theorem 3.8 for the precise (more general) statement. This amounts to showing the following.

Theorem A. *Let G be a finitely presented LERF group and let R be an integral domain. An epimorphism $\varphi\colon G \to \mathbb{Z}$ is algebraically fibred if and only if for every epimorphism $\alpha\colon G \twoheadrightarrow Q$ with finite image the associated first twisted Alexander polynomial over R is non-zero.*

Here, a group is *algebraically fibred* if it admits an epimorphism to $\mathbb{Z}$ with a finitely generated kernel. Also, we are talking about vanishing of Alexander polynomials over arbitrary integral domains, which might seem worrying, as the definition of the polynomial requires R to be a UFD. It does however make sense to talk about vanishing even when the polynomial is itself not well-defined, see Definition 2.8.

We use the above to show that for finitely presented LERF groups, algebraic fibring is a profinite property.

Theorem B. *Let G_A and G_B be finitely presented LERF groups with isomorphic profinite completions. The group G_A is algebraically fibred if and only if G_B is.*

Again, this is really a corollary of the more general Corollary 4.14 combined with Remark 3.9.

An even more general (and more technical) result is given by Theorem 4.12, where we deal with algebraic semi-fibring of higher degree (see Definition 2.4). It allows us to show the following.

Theorem C. *Let $\mathbb{F}$ be a finite field. Let G_A and G_B be profinitely isomorphic finite products of limit groups. The group G_A is $\mathsf{FP}_n(\mathbb{F})$-semi-fibred if and only if G_B is.*

Theorem B finds another application in the study of profinite rigidity of Poincaré duality groups.

Theorem D. *Let G_A be a LERF PD_3-group. Let G_B be the fundamental group of a closed connected hyperbolic 3-manifold. If $\widehat{G_A} \cong \widehat{G_B}$, then G_A is the fundamental group of a closed connected hyperbolic 3-manifold.*

Finally, Theorem 5.11 implies that for a cohomologically good RFRS group G of type F, the profinite completion of G detects the degree of acyclicity of G with coefficients in the skew-field $\mathcal{D}_{\mathbb{F}G}$ introduced by Jaikin-Zapirain; here $\mathbb{F}$ is a finite field. The skew-field $\mathcal{D}_{\mathbb{F}G}$ can be thought of as an analogue of the Linnell skew-field in positive characteristic, and hence can be used to define a positive-characteristic version of ℓ^2-homology.

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2. Preliminaries

Throughout, all rings are associative and unital, and ring morphisms preserve units. All modules are left-modules, unless stated otherwise. In particular, resolutions will be left-resolutions, and hence coefficients in homology will be right-modules (and quite often bimodules).

2.1. **Bieri–Neumann–Strebel invariants.** Let R be a ring, G a group, and $\varphi\colon G \to \mathbb{R}$ a non-trivial homomorphism. Observe that

$$G_\varphi = \{g \in G \mid \phi(g) \geqslant 0\}$$

is a monoid.

Definition 2.1 (Homological finiteness properties)**.** We say that a monoid M is of *type* $\mathsf{FP}_n(R)$ if the trivial M-module R admits a resolution $C_\bullet$ by projective RM-modules in which C_i is finitely generated for all $i \leqslant n$.

Since every group is a monoid, the definition readily applies to groups as well.

The definition above is standard; we will sporadically mention also other standard finiteness properties, like type $\mathsf{FP}(R)$ and F. Note that G is of type $\mathsf{FP}_1(R)$ if and only if it is finitely generated, and if it is finitely presented then it is of type $\mathsf{FP}_2(R)$ for every ring R.

Definition 2.2. We say that φ lies in the *nth BNS invariant over R*, and write $\varphi \in \Sigma^n(G; R)$, if G_φ is of type $\mathsf{FP}_n(R)$.

We set $\Sigma^\infty(G; R) = \bigcap_n \Sigma^n(G; R)$.

The first BNS invariant $\Sigma^1(G; R) = \Sigma^1(G)$ is independent of R. It was introduced by Bieri–Neumann–Strebel in [BNS87]. The higher (homological) invariants defined above were introduced by Bieri–Renz [BR88] for $R = \mathbb{Z}$. The definition for general R appears for example in the work of Fisher [Fis21]. Fisher's paper also contains the following straight-forward generalisation of the work of Bieri–Renz.

Theorem 2.3 ([Fis21, Theorem 6.5], [BR88, Theorem 5.1])**.** *Suppose that $\varphi\colon G \to \mathbb{Z}$ is a non-trivial homomorphism. The kernel* $\ker\varphi$ *is of type* $\mathsf{FP}_n(R)$ *if and only if* $\{\varphi, -\varphi\} \subseteq \Sigma^n(G; R)$*.*

Definition 2.4. A non-trivial character $\varphi\colon G \to \mathbb{Z}$ is $\mathsf{FP}_n(R)$-*fibred* if $\ker\varphi$ is of type $\mathsf{FP}_n(R)$. An $\mathsf{FP}_1(R)$-fibred character will be also called *algebraically fibred*; this last notion is independent of R.

Similarly, an integral character in $\Sigma^n(G;R) \cup -\Sigma^n(G;R)$ will be called $\mathsf{FP}_n(R)$-*semi-fibred*, and a character in $\Sigma^1(G) \cup -\Sigma^1(G)$ will be called *algebraically semi-fibred*.

A group G will be called *algebraically fibred* if it admits an algebraically fibred character.

The invariant $\Sigma^1(G)$ admits a number of alternative definitions. Let us now discuss one of them.

Definition 2.5. Let B be a group, let $A, C \leqslant B$, and suppose that there exists an isomorphism $\iota\colon A \to C$. The *HNN extension* $B{*_\iota}$ with *base group* B and *associated subgroups* A and C is defined by

$$B{*_\iota} = B * \langle t \rangle / \langle\!\langle \{t^{-1}at = \iota(a) : a \in A\} \rangle\!\rangle.$$

The HNN extension is *ascending* if $C = B$ and *descending* if $A = B$. If it is ascending but not descending, it is *properly ascending*.

Proposition 2.6 ([Bro87]). *Let G be a finitely generated group. An epimorphism $\varphi\colon G \to \mathbb{Z}$ lies in $\Sigma^1(G)$ if and only if there exists an isomorphism $\rho\colon G \to B{*_\iota}$ where B is finitely generated, the HNN extension $B{*_\iota}$ is descending, and φ is equal to the composition of ρ with the quotient map $B{*_\iota} \to B *_\iota / \langle\!\langle B \rangle\!\rangle = \langle t \rangle = \mathbb{Z}$.*

An observant reader will notice that Brown's original statement uses ascending, rather than descending HNN extensions. This has to do with left/right conventions for modules used in the definition of $\Sigma^1(G)$.

2.2. **Twisted Alexander polynomials.** The following definitions are taken from Friedl and Vidussi's survey [FV11a]. However, we have taken liberty to phrase them in terms of group homology as opposed to the homology of a topological space with twisted coefficients.

Let R be an integral domain and $R[t^{\pm 1}]$ the ring of Laurent polynomials over R in an indeterminate t. Let $\alpha\colon G \twoheadrightarrow Q$ be a finite quotient of G. This induces an RG-bimodule structure on the free R-module RQ induced by left and right multiplication precomposed with α – another way to say it is that RQ is a quotient ring of RG, and this way becomes an RG-bimodule. Let $\varphi \in H^1(G;\mathbb{Z})$ be a cohomology class considered as a homomorphism $\varphi\colon G \to \mathbb{Z}$. Consider $RQ[t^{\pm 1}]$ equipped with the RG-bimodule structure given by

$$g.x = t^{\varphi(g)}\alpha(g)x, \quad x.g = xt^{\varphi(g)}\alpha(g)$$

for $g \in G, x \in RQ[t^{\pm 1}]$. Note that $RQ[t^{\pm 1}] = R(\mathbb{Z} \times Q)$, and the action is multiplication precomposed with $\varphi \times \alpha$, as above.

For $n \in \mathbb{Z}$, we define the *nth twisted (homological) Alexander module of φ and α* to be $H_n(G; RQ[t^{\pm 1}])$, where $RQ[t^{\pm 1}]$ has the non-trivial module structure described above. Observe that $H_n(G; RQ[t^{\pm 1}])$ also has the structure of a left $R[t^{\pm 1}]$-module. We will denote the module by $H^{\varphi,\alpha}_{n,R}$. If G is of type $\mathsf{FP}_n(R)$, then the nth twisted Alexander module is a finitely generated $R[t^{\pm 1}]$-module. Moreover, it is zero whenever $n < 0$ or n is greater than the cohomological dimension of G over R.

More generally, given two group homomorphisms $\alpha\colon G \to Q$ and $\varphi\colon G \to Z$, we will sometimes use $H^{\varphi,\alpha}_{n,R}$ to denote $H_n(G; R(Z \times Q))$ with the RG-bimodule structure on $R(Z \times Q)$ being multiplication precomposed with $\varphi \times \alpha$.

For any integral domain S and any finitely generated S-module M, define the *rank* of M to be $\operatorname{rk}_S M = \dim_{\mathrm{Frac}(S)} \mathrm{Frac}(S) \otimes_S M$, where $\mathrm{Frac}(S)$ denotes the classical field of fractions (that is, the Ore localisation) of S. When S is additionally a UFD, the *order* of M is the greatest common divisor of all maximal minors in a presentation matrix of M with finitely many columns. The order of M is well-defined up to a unit of S and depends only on the isomorphism type of M.

Suppose that G is of type $\mathsf{FP}_n(R)$, with R being a UFD. The *nth twisted Alexander polynomial* $\Delta^{\varphi,\alpha}_{n,R}(t)$ over R with respect to φ and α is defined to be the order of the nth twisted (homological) Alexander module of φ and α, treated as a left $R[t^{\pm 1}]$-module. Note that $R[t^{\pm 1}]$ is a UFD since R is.

Since we will be concerned with the vanishing of $\Delta^{\varphi,\alpha}_{n,R}(t)$, let us record a number of equivalent statements. From now on we drop the requirement on R being a UFD.

Lemma 2.7. *Let R be an integral domain, and let $F = \mathrm{Frac}(R)$. The following are equivalent:*

(1) $\operatorname{rk}_{R[t^{\pm 1}]} H^{\varphi,\alpha}_{n,R} = 0$*;*

(2) $H_{n,R}^{\varphi,\alpha}$ *is a torsion* $R[t^{\pm 1}]$*-module.*
(3) $H_{n,F}^{\varphi,\alpha}$ *is a torsion* $F[t^{\pm 1}]$*-module.*
(4) $H_{n,F}^{\varphi,\alpha}$ *is a finitely generated* F*-module.*

If additionally R *is a UFD, then these are equivalent to*

(5) $\Delta_{n,R}^{\varphi,\alpha}(t) \neq 0$.

Sketch proof. We offer only a sketch, since these equivalences are standard.

Items (2) and (3) are equivalent since F is a flat R-module. Items (3), (4), and (1) are equivalent thanks to the classification theorem for finitely generated modules over a PID, since $F[t^{\pm 1}]$ is a PID; one also needs to note that $\mathrm{Frac}(R[t^{\pm 1}]) = \mathrm{Frac}(F[t^{\pm 1}])$.

The equivalence of (5) with the other ones is explained in [Tur01, Remark 4.5.2]. □

Definition 2.8. Let R be an integral domain, $\varphi\colon G \to \mathbb{Z}$ be a homomorphism, and $\alpha\colon G \twoheadrightarrow Q$ be a homomorphism with finite image. We say that φ has *non-vanishing nth Alexander polynomial twisted by* α if $\mathrm{rk}_{R[t^{\pm 1}]} H_{n,R}^{\varphi,\alpha} = 0$. If this holds for $\alpha = \mathrm{tr}\colon G \to \{1\}$, we say that the *nth Alexander polynomial in dimension n* does not vanish; if the statement holds for all choices of α, we say that φ has *non-vanishing nth twisted Alexander polynomials.*

Lemma 2.7 shows that in this definition we may replace R by $\mathrm{Frac}(R)$.

Lemma 2.9. *The nth Alexander polynomial of* φ *twisted by* α *vanishes if and only if the nth (untwisted) Alexander polynomial of* $\varphi|_{\ker \alpha}$ *vanishes. Moreover, if* R *is a UFD then the corresponding twisted Alexander polynomials are equal.*

Proof. We need to compare the $R[t^{\pm 1}]$-modules $H_n(G; RQ[t^{\pm 1}])$ and $H_n(\ker \alpha; R[t^{\pm 1}])$. Shapiro's lemma shows that these modules are isomorphic, since $RQ[t^{\pm 1}]$ is isomorphic to the induced right RG-module of the right $R(\ker \alpha)$-module $R[t^{\pm 1}]$. □

The following result is well known for 3-manifolds and has appeared in several places [KL99, CR12, GKM05, FK06]; in fact, it appears to date back to work of Milnor [Mil68]. We include a proof in the group theoretic setting for completeness.

Proposition 2.10. *Let R be an integral domain. Let G be a group of type $\mathsf{FP}_n(R)$ and let $\varphi\colon G \to \mathbb{Z}$ be a non-trivial character. If φ is $\mathsf{FP}_n(R)$-fibred, then its kth twisted Alexander polynomials do not vanish for $k \leqslant n$.*

Proof. Since φ is $\mathsf{FP}_n(R)$-fibred, G splits as a semi-direct product $A \rtimes \mathbb{Z}$ with A of type $\mathsf{FP}_n(R)$. Now, let $\alpha\colon G \twoheadrightarrow Q$ be an epimorphism of G onto a finite group and let $RQ[t^{\pm 1}]$ be the right RG-module with action given by φ and α. Applying [Bro94, III.6.2 and III.8.2] yields that $H_\bullet(G; RQ[t^{\pm 1}]) \cong H_\bullet(A; RQ)$ as R-modules. Now, since A is of type $\mathsf{FP}_n(R)$ and Q is finite it follows that $H_k(A; RQ)$ for $k \leqslant n$ is a finitely generated R-module. Such a module cannot contain a copy of $R[t^{\pm 1}]$, and therefore $H_k(G; RQ[t^{\pm 1}])$ is a torsion $R[t^{\pm 1}]$-module. We are done by Lemma 2.7. $\square$

Proposition 2.11. *Let G be a group of type $\mathsf{FP}_n(R)$, and let $\varphi\colon G \to \mathbb{Z}$ be an $\mathsf{FP}_n(R)$-semi-fibred character. The kth twisted Alexander polynomials of φ are non-zero for all $k \leqslant n$.*

Proof. Since G is of type $\mathsf{FP}_n(R)$, we find a projective resolution $C_\bullet$ of the trivial G-module R with C_k a finitely generated RG-module for every $k \leqslant n$. We replace φ by $-\varphi$ if needed, and assume that $\varphi \in \Sigma^n(G; R)$; note that this replacement does not affect the vanishing of twisted Alexander polynomials.

Using Fisher's version of Sikorav's theorem [Fis21, Theorem 5.3], we find a partial chain contraction for $C_\bullet$ over the Novikov ring $\mathrm{Nov}(RG, \varphi)$ in the following sense: Denote the differentials of $C_\bullet$ by $\partial_i\colon C_i \to C_{i-1}$. We find $\mathrm{Nov}(RG, \varphi)$-module morphisms

$$A_i\colon \mathrm{Nov}(RG, \varphi) \otimes_{RG} C_i \to \mathrm{Nov}(RG, \varphi) \otimes_{RG} C_{i+1}$$

such that for every $i \leqslant n$ we have $A_{i-1}\partial'_i + \partial'_{i+1}A_i = \mathrm{id}$ where $\partial'_i = \mathrm{id}_{\mathrm{Nov}(RG,\varphi)} \otimes_{RG} \partial_i$, and $A_{-1} = 0, \partial'_{-1} = 0$. Here the Novikov ring $\mathrm{Nov}(RG, \varphi)$ is the ring of twisted Laurent power series with coefficients in $R(\ker \varphi)$ and with variable $t \in G$ with $\varphi(t) = 1$, where the twisting is given by the conjugation action of t on $\ker \varphi$; multiplication in $\mathrm{Nov}(RG, \varphi)$ induces a right RG-module structure on $\mathrm{Nov}(RG, \varphi)$.

Now let $\alpha\colon G \twoheadrightarrow Q$ be an epimorphism with Q finite. Dividing G by the normal subgroup $K = \ker\alpha \cap \ker\varphi$ induces a ring morphism

$$\beta\colon \mathrm{Nov}(RG, \varphi) \to \mathrm{Nov}(R(G/K), \psi)$$

where $\psi\colon G/K \to \mathbb{Z}$ is induced by φ. Applying β to the entries of the matrices A_i shows that

$$H_i(G; \mathrm{Nov}(R(G/K), \psi)) = 0$$

for all $i \leqslant n$.

The ring $\mathrm{Nov}(R(G/K), \psi)$ is isomorphic to $\bigoplus_Q \mathrm{Nov}(R(\ker\alpha/K), \psi)$ as an $R(\ker\alpha/K)$-module, and hence also as an $R(\ker\alpha)$-module, and so

$$H_i\big(\ker\alpha; \mathrm{Nov}(R(\ker\alpha/K), \psi)\big) = 0$$

for all $i \leqslant n$. Arguing with chain contractions as before, we see that

$$H_i\big(\ker\alpha; \mathrm{Nov}(\mathrm{Frac}(R)(\ker\alpha/K), \psi)\big) = 0$$

for all $i \leqslant n$.

Now, $\ker\alpha/K \cong \mathbb{Z}$, and therefore $\mathrm{Nov}(\mathrm{Frac}(R)(\ker\alpha/K), \psi))$ is the field of Laurent power series in a single variable t and coefficients in $\mathrm{Frac}(R)$, where $t \in \ker\alpha$ is mapped by ψ to a generator of $\mathbb{Z}$. This field contains the field $R(t)$ of rational functions in a single variable and coefficients in R in the obvious way. Since $R(t)$ is a right $R(\ker\alpha)$-submodule of $\mathrm{Nov}(\mathrm{Frac}(R)(\ker\alpha/K), \psi))$, and since $\mathrm{Nov}(\mathrm{Frac}(R)(\ker\alpha/K), \psi))$ is a flat $R(t)$-module as both are skew-fields, we conclude that

$$0 = H_i(\ker\alpha; R(t)).$$

Now, using flatness of localisations, we obtain

$$H_i(\ker\alpha; R(t)) = H_i(\ker\alpha; R[t^{\pm 1}]) \otimes_{R[t^{\pm 1}]} R(t)$$

and therefore $H_i(\ker\alpha; R[t^{\pm 1}])$ is a torsion $R[t^{\pm 1}]$-module. We are now done thanks to Lemmata 2.7 and 2.9. □

Example 2.12. The Baumslag–Solitar group

$$\mathrm{BS}(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle$$

has $H^1(G;\mathbb{R}) \cong \mathbb{R}$ with basis given by the character

$$\varphi\colon \mathrm{BS}(1,n) \twoheadrightarrow \langle t\rangle \cong \mathbb{Z}$$

killing a. The BNS invariant $\Sigma^1(\mathrm{BS}(1,n))$ consists only of the ray $\{\lambda\varphi \mid \lambda \in (0,\infty)\}$. It follows that for every integral domain R and every finite quotient $\alpha\colon \mathrm{BS}(1,n) \twoheadrightarrow Q$, the twisted Alexander polynomials do not vanish. (In fact, the polynomials can be computed by hand rather easily.) Note that $\mathrm{BS}(1,n)$ splits as $\mathbb{Z}[1/n] \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts as multiplication by n, so $\ker(\varphi)$ is not finitely generated.

3. TAP groups

3.1. The definition.

Definition 3.1. Let R be a integral domain. We say that a group G of type $\mathsf{FP}_n(R)$ is in the class $\mathsf{TAP}_n(R)$ if for every non-trivial character $\varphi \in H^1(G;\mathbb{Z})$ the following property holds:

> *φ is $\mathsf{FP}_n(R)$-semi-fibred if and only if for each $i \leqslant n$ its twisted ith Alexander polynomials do not vanish.*

We allow $n = \infty$ in the above definition.

The definition is best motivated and explained by the following slogan: *"A group is in $\mathsf{TAP}_n(R)$ if and only if twisted Alexander polynomials detect algebraic semi-fibring over R up to dimension n".*

Note that in view of Example 2.12 it is more natural to use semi-fibring rather than fibring in the definition above.

Example 3.2. Theorem 1.1 by Friedl–Vidussi shows that fundamental groups of compact, orientable, connected 3-manifolds with empty or toroidal boundary are in $\mathsf{TAP}_1(R)$, and hence in $\mathsf{TAP}_\infty(R)$, since the first BNS invariants of compact 3-manifold groups are symmetric [BNS87, Corollary F], and since finitely generated fundamental groups of 3-manifolds are of type F_∞, and therefore $\mathsf{FP}_\infty(R)$ over every R – this follows from Scott's compact core theorem [Sco73].

Example 3.3. A non-example is given by $G = S \wr \mathbb{Z}$ where S is an infinite simple group. Note that such a group has an obvious map $\varphi\colon G \twoheadrightarrow \mathbb{Z}$ and this map is a basis for $H^1(G;\mathbb{R}) \cong \mathbb{R}$. The group G admits an automorphism that acts as on $H^1(G;\mathbb{R})$ as minus the identity, and hence the BNS invariants of G must be symmetric. Therefore $\Sigma^1(G; R)$ is empty since $\ker\varphi = \bigoplus_{\mathbb{Z}} S$ is not finitely generated. Now, every finite quotient of G is cyclic and the corresponding kernel is isomorphic to $S^n \wr \mathbb{Z}$ for some n; the Alexander polynomial of such a group is equal to 1, since the relevant R-module is $H_1(S^n \wr \mathbb{Z}; R[t^{\pm 1}]) \cong H_1(\bigoplus_{\mathbb{Z}} S^n; R) = 0$. This shows that G is not in $\mathsf{TAP}_1(R)$ for any R.

Example 3.4. Another non-example is provided by every group that admits a character that is $\mathsf{FP}_2(\mathbb{Q})$-semi-fibred without being $\mathsf{FP}_2(\mathbb{Z})$-semi-fibred. Such a group cannot be in $\mathsf{TAP}_2(\mathbb{Z})$, since if it were then the character would have non-vanishing twisted second Alexander polynomials over $\mathbb{Q}$ by Proposition 2.11, and hence over $\mathbb{Z}$ by Lemma 2.7, and then $\mathsf{TAP}_2(\mathbb{Z})$ would show that the character is $\mathsf{FP}_2(\mathbb{Z})$-semi-fibred. An explicit example of a group satisfying the requirement is every RAAG based on a triangulation of the real projective plane; the character will then be the Bestvina–Brady character.

We will be primarily interested in profinite aspects of TAP groups, but the property has also other uses.

Italiano–Martelli–Migliorini in [IMM20] introduced a finite-volume hyperbolic 7-manifold whose fundamental group maps onto $\mathbb{Z}$ with finitely presented kernel. Fisher [Fis22] showed that by passing to a suitable finite cover, one obtains a finite-volume hyperbolic 7-manifold M with $G = \pi_1(M)$ and an epimorphism $\varphi\colon G \to \mathbb{Z}$ with kernel that is finitely presented and of type $\mathsf{FP}(\mathbb{Q})$.

Suppose that G lies in $\mathsf{TAP}_7(\mathbb{Z})$ and that

$$\Sigma^7(G; \mathbb{Z}) = -\Sigma^7(G; \mathbb{Z}).$$

Since $\varphi \in \Sigma^7(\pi_1(M); \mathbb{Q})$, we see that the twisted Alexander polynomials of M over $\mathbb{Q}$ do not vanish in dimensions 1 to 7. This means that the polynomials over $\mathbb{Z}$ do not vanish either, and since G is in $\mathsf{TAP}_7(\mathbb{Z})$ we conclude that $\varphi \in \Sigma^7(\pi_1(M); \mathbb{Z})$. Since the BNS invariant is also assumed to be symmetric, $\ker\varphi$ is finitely presented and of type $\mathsf{FP}_7(\mathbb{Z})$, and hence is of type F. If one now had a version of Farrell's theorem [Far72] for manifolds with boundary, one could conclude that M fibres over the circle.

3.2. **Almost finitely presented LERF groups are $\mathsf{TAP}_1(R)$.** Now that we have defined TAP, let us introduce the class of groups whose TAPness we want to establish.

Definition 3.5. Let G be a group. A subgroup $A \leqslant G$ is *separable* if for every $g \in G \smallsetminus A$ there exists an epimorphism $\alpha\colon G \twoheadrightarrow Q$ with Q finite such that

$$\alpha(g) \notin \alpha(A).$$

A group G is *LERF* (or *locally extended residually finite*, or *subgroup separable*) if every finitely generated subgroup is separable.

We will need some standard terminology related to graph-of-groups decompositions.

Definition 3.6. We say that a group G *splits over a subgroup* A if G decomposes as a reduced graph of groups with a single edge and edge group A. Recall that a graph of groups is *reduced* if every edge both of whose attaching maps are isomorphisms is a loop.

We are ready to state our main technical tool. The HHN extension case is a variation on the proofs from [FV08].

Proposition 3.7. *Let G be a finitely generated group that splits over a separable subgroup. Let $\varphi\colon G \to \mathbb{Z}$ be a non-zero character that vanishes on the edge group. If for some integral domain R the first twisted Alexander polynomials do not vanish, then the splitting has only one vertex and φ is algebraically fibred with kernel equal to the edge group.*

Proof. We need to consider two cases, depending on whether the splitting is an HNN extension or an amalgamated free product.

Suppose first that G splits as an HNN extension. If both edge maps are isomorphisms, then the edge group is a normal subgroup, and quotienting by it yields $\mathbb{Z}$. Hence φ is algebraically fibred with kernel equal to the edge group, as claimed. Suppose now that at least one of the attaching maps is not a surjection. Let A denote the image of this map, and let B denote the vertex group.

Let $\alpha\colon G \twoheadrightarrow Q$ be an epimorphism with finite image. Consider the Mayer–Vietoris sequence for an HNN-extension (see for instance

[Bro94, Chapter VII.9]) with non-trivial coefficients $RQ[t^{\pm 1}]$ as in Section 2.2, where the action of A and B on the module is inherited from G. The sequence takes the following form:

$$\begin{tikzcd} & & \cdots \arrow[r] & H_1(G; RQ[t^{\pm1}]) \arrow[dll] \\ H_0(A; RQ[t^{\pm1}]) \arrow[r] & H_0(B; RQ[t^{\pm1}]) \arrow[r] & H_0(G; RQ[t^{\pm1}]) \arrow[dll] & \\ 0 & & & \end{tikzcd}$$

Since $A \leqslant \ker\varphi$, we have a right A-module isomorphism

$$RQ[t^{\pm1}] = R[t^{\pm1}] \otimes_R RQ$$

where the action of $g \in A$ on $R[t^{\pm1}] \otimes_R RQ$ is the diagonal action given by right-multiplication by $\alpha(g)$ on RQ and the trivial action of $R[t^{\pm1}]$. We also have an R-module isomorphism

$$H_0(A; R[t^{\pm1}] \otimes_R RQ) \cong R[t^{\pm1}] \otimes_R (RQ)_A$$

by the definition of zeroth homology, where $(-)_A$ denotes A-coinvariants.

By assumption, $H_1(G; RQ[t^{\pm1}])$ is $R[t^{\pm1}]$-torsion and it is clear that $H_0(G; RQ[t^{\pm1}])$ is $R[t^{\pm1}]$-torsion (see for instance [FV08, Lemma 4.4]). Applying these observations in the trivial case $\alpha = \mathrm{tr}, Q = \{1\}$, we see that $H_0(B; R[t^{\pm1}])$ must contain a copy of $R[t^{\pm1}] \otimes_R R_A = R[t^{\pm1}]$. If $\varphi|_B \neq 0$, then it is immediate that $H_0(B; R[t^{\pm1}]) = (R[t^{\pm1}])_B$ is a torsion $R[t^{\pm1}]$-module, yielding a contradiction. We conclude that $\varphi|_B = 0$, and hence we have $H_0(B; RQ[t^{\pm1}]) \cong R[t^{\pm1}] \otimes_R (RQ)_B$ for all α and Q.

Using the fact that A is separable, we produce an epimorphism $\alpha\colon G \twoheadrightarrow Q$ with finite image such that $\alpha(A)$ is a proper subgroup of $\alpha(B)$. Let $F = \mathrm{Frac}(R)$. Note that $F(t)$, the field of rational functions, is a flat $R[t^{\pm1}]$-module. Tensoring the Mayer–Vietoris sequence above (with this choice of α) with $F(t)$ over $R[t^{\pm1}]$ we see that

$$\dim_{F(t)} F(t) \otimes_R (RQ)_A = \dim_{F(t)} F(t) \otimes_R (RQ)_B.$$

Observe that $(RQ)_A$ is a free right R-module of rank $|Q : \alpha(A)|$, and similarly for $(RQ)_B$. The dimensions above pick up exactly the R-rank, and so we may conclude that

$$|Q : \alpha(A)| = |Q : \alpha(B)|,$$

contradicting $|\alpha(A)| < |\alpha(B)|$.

If G splits as an amalgamated free product, the edge group A must be a proper subgroup of the vertex groups B and B', since otherwise the graph of groups would not be reduced.

We now consider the Mayer–Vietoris sequence for a free product with amalgamation:

$$\begin{tikzcd}
 & & \cdots \arrow[r] & H_1(G; RQ[t^{\pm 1}]) \arrow[dlll] \\
H_0(A; RQ[t^{\pm 1}]) \arrow[r] & H_0(B; RQ[t^{\pm 1}]) \oplus H_0(B'; RQ[t^{\pm 1}]) \arrow[rr] & & H_0(G; RQ[t^{\pm 1}]) \arrow[dlll] \\
0 & & &
\end{tikzcd}$$

Arguing as before with $\alpha = \mathrm{tr}$, we first see that φ must vanish on precisely one of the vertex groups, say B – it cannot vanish on both since $\varphi \neq 0$. As before, we produce $\alpha\colon G \twoheadrightarrow Q$ such that $\alpha(A) < \alpha(B)$. After tensoring with $F(t)$ over $R[t^{\pm 1}]$ we obtain an isomorphism between $F(t) \otimes_{R[t^{\pm 1}]} H_0(A; RQ[t^{\pm 1}])$ and

$$\big(H_0(B; RQ[t^{\pm 1}]) \otimes_{R[t^{\pm 1}]} F(T)\big) \oplus \big(H_0(B'; RQ[t^{\pm 1}]) \otimes_{R[t^{\pm 1}]} F(t)\big).$$

Since $\varphi|_{B'}$ is non-trivial, the $R[t^{\pm 1}]$-module $H_0(B'; RQ[t^{\pm 1}])$ is torsion as before, and hence

$$F(t) \otimes_{R[t^{\pm 1}]} H_0(B'; RQ[t^{\pm 1}]) = 0.$$

Using dimensions over $F(t)$ we conclude that $|\alpha(A)| = |\alpha(B)|$, as before. This is a contradiction. $\square$

We are now ready for our first main theorem.

Theorem 3.8. *If G is a LERF group of type $\mathsf{FP}_2(S)$ for some commutative ring S, then G is in $\mathsf{TAP}_1(R)$ for every integral domain R.*

Proof. Let $\varphi\colon G \to \mathbb{Z}$ be a non-trivial character. We aim to show that φ is algebraically fibred if and only if for every epimorphism onto a finite group $\alpha\colon G \twoheadrightarrow Q$ the corresponding twisted Alexander polynomial does not vanish. The 'if' direction is given by Proposition 2.11. For the converse suppose that the twisted Alexander polynomials of φ are non-zero.

Since G is of type $\mathsf{FP}_2(S)$, by [BS78, Theorem A] there exist finitely generated subgroups $A, B, C \leqslant G$ with $A, C \leqslant B$, and an isomorphism $\iota\colon A \to C$, such that G splits as an HNN-extension $B*_\iota$, and dividing by B coincides with φ.

Since A is finitely generated and G is LERF, we see that A is separable. The result now follows from Proposition 3.7. □

Remark 3.9. The proof of the above result together with Proposition 2.11 show that $\Sigma^1(G) = -\Sigma^1(G)$. This is a well-known fact that can be proved directly using Proposition 2.6.

Proposition 3.7 can also be used in the setting of graphs of groups.

Theorem 3.10. *Let R be a integral domain. Let G be a finitely generated fundamental group of a finite reduced graph of groups $\mathcal{G}$. Let $\varphi \in H^1(G;\mathbb{Z})$ be a non-zero character and suppose that G is LERF. If the first twisted Alexander polynomials of φ do not vanish, then for every finitely generated edge group A precisely one of the following holds:*

(1) either $G = A \rtimes \mathbb{Z}$ with φ being the projection map,
(2) or $\varphi|_A \neq 0$.

Proof. Consider an edge e with a finitely generated group A. The proof splits into two cases.

If e is non-separating, then we may collapse all the other edges and obtain a splitting of G as an HNN extension with edge group A. Now, Proposition 3.7 tells us that if $\varphi|_A = 0$, then φ is algebraically fibred with kernel A, that is, $G = A \rtimes \mathbb{Z}$.

If e is a separating edge, then G splits as a free product amalgamated over A. Proposition 3.7 tells us that $\varphi|_A \neq 0$. □

3.3. **Products of $\mathsf{TAP}_1(R)$ groups.** We will now discuss the structure of the BNS invariants for products of groups. When working over fields, this structure is completely understood in terms of BNS invariants of factors; over general commutative rings all we have is an inequality. To understand the inequality, recall that we have defined the BNS invariants $\Sigma^n(G;R)$ as subsets of $H^1(G;\mathbb{R}) \smallsetminus \{0\}$. This in particular applies to $\Sigma^0(G;R)$. For a subset $U \subseteq H^1(G;\mathbb{R})$ we denote the complement by $U^c = H^1(G;\mathbb{R}) \smallsetminus U$. In particular, we have $\Sigma^0(G;R)^c = \{0\}$.

When $G = G_1 \times G_2$, we have $H^1(G;\mathbb{R}) = H^1(G_1;\mathbb{R}) \oplus H^1(G_2;\mathbb{R})$. Given subsets $U_i \subseteq H^1(G_i;\mathbb{R})$ we define their *join* to be

$$U_1 * U_2 = \{tu_1 + (1-t)u_2 \mid u_i \in U_i, t \in [0,1]\}.$$

The following inequality is due to Meinert; see [BG10] for the history of this and [Geh98] for a proof. The "moreover" is due to Bieri–Geoghegan [BG10] and for $R = \mathbb{Z}$ the inequality can be strict [Sch08].

Theorem 3.11 (Meinert's inequality)**.** *Let G_1 and G_2 be groups of type $\mathsf{FP}_n(R)$ where R is a commutative ring, and let $G = G_1 \times G_2$. Then*

$$\Sigma^n(G;R)^c \subseteq \bigcup_{p=0}^{n} \Sigma^p(G_1;R)^c * \Sigma^{n-p}(G_2;R)^c.$$

Moreover, equality holds if R is a field.

Proposition 3.12. *Let R be a integral domain and let G_1 and G_2 be finitely generated groups. If G_i is in $\mathsf{TAP}_1(R)$ for $i = 1,2$, then $G_1 \times G_2$ is in $\mathsf{TAP}_1(R)$.*

Proof. Let $G = G_1 \times G_2$. Suppose $\varphi\colon G \twoheadrightarrow \mathbb{Z}$ is not algebraically semi-fibred and is non-zero. We need to show that there exists a finite quotient $\alpha\colon G \twoheadrightarrow Q$ such that the corresponding twisted Alexander polynomial vanishes.

By Meinert's inequality, we have

$$\varphi \in (\Sigma^1(G_1;R)^c * \{0\}) \cup (\{0\} * \Sigma^1(G_2;R)^c).$$

In particular, for exactly one $i \in \{1,2\}$ we have $\varphi|_{G_i} = 0$. Suppose without loss of generality that $i = 2$.

Now, we have a splitting $\ker(\varphi) = \ker(\varphi|_{G_1}) \times G_2$. Since G_1 lies in $\mathsf{TAP}_1(R)$, there exists a finite quotient $\alpha_1\colon G_1 \twoheadrightarrow Q$ such that the module $H_{1,R}^{\varphi|_{G_1},\alpha_1}$ is not $R[t^{\pm 1}]$-torsion, and hence contains a free $R[t^{\pm 1}]$-module. Let F denote $\operatorname{Frac}(R)$. Since F is a flat R-module, and since $\dim_F F \otimes_R R[t^{\pm 1}] = \infty$, we immediately see that

$$\dim_F F \otimes_R H_{1,R}^{\varphi|_{G_1},\alpha_1} = \infty.$$

Define $\alpha\colon G \twoheadrightarrow Q$ to be the composite $G \twoheadrightarrow G_1 \twoheadrightarrow Q$. Applying Shapiro's lemma (as in the proof of Lemma 2.9), and then [Bro94, III.6.2 and III.8.2], gives isomorphisms of R-modules

$$H_{1,R}^{\varphi,\alpha} \cong H_{1,R}^{\varphi|_{\ker\alpha},\mathrm{tr}} \cong H_1(\ker(\varphi)\cap\ker(\alpha);R),$$

but $\ker(\varphi)\cap\ker(\alpha) \cong (\ker(\varphi|_{G_1})\cap\ker(\alpha_1))\times G_2$. It follows that we can compute $H_{1,R}^{\varphi,\alpha}$ by the Künneth spectral sequence (note that R is not necessarily a PID so we cannot use the Künneth formula). We have

$$\mathrm{Tor}_0^R(H_{1,R}^{\varphi|_{G_1},\alpha_1},R) \cong H_{1,R}^{\varphi|_{G_1},\alpha_1}\otimes_R R \cong H_{1,R}^{\varphi|_{G_1},\alpha_1} \leqslant H_{1,R}^{\varphi,\alpha}$$

as R-modules. We conclude that

$$\dim_F F\otimes_R H_{1,R}^{\varphi,\alpha} = \infty.$$

Using flatness again we get

$$\dim_F H_{1,F}^{\varphi,\alpha} = \infty,$$

and hence the first Alexander polynomials twisted by α over F and over R vanish by Lemma 2.7. $\square$

3.4. Products of limit groups are $\mathsf{TAP}_\infty(\mathbb{F})$.

Theorem 3.13. *Let $\mathbb{F}$ be a field and let $G=\prod_{i=1}^n G_i$ be a product of limit groups. Then, G is in $\mathsf{TAP}_\infty(\mathbb{F})$.*

Proof. By [Wil08] limit groups are LERF, and by [BF09, Exercise 13] limit groups are of type F, and hence $\mathsf{FP}_2(\mathbb{Z})$. It follows that products of limit groups are $\mathsf{TAP}_1(\mathbb{F})$ by Theorem 3.8 and Proposition 3.12.

Let $\varphi\colon G\twoheadrightarrow\mathbb{Z}$ be a character which is $\mathsf{FP}_{k-1}(\mathbb{F})$-semi-fibred but not $\mathsf{FP}_k(\mathbb{F})$-semi-fibred for some $2\leqslant k\leqslant n$. If no such k exists, then we are done by Proposition 2.11. The same result tells us that all twisted Alexander polynomials of φ in dimension at most $k-1$ will vanish. We need to exhibit a non-vanishing one in dimension k. Lemma 2.9 tells us that it is enough to find such a non-vanishing twisted polynomial for some normal finite-index subgroup of G.

We may assume that if some G_i is abelian then $\varphi|_{G_i}=0$. Otherwise, φ would be $\mathsf{FP}_\infty(\mathbb{F})$-semi-fibred by Meinert's inequality. After passing to a finite index normal subgroup $H\times K$ with $H=\prod_{i=1}^p H_i, K=\prod_{j=1}^q K_j$, $p+q=n$, $H_i\trianglelefteq G_i$, and $K_j=G_{q+j}$, we may assume that $\varphi|_{H_i}$ is surjective and $\varphi|_{K_j}=0$. Let ψ denote the restriction of φ to H. By [BHMS09, Theorem 7.2] (note that the result is only stated for $\mathbb{Q}$ but by the paragraph after Theorem C loc. cit. it holds for arbitrary fields) we have that $H_p(\ker\psi;\mathbb{F})$ has infinite dimension over $\mathbb{F}$ (here we are using the fact that ψ vanishes on abelian factors). It follows from Lemma 2.7 that the twisted Alexander polynomial of G associated to $\alpha\colon G \twoheadrightarrow G/(H\times K)$ vanishes.

We have found a vanishing Alexander polynomial in dimension p. Note that $p \geqslant k$ since φ is $\mathsf{FP}_{k-1}(\mathbb{F})$-semi-fibred. Meinert's inequality tells us that $\Sigma^{p-1}(G;R)^c$ is the union of joins of the form

$$\Sigma^{m_1}(G_1;R)^c * \cdots * \Sigma^{m_n}(G_1;R)^c$$

with $\sum m_i = p-1$. Each such join must therefore have at most $p-1$ factors with $n_i > 0$, and hence characters lying in such a join must vanish on all but at most $p-1$ factors G_i. But φ does not vanish on p factors, and hence $\varphi \in \Sigma^{p-1}(G;R)$. Hence $p-1 \leqslant k-1$, and therefore $p=k$. We have now shown that the first dimension in which a twisted Alexander polynomial vanishes is equal to the first dimension in which φ is not semi-fibred. This proves the claim. □

4. Profinite rigidity of fibring

Definition 4.1. Let G be a group, R be a ring, and let $\mathcal{C}$ be a directed system of normal finite-index subgroups of G. We set

$$\widehat{G}_{\mathcal{C}} = \varprojlim_{U\in\mathcal{C}} G/U$$

and

$$R[\![G]\!]_{\mathcal{C}} = \varprojlim_{U\in\mathcal{C}} R(G/U).$$

When $\mathcal{C}$ consists of all normal subgroups of finite index, we write $\widehat{G}$ for $\widehat{G}_{\mathcal{C}}$, $R[\![G]\!]$ for $R[\![G]\!]_{\mathcal{C}}$, and call them respectively the *profinite completion* and the *completed group ring.*

Note that $\widehat{\mathbb{Z}}$ is a ring with the obvious multiplication.

The groups $\widehat{G}$ and more generally $\widehat{G}_{\mathcal{C}}$ carry a natural compact topology obtained as the limit of the discrete topology on G/U. Whenever we will use this topology, we will state it explicitly, as we do below.

Definition 4.2. Let G be a residually finite group. We say that G is *n-good* if for all $0 \leqslant j \leqslant n$ and all $\mathbb{Z}G$-modules M that are finite as sets, the map

$$H^n_{\mathrm{cont}}(\widehat{G};M) \to H^n(G;M)$$

induced by the inclusion $G \to \widehat{G}$ is an isomorphism. Here, H^*_{cont} denotes *continuous group cohomology* which is defined analogously to ordinary group cohomology except for the following modifications: First, we require M to be a topological $\widehat{G}$-module, that is, M carries a (possibly discrete) topology and the $\widehat{G}$-action on M is continuous, and secondly, the cochain groups $C^\bullet_{\text{cont}}(\widehat{G}; M)$ consist of continuous functions $\widehat{G}^n \to M$.

A group that is n-good for all n is called *cohomologically good*, or *good in the sense of Serre*.

Remark 4.3. It is very easy to see that every residually finite group is 1-good.

Proposition 4.4 ([GJZZ08, Lemma 3.2])**.** *Finite-index subgroups of n-good groups are themselves n-good.*

The above proposition is stated in a slightly less general way in the paper of Grunewald–Jaikin-Zapirain–Zalesskii [GJZZ08], but the proof gives precisely what we claim above.

The following result is a slight variation on a theorem of Kochloukova and Zalesskii. The only difference consists of replacing the assumption of G being type FP_∞ with the assumption of G being type FP_n. The proof is very similar but we include it to highlight the differences.

Proposition 4.5. [KZ08, Theorem 2.5] *Let G be a group of type $\mathsf{FP}_n(\mathbb{Z})$ and let $\mathcal{C}$ be a directed system of finite index normal subgroups. Suppose that for a fixed prime p and for all $1 \leqslant i \leqslant n$ we have*

$$\varprojlim_{U \in \mathcal{C}} H_i(U; \mathbb{Z}/p\mathbb{Z}) = 0.$$

Then, for all $m \geqslant 1$ and $1 \leqslant i \leqslant n$ we have

$$\operatorname{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, (\mathbb{Z}/p^m\mathbb{Z})[\![G]\!]_{\mathcal{C}}) = 0 \text{ and } \operatorname{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}_p[\![G]\!]_{\mathcal{C}}) = 0$$

where $\mathbb{Z}_p$ denotes the p-adic integers.

In both the statement above and the proof below, we stay in the abstract category, that is we do not require any continuity, and homology is taken without closing images.

Proof. Let $P_\bullet$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for $i \leqslant n$. Let $P_\bullet^{(m)} = (\mathbb{Z}/p^m\mathbb{Z})[\![G]\!]_{\mathcal{C}} \otimes_{\mathbb{Z}G} P_\bullet$. By

[KZ08, Lemma 2.1] we have $H_i(P_\bullet^{(1)}) \cong \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, (\mathbb{Z}/p\mathbb{Z})[\![G]\!]_{\mathcal{C}}) = 0$ for $1 \leqslant i \leqslant n$.

The short exact sequence of right $\mathbb{Z}G$-modules

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z})[\![G]\!]_{\mathcal{C}} \longrightarrow (\mathbb{Z}/p^m\mathbb{Z})[\![G]\!]_{\mathcal{C}} \longrightarrow (\mathbb{Z}/p^{m-1}\mathbb{Z})[\![G]\!]_{\mathcal{C}} \longrightarrow 0$$

induces a long exact sequence in homology containing sequences

$$H_i(G; (\mathbb{Z}/p\mathbb{Z})[\![G]\!]_{\mathcal{C}}) \to H_i(G; (\mathbb{Z}/p^m\mathbb{Z})[\![G]\!]_{\mathcal{C}}) \to H_i(G; (\mathbb{Z}/p^{m-1}\mathbb{Z})[\![G]\!]_{\mathcal{C}})$$

exact in the middle term. This latter sequence implies via an easy induction that

$$\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, (\mathbb{Z}/p^m\mathbb{Z})[\![G]\!]_{\mathcal{C}}) = 0$$

for $1 \leqslant i \leqslant n$, and so $P_\bullet^{(m)}$ is exact up to dimension n. It also shows that $H_{n+1}(G; (\mathbb{Z}/p^m\mathbb{Z})[\![G]\!]_{\mathcal{C}}) \to H_{n+1}(G; (\mathbb{Z}/p^{m-1}\mathbb{Z})[\![G]\!]_{\mathcal{C}})$ is a surjection.

For every m we have an obvious chain map $P_\bullet^{(m+1)} \to P_\bullet^{(m)}$. Let $Q_\bullet = \varprojlim_m P_\bullet^{(m)}$ where the limit is taken along these maps. By [Wei94, Proposition 3.5.7 and Theorem 3.5.8], the complex $Q_\bullet$ is exact up to dimension n and by construction $Q_\bullet \cong \mathbb{Z}_p[\![G]\!]_{\mathcal{C}} \otimes_{\mathbb{Z}G} P_\bullet$. Therefore $H_i(Q_\bullet) \cong \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}; \mathbb{Z}_p[\![G]\!]_{\mathcal{C}}) = 0$ for $1 \leqslant i \leqslant n$. $\square$

The next result is due to Jaikin-Zapirain; we have weakened the original assumption of type FP_∞ to FP_n. The proof goes through verbatim after substituting Proposition 4.5 for Jaikin-Zapirain's use of [KZ08, Theorem 2.5].

Proposition 4.6. [JZ20, Proposition 3.1] *Let G be a group of type $\mathsf{FP}_n(\mathbb{Z})$ and let $(F_\bullet, \partial_\bullet)$ be a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ which is finitely generated up to dimension n, and in which $F_0 = \mathbb{Z}G$. Then G is n-good if and only if the induced sequence*

$$\cdots \xrightarrow{\widehat{\partial_{n+1}}} \widehat{F_n} \xrightarrow{\widehat{\partial_n}} \cdots \xrightarrow{\widehat{\partial_2}} \widehat{F_1} \xrightarrow{\widehat{\partial_1}} \widehat{F_0} \xrightarrow{\widehat{\partial_0}} \widehat{\mathbb{Z}}$$

is exact up to dimension n, where $(\widehat{F}_\bullet, \widehat{\partial}_\bullet)$ is obtained from $(F_\bullet, \partial_\bullet)$ by tensoring with $\widehat{\mathbb{Z}}[\![G]\!]$ over $\mathbb{Z}G$.

We are next going to state a result of Liu. First we need to introduce some notation. Recall that

$$H_{n,R}^{\varphi,\alpha} = H_n\big(G; R(Q \times Z)\big)$$

where $\alpha\colon G \to Q$ and $\varphi\colon G \to Z$ are homomorphisms, and $R(Q \times Z)$ is a right RG-module via $(q, z).g = (q\alpha(g), z\varphi(g))$ with $(g, q, z) \in G \times Q \times Z$. We also treat $R(Q \times Z)$ as an RZ module via the inclusion $Z \to Q \times Z$.

Now suppose that $Z \in \{\mathbb{Z}, \widehat{\mathbb{Z}}\}$, so that $\widehat{Z} = \widehat{\mathbb{Z}}$. Let $\widehat{G}$ be the profinite completion of G, and let $\widehat{\alpha}\colon \widehat{G} \to Q$ and $\widehat{\varphi}\colon \widehat{G} \to \widehat{\mathbb{Z}}$ be the completions of the morphisms from before. Note that $Q = \widehat{Q}$ since Q is finite. Let $R = \mathbb{F}$ be a finite field. We let

$$\widehat{H}^{\widehat{\varphi},\widehat{\alpha}}_{n,\mathbb{F}} = H^{\mathrm{prof}}_n(\widehat{G}; \mathbb{F}[\![Q \times \widehat{\mathbb{Z}}]\!])$$

where H^{prof}_* denotes *profinite homology*, as defined in [RZ10]. Observe that $\mathbb{F}[\![Q \times \widehat{\mathbb{Z}}]\!] = \mathbb{F}Q[\![\widehat{\mathbb{Z}}]\!]$ has a structure of an $\mathbb{F}[\![\widehat{\mathbb{Z}}]\!]$ module, and hence so does $\widehat{H}^{\widehat{\varphi},\widehat{\alpha}}_{n,\mathbb{F}}$.

Proposition 4.7. [Liu20, Proposition 4.6] *Let G be a group which is n-good and of type $\mathsf{FP}_n(\mathbb{Z})$. Let $\mathbb{F}$ be a finite field. Let $\alpha\colon G \twoheadrightarrow Q$ be a finite quotient of G. Denote by $\widehat{\alpha}\colon \widehat{G} \twoheadrightarrow Q$ the completion of α.*

1. *Let $\varphi\colon G \to \widehat{\mathbb{Z}}$ be a group homomorphism, and let $\widehat{\varphi}\colon \widehat{G} \to \widehat{\mathbb{Z}}$ denote its completion. If the annihilator of $H^{\varphi,\alpha}_{n,\mathbb{F}}$ in $\mathbb{F}\widehat{\mathbb{Z}}$ is non-zero, then the annihilator of $\widehat{H}^{\widehat{\varphi},\widehat{\alpha}}_{n,\mathbb{F}}$ is non-zero in $\mathbb{F}[\![\widehat{\mathbb{Z}}]\!]$.*
2. *Let $\varphi, \psi\colon G \to \widehat{\mathbb{Z}}$ be group homomorphisms and suppose that* $\ker(\psi)$ *contains* $\ker(\varphi)$. *If $H^{\psi,\alpha}_{n,\mathbb{F}}$ has a non-zero annihilator in $\mathbb{F}\widehat{\mathbb{Z}}$, then $H^{\varphi,\alpha}_{n,\mathbb{F}}$ has a non-zero annihilator in $\mathbb{F}\widehat{\mathbb{Z}}$.*
3. *Let Γ be a profinite group, let $\Psi\colon \Gamma \to \widehat{G}$ be a continuous epimorphism and let $\psi\colon G \to \widehat{\mathbb{Z}}$ be a group homomorphism. Let $\widehat{\alpha}'$ and $\widehat{\psi}'$ denote the pullbacks $\widehat{\alpha} \circ \Psi$ and $\widehat{\psi} \circ \Psi$. If $\widehat{H}^{\widehat{\varphi}',\widehat{\alpha}'}_{n,\mathbb{F}}$ has a non-zero annihilator in $\mathbb{F}[\![\widehat{\mathbb{Z}}]\!]$, then $\widehat{H}^{\widehat{\varphi},\widehat{\alpha}}_{n,\mathbb{F}}$ has a non-zero annihilator in $\mathbb{F}[\![\widehat{\mathbb{Z}}]\!]$.*
4. *Let $\varphi\colon G \to \mathbb{Z}$ be a group homomorphism. The module $\widehat{H}^{\widehat{\varphi},\widehat{\alpha}}_{n,\mathbb{F}}$ has a non-zero annihilator in $\mathbb{F}[\![\widehat{\mathbb{Z}}]\!]$ if and only if $H^{\varphi,\alpha}_{n,\mathbb{F}}$ has finite dimension over $\mathbb{F}$.*

Note that we have weakened the hypotheses 'cohomologically good and type FP_∞' in [Liu20, Proposition 4.6] to 'n-good and type FP_n'. To make the adjustment we simply substitute the use of [JZ20, Proposition 3.1] in the proof of [Liu20, Proposition 4.6] with Proposition 4.6.

Definition 4.8. Let H_A and H_B be a pair of finitely generated $\mathbb{Z}$-modules. Let $\Phi\colon \widehat{H_A} \to \widehat{H_B}$ be a continuous homomorphism of the profinite completions. We define the *matrix coefficient module*

$$\mathrm{MC}(\Phi; H_A, H_B)$$

(or simply $\mathrm{MC}(\Phi)$ if there is no chance of confusion) for Φ with respect to H_A and H_B to be the smallest $\mathbb{Z}$-submodule L of $\widehat{\mathbb{Z}}$ such that $\Phi(H_A)$ lies in the submodule $H_B \otimes_{\mathbb{Z}} L$ of $\widehat{H_B}$. We denote by

$$\Phi^{\mathrm{MC}}\colon H_A \to H_B \otimes_{\mathbb{Z}} \mathrm{MC}(\Phi)$$

the homomorphism uniquely determined by the restriction of Φ to H_A.

By [Liu20, Proposition 3.2(1)], the $\mathbb{Z}$-module $\mathrm{MC}(\Phi; H_A, H_B)$ is a non-zero finitely generated free $\mathbb{Z}$-module.

Definition 4.9. We define $\epsilon \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{MC}(\Phi), \mathbb{Z})$ by picking a free generating set for $\mathrm{MC}(\Phi)$ and sending every generator to either 0 or 1 in such a way that following ϵ with the natural projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ coincides with the restriction of the natural projection $\widehat{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$ applied to $\mathrm{MC}(\Phi)$. The definition of ϵ depends on the choice of a basis for $\mathrm{MC}(\Phi)$.

The *ϵ-specialisation* of Φ refers to the composite homomorphism

$$H_A \xrightarrow{\Phi^{\mathrm{MC}}} H_B \otimes_{\mathbb{Z}} \mathrm{MC}(\Phi) \xrightarrow{1\otimes\epsilon} H_B \otimes_{\mathbb{Z}} \mathbb{Z} = H_B,$$

denoted by $\Phi_\epsilon\colon H_A \to H_B$. The *dual ϵ-specialisation* of Φ refers to the homomorphism $\Phi^\epsilon\colon \mathrm{Hom}_{\mathbb{Z}}(H_B, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_A, \mathbb{Z})$ precomposing with Φ_ϵ.

Lemma 4.10. *If Φ is an isomorphism, then the images of Φ_ϵ and Φ^ϵ are of finite index in their respective codomains.*

Proof. Let b denote the rank of H_B. We have a natural epimorphism $\rho\colon H_B \to (\mathbb{Z}/2\mathbb{Z})^b$ that extends to $\widehat{\rho}\colon H_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \to (\mathbb{Z}/2\mathbb{Z})^b$. By construction, $\rho \circ \Phi_\epsilon = \widehat{\rho} \circ \Phi$. Let us assume that Φ is an isomorphism. Since $\widehat{\rho}$ is clearly surjective, we conclude that $\rho \circ \Phi_\epsilon$ is surjective. Pick a basis of $(\mathbb{Z}/2\mathbb{Z})^b$, and lift it via ρ to a set $v_1, \ldots, v_b \in \mathrm{Im}\,\Phi_\epsilon$. Suppose that the elements $v_1, \ldots, v_b$ are $\mathbb{Z}$-linearly dependent. By removing the common factors of 2 from the coefficients, we may assume that we have

$$\sum_{i=1}^{b} \lambda_i v_i = 0$$

with $\lambda_i \in \mathbb{Z}$ and with at least one λ_i odd. Applying ρ to this formula contradicts the fact that we started with a basis for $(\mathbb{Z}/2\mathbb{Z})^b$. Hence $v_1, \ldots, v_b$ are $\mathbb{Z}$-linearly independent, and hence by tensoring with $\mathbb{Q}$ we see that $\operatorname{Im}\Phi_\epsilon$ is of finite index in H_B.

The result for Φ^ϵ follows immediately, since we have just shown that $\Phi_\epsilon \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}$ is surjective, and hence an isomorphism, since H_A and H_B have the same rank. □

Definition 4.11. Let G_A and G_B be finitely generated groups and let $\Psi\colon \widehat{G}_A \to \widehat{G}_B$ be an isomorphism of profinite completions. Let H_A and H_B be the torsion-free parts of the abelianisations of, respectively, G_A and G_B; let ab denote both of the free abelianisation maps. Note that Ψ induces $\Psi_1\colon \widehat{H_A} \to \widehat{H_B}$. Pick $\epsilon \in \operatorname{Hom}_{\mathbb{Z}}(\mathrm{MC}(\Psi_1), \mathbb{Z})$ as in Definition 4.9. Given $\varphi \in H^1(G_B;\mathbb{Z})$ we define

$$\psi = \Psi_1^\epsilon(\varphi \circ \mathrm{ab}^{-1}) \circ \mathrm{ab} \in H^1(G_A;\mathbb{Z})$$

to be the *ϵ-pullback* of φ.

Theorem 4.12. *Let n be a positive integer. Let G_A and G_B be n-good groups of type $\mathsf{FP}_n(\mathbb{Z})$, and suppose that G_B is in $\mathsf{TAP}_n(\mathbb{F})$, where $\mathbb{F}$ is a finite field. Let $\Psi\colon \widehat{G}_A \to \widehat{G}_B$ be an isomorphism of profinite completions and let $\varphi \in H^1(G_B;\mathbb{Z})$. If for every $i \leqslant n$ the ϵ-pullback $\psi \in H^1(G_A;\mathbb{Z})$ of φ has non-vanishing ith twisted Alexander polynomials over $\mathbb{F}$, then φ is $\mathsf{FP}_n(\mathbb{F})$-semi-fibred.*

Proof. Note that Ψ is continuous by the work of Nikolov–Segal [NS07a, NS07b]. Let $\widehat{\rho}\colon G_A \to \widehat{\mathbb{Z}}$ denote the composite

$$G_A \rightarrowtail \widehat{G}_A \xrightarrow{\Psi} \widehat{G}_B \xrightarrow{\widehat{\varphi}} \widehat{\mathbb{Z}},$$

where $\widehat{\varphi}$ is the completion of φ. Observe that $\mathrm{Ker}(\psi)$ contains $\mathrm{Ker}(\hat{\rho})$. Indeed, $\hat{\rho}$ factorises as the top composite and ψ as the bottom composite

$$\begin{tikzcd}
 & & & & & \widehat{\mathbb{Z}} \\
G_A \arrow[r] & H_A \arrow[r, "\Psi_1^{\mathrm{MC}}"] & H_B \otimes_{\mathbb{Z}} \mathrm{MC}(\Psi_1) \arrow[r, "\varphi\otimes 1"] & \mathbb{Z} \otimes_{\mathbb{Z}} \mathrm{MC}(\Psi_1) \arrow[r, "="] & \mathrm{MC}(\Psi_1) \arrow[ur, tail] \arrow[dr, "\epsilon"] & \\
 & & & & & \mathbb{Z},
\end{tikzcd}$$

so clearly ψ vanishes on everything $\widehat{\rho}$ vanishes on.

Let $\beta\colon G_B \twoheadrightarrow Q$ be a finite quotient with completion $\widehat{\beta}$, and let $\alpha\colon G_A \twoheadrightarrow Q$ denote the composite $G_A \rightarrowtail \widehat{G}_A \xrightarrow{\Psi} \widehat{G}_B \xrightarrow{\widehat{\beta}} Q$. Let $i \leqslant n$. By assumption, the homology group $H^{\psi,\alpha}_{i,\mathbb{F}}$ is $\mathbb{F}\mathbb{Z}$-torsion, and hence

$$0 = \mathrm{Frac}(\mathbb{F}\mathbb{Z}) \otimes_{\mathbb{F}\mathbb{Z}} H^{\psi,\alpha}_{i,\mathbb{F}} = H_i(G_A; \mathrm{Frac}(\mathbb{F}\mathbb{Z})Q)$$

for $i \leqslant n$, where the second equality comes from the fact that localisations are flat, and that $\mathrm{Frac}(\mathbb{F}\mathbb{Z})Q$ is the localisation of $\mathbb{F}(\mathbb{Z} \times Q)$ at $\mathbb{F}(\mathbb{Z} \times \{1\}) \smallsetminus \{0\}$.

Since G_A is of type $\mathsf{FP}_n(\mathbb{Z})$, we find a free resolution $C_\bullet$ of $\mathbb{Z}$ with C_i finitely generated; let $\partial_i\colon C_i \to C_{i-1}$ denote the differentials of $C_\bullet$. The acyclicity above allows us to construct $\mathrm{Frac}(\mathbb{F}\mathbb{Z})Q$-module maps

$$d_i\colon \mathrm{Frac}(\mathbb{F}\mathbb{Z})Q \otimes_{\mathbb{Z}G} C_i \to \mathrm{Frac}(\mathbb{F}\mathbb{Z})Q \otimes_{\mathbb{Z}G} C_{i+1}$$

for $i \leqslant n$ with

$$d_{i-1} \circ \partial_i + \partial_{i+1} \circ d_i = \mathrm{id},$$

where we now view ∂_i as $\mathrm{id}_{\mathrm{Frac}(\mathbb{F}\mathbb{Z})Q} \otimes \partial_i$. Since the modules

$$\mathrm{Frac}(\mathbb{F}\mathbb{Z})Q \otimes_{\mathbb{Z}G} C_i$$

are finitely generated, by multiplying the maps d_i by the common denominator of all the entries of the matrices representing the maps d_i, we arrive at the existence of $\mathbb{F}(\mathbb{Z} \times Q)$-module maps

$$d_i'\colon \mathbb{F}(\mathbb{Z} \times Q) \otimes_{\mathbb{Z}G} C_i \to \mathbb{F}(\mathbb{Z} \times Q) \otimes_{\mathbb{Z}G} C_{i+1}$$

with

$$d_{i-1}' \circ \partial_i + \partial_{i+1} \circ d_i'$$

being equal to the right-multiplication by some

$$z \in \mathbb{F}(\mathbb{Z} \times \{1\}) \smallsetminus \{0\}.$$

Again, we have to interpret the differentials ∂_i in a suitable way. Crucially, since $\mathbb{F}\mathbb{Z}$ is central in $\mathbb{F}(\mathbb{Z}\times Q)$, right-multiplication by z coincides with left-multiplication by z.

Let $\psi' \colon G_A \to \widehat{\mathbb{Z}}$ denote ψ followed by the natural embedding $\mathbb{Z} \to \widehat{\mathbb{Z}}$. The maps d_i' can be easily extended to maps

$$\mathbb{F}(\widehat{\mathbb{Z}} \times Q) \otimes_{\mathbb{Z}G} C_i \to \mathbb{F}(\widehat{\mathbb{Z}} \times Q) \otimes_{\mathbb{Z}G} C_{i+1}$$

immediately yielding that $H_{i,\mathbb{F}}^{\psi',\alpha}$ is $\mathbb{F}\mathbb{Z}$-torsion, and hence $\mathbb{F}\widehat{\mathbb{Z}}$-torsion. Still, $\ker(\widehat{\rho}) \leqslant \ker(\psi')$. Applying Proposition 4.7(2), (1), (3), and (4) in the given order, we see that $H_{i,\mathbb{F}}^{\varphi,\beta}$ is a finite dimensional $\mathbb{F}$-module, and hence a torsion $\mathbb{F}\mathbb{Z}$-module. Since β was arbitrary and $G_B \in \mathsf{TAP}_n(\mathbb{F})$, it follows that φ is $\mathsf{FP}_n(\mathbb{F})$-semi-fibred. $\square$

Corollary 4.13. *Let n be a positive integer. Let G_A and G_B be n-good groups of type $\mathsf{FP}_n(\mathbb{Z})$ with isomorphic profinite completions. Suppose that G_A lies in $\mathsf{TAP}_n(\mathbb{F})$, where $\mathbb{F}$ is a finite field. The group G_A is $\mathsf{FP}_n(\mathbb{F})$-semi-fibred if G_B is.*

Proof. Let $\psi \colon G_B \to \mathbb{Z}$ be a non-trivial $\mathsf{FP}_n(\mathbb{F})$-semi-fibred character; observe that this statement remains unchanged if we replace ψ by a positive scalar multiple. By Proposition 2.11, the twisted Alexander polynomials of ψ over $\mathbb{F}$ do not vanish. Lemma 4.10 gives us a bijection between positive scalar multiples of characters in $H^1(G_A;\mathbb{Z})$ and $H^1(G_B;\mathbb{Z})$, and hence, in particular, we find a non-trivial character $\varphi \colon G_A \to \mathbb{Z}$ such that ψ is its ϵ-pullback (up to multiplication by a positive scalar). Theorem 4.12 shows that φ is $\mathsf{FP}_n(\mathbb{F})$-semi-fibred. $\square$

We may summarise the above by saying that being $\mathsf{FP}_n(\mathbb{F})$-semi-fibred is a profinite property among n-good groups of type $\mathsf{FP}_n(\mathbb{Z})$ in $\mathsf{TAP}_n(\mathbb{F})$.

Using Remark 4.3 we obtain the following crisper formulation for $n = 1$.

Corollary 4.14. *Let G_A and G_B be finitely generated groups with isomorphic profinite completions. Suppose that G_A lies in $\mathsf{TAP}_1(\mathbb{F})$, where $\mathbb{F}$ is a finite field. If G_B is algebraically semi-fibred, then so is G_A.*

5. Applications

5.1. Products of LERF groups.

Theorem 5.1. *Let G_A and G_B be groups such that all of the following hold:*

- *G_A is finitely generated;*
- *G_B is a product of LERF groups and is of type $\mathsf{FP}_2(R)$ for some ring R;*
- *there is an isomorphism $\widehat{G_B} \to \widehat{G_A}$.*

If G_A is algebraically semi-fibred, then G_B is algebraically fibred.

Proof. The group G_B is in $\mathsf{TAP}_1(\mathbb{F})$ for every finite field $\mathbb{F}$ by Theorem 3.8 and Proposition 3.12 – we are also using the fact that each of the factors of G_B is itself of type $\mathsf{FP}_2(R)$, which is easy to see. Now we use Corollary 4.14 and see that G_B is algebraically semi-fibred. But the first BNS invariant of LERF groups is symmetric by Remark 3.9. It follows from Meinert's inequality that products of LERF groups also have symmetric first BNS invariant, and hence that G_B is algebraically fibred. □

The following is restating Theorem C from the introduction.

Theorem 5.2. *Let $\mathbb{F}$ be a finite field. Let G_A and G_B be profinitely isomorphic finite products of limit groups. The group G_A is $\mathsf{FP}_n(\mathbb{F})$-semi-fibred if and only if G_B is.*

Proof. By Theorem 3.13, finite products of limit groups are $\mathsf{TAP}_\infty(\mathbb{F})$; they are also of type F, as mentioned before. The result now follows from Corollary 4.13. Indeed, limits groups are cohomologically good by [GJZZ08, Theorem 1.3] and so products of them are cohomologically good by [Lor08, Theorem 2.5]. □

5.2. Poincaré duality groups.

We now turn our attention to PD_3-groups, that is, Poincaré duality groups in dimension 3. For an introduction to this topic, see [Hil20b].

Theorem 5.3. *Let G_A be a PD_3-group in $\mathsf{TAP}_1(\mathbb{F})$ for some finite field $\mathbb{F}$. Let G_B be a finitely generated algebraically fibred group. If $\widehat{G_A} \cong \widehat{G_B}$, then G_A is the fundamental group of a closed connected 3-manifold.*

Proof. By [Hil20a, Theorem 5] and Proposition 2.6 we have that $\Sigma^1(G_A) = -\Sigma^1(G_A)$. By Corollary 4.14, G_A is algebraically fibred. Hence,

$$G_A = K \rtimes \mathbb{Z}$$

for some finitely generated subgroup K. It follows from a result of Strebel [Str77] (see [Hil02, Theorem 1.19] for an explanation), that K has cohomological dimension at most 2 and hence is a PD_2-group. In particular, by [EM80] (see also [KK21]) the group K is isomorphic to the fundamental group of a closed surface. Since every outer automorphism of K is realised by a mapping class of the underlying surface by the Dehn–Nielsen–Baer theorem, we conclude that G_A is the fundamental group of a closed connected 3-manifold. □

The following is restating Theorem D from the introduction.

Corollary 5.4. *Let G_A be a LERF PD_3-group. Let G_B be the fundamental group of a closed connected hyperbolic 3-manifold. If $\widehat{G_A} \cong \widehat{G_B}$, then G_A is the fundamental group of a closed connected hyperbolic 3-manifold.*

Proof. By [Hil20a, Theorem 5] and Proposition 2.6, for every finite index subgroup $G'_A \leqslant G_A$ we have that $\Sigma^1(G'_A) = -\Sigma^1(G'_A)$. Let H_B be a finite index subgroup of G_B that is algebraically fibred – the existence of such a subgroup is guaranteed by Agol's theorem [Ago13]. Let H_A be the corresponding finite index subgroup of G_A; we still have $\widehat{H_A} \cong \widehat{H_B}$. The group H_A is still a PD_3-group by [JW72, Theorem 2]. It is immediate that H_A is LERF. Since all PD_3-groups are of type $\mathsf{FP}(\mathbb{Z})$, we conclude, using Theorem 3.8, that H_A is $\mathsf{TAP}_1(\mathbb{F})$ for every finite field. Theorem 5.3 now shows that H_A is the fundamental group of a connected compact 3-manifold. By [Hil20b, Lemma 8.2], the group G_A is also a fundamental group of a connected compact 3-manifold M. The manifold M is hyperbolic by [WZ17]. □

5.3. **RFRS groups and agrarian Betti numbers.** The following definition is due to Agol [Ago08] and played a crucial role in solving the Virtual Fibring Conjecture for hyperbolic 3-manifolds.

Definition 5.5. Let G be a group. We say that G is *residually finite rationally solvable (RFRS)* if

(1) there is a chain of finite index normal subgroups

$$G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \cdots$$

of G such that $\bigcap_{\mathbb{N}} G_i = \{1\}$;

(2) $\ker\big(G_i \to H_1(G_i;\mathbb{Q})\big) \leqslant G_{i+1}$ for $i \geqslant 0$.

Definition 5.6. A group G is *indicable* if G is trivial or admits an epimorphism to $\mathbb{Z}$. We say that G is *locally indicable* if every finitely generated subgroup of G is indicable.

Note that RFRS groups are locally indicable.

Definition 5.7. Let R and $\mathcal{D}$ be skew-fields, let G be a locally indicable group, and let $\psi\colon RG \to \mathcal{D}$ be a ring homomorphism. The pair $(\mathcal{D},\psi)$ is *Hughes-free* if

(1) $\mathcal{D}$ is generated by $\psi(RG)$ as a skew-field, that is, $\langle \psi(RG)\rangle = \mathcal{D}$;

(2) for every finitely generated subgroup $H \leqslant G$, every normal subgroup $N \lhd H$ with $H/N \cong \mathbb{Z}$, and every set of elements $h_1, \ldots, h_n \in H$ lying in distinct cosets of N, the sum

$$\langle \psi(RN)\rangle \cdot \psi(h_1) + \cdots + \langle \psi(RN)\rangle \cdot \psi(h_n)$$

is direct.

By [Hug70], if such a pair $(\mathcal{D},\psi)$ exists, then $\mathcal{D}$ is unique up to RG-algebra isomorphism. In this case we denote $\mathcal{D}$ by $\mathcal{D}_{RG}$.

(Like the property, the Hughes mentioned here and the first author are free of any of relation.)

The following result is due to Jaikin-Zapirain.

Proposition 5.8. [JZ21, Corollary 1.3] *If G is a RFRS group and R is a skew-field, then $\mathcal{D}_{RG}$ exists and it is the universal division ring of fractions of RG.*

Definition 5.9. A group G is *agrarian* over a ring R if there exists a skew-field $\mathcal{D}$ and a monomorphism $\psi\colon RG \rightarrowtail \mathcal{D}$ of rings. If G is agrarian over R, then we define the *agrarian $\mathcal{D}$-homology* to be

$$H_p^{\mathcal{D}}(G) = \operatorname{Tor}_p^{RG}(R,\mathcal{D})$$

where R is the trivial RG-module and $\mathcal{D}$ is viewed as an $\mathcal{D}$-RG-bimodule via the embedding $RG \rightarrowtail \mathcal{D}$. Since modules over a skew-field have a canonical dimension function taking values in $\mathbb{N} \cup \{\infty\}$ we may define

$$b_p^{\mathcal{D}}(G) = \dim_{\mathcal{D}} H_p^{\mathcal{D}}(G).$$

When G is RFRS, by the previous proposition, we have (up to RG-isomorphism) a canonical choice $\mathcal{D}_{RG}$ of $\mathcal{D}$ for each skew-field R.

Theorem 5.10. *Let R be a skew-field and let $n \in \mathbb{N}$. Let G be a virtually RFRS group of type $\mathsf{FP}_n(R)$. The following are equivalent:*

(1) $b_p^{\mathcal{D}_{RG}}(G) = 0$ for all $p \leqslant n$;
(2) G is virtually $\mathsf{FP}_n(R)$-fibred;
(3) G is virtually $\mathsf{FP}_n(R)$-semi-fibred.

Proof. The equivalence of the first two items is [Fis21, Theorem 6.6]. The implication (2) $\Rightarrow$ (3) is clear, so let us prove (3) $\Rightarrow$ (1).

By [Fis21, Lemma 6.3], the numbers $b_p^{\mathcal{D}_{RG}}(G)$ scale with the index when passing to finite-index subgroups. Thus, we may assume without loss of generality that G itself is $\mathsf{FP}_n(R)$-semi-fibred. In particular, let $\varphi \in \Sigma^n(G; R)$ witness this semi-fibration. By [Fis21, Lemma 5.3], we have

$$\operatorname{Tor}_i^{RG}(R, \operatorname{Nov}(RG, \varphi)) = 0$$

for all $0 \leqslant i \leqslant n$.

Let $\mathbb{K}$ be the skew-field of twisted Laurent series with variable t and coefficients in the skew-field $\mathcal{D}_{R(\ker \varphi)}$; the variable t is an element of G with $\varphi(t) = 1$, a generator of $\mathbb{Z}$, and the twisting extends the conjugation action of t on $\ker \varphi$ to $\mathcal{D}_{R(\ker \varphi)}$ – such an extension is possible since $\mathcal{D}_{R(\ker \varphi)}$ is Hughes free, see [JZ21] for an explanation of this fact. The skew-field $\mathbb{K}$ contains $\operatorname{Nov}(RG, \varphi)$, since the latter can also be viewed as a ring of twisted Laurent series in t with coefficients in $R(\ker \varphi)$, with the twisting described above. Hence, using chain contractions, we see that

$$\operatorname{Tor}_i^{RG}(R, \mathbb{K}) = 0$$

for all $0 \leqslant i \leqslant n$.

Now, Hughes-freeness of $\mathcal{D}_{RG}$ tells us that it is isomorphic as an RG-module to the division closure of the twisted Laurent polynomial ring $R(\ker \varphi)[t^{\pm 1}]$ in $\mathbb{K}$, where we identify the rings $R(\ker \varphi)[t^{\pm 1}]$ and RG using the group isomorphism $(\ker \varphi) \rtimes \mathbb{Z} = G$. This endows $R(\ker \varphi)[t^{\pm 1}]$

with an RG-bimodule structure. Hence, we may view $\mathcal{D}_{RG}$ as a subring of $\mathbb{K}$, and view $\mathbb{K}$ as a $\mathcal{D}_{RG}$-module. Since both rings are skew-fields, the module is flat. We conclude that

$$\operatorname{Tor}_i^{RG}(R, \mathcal{D}_{RG}) = 0$$

for all $0 \leqslant i \leqslant n$, as claimed. □

Theorem 5.11. *Let $n \in \mathbb{N} \cup \{\infty\}$, and let $\mathbb{F}$ be a finite field. Let G_A and G_B be n-good virtually RFRS groups of type $\mathsf{FP}_n(\mathbb{F})$ and suppose that $\widehat{G}_A \cong \widehat{G}_B$. Suppose that every finite index subgroup of G_A and G_B is in $\mathsf{TAP}_n(\mathbb{F})$. We have*

$$\min\{j \leqslant n \mid b_j^{\mathcal{D}_{\mathbb{F}G_A}}(G_A) \neq 0\} = \min\{j \leqslant n \mid b_j^{\mathcal{D}_{\mathbb{F}G_B}}(G_B) \neq 0\}$$

where we take the minimum of the empty set to be ∞.

Proof. We first assume that $n \in \mathbb{N}$. Since we are concerned with virtual properties we may assume without loss of generality that G_A and G_B are RFRS, n-good, of type $\mathsf{FP}_n(\mathbb{Z})$, and all finite-index subgroups of G_A and G_B are in $\mathsf{TAP}_n(\mathbb{F})$; we have used Proposition 4.4 here.

Suppose that $b_j^{\mathcal{D}_{\mathbb{F}G_A}}(G_A) = 0$ for $j \leqslant m$ for some $m \leqslant n$. The group G_A is virtually $\mathsf{FP}_m(\mathbb{F})$-fibred by Theorem 5.10. We may pass to further finite index subgroups of G_A and G_B and assume that G_A is $\mathsf{FP}_m(\mathbb{F})$-fibred. By Corollary 4.13, the group G_B is $\mathsf{FP}_m(\mathbb{F})$-semi-fibred, and hence

$$b_j^{\mathcal{D}_{\mathbb{F}G_B}}(G_B) = 0$$

for $j \leqslant m$ by Theorem 5.10. This shows an inequality between the minima in the statement. The argument is symmetric in G_A and G_B, and hence we also obtain the converse inequality.

Now suppose that $n = \infty$. If both of the minima in the statement are ∞, then we are done. Without loss of generality let us suppose that the left-hand side one is equal to $m < \infty$. We observe that G_A and G_B satisfy the hypothesis of our theorem for $n = m$, and hence the right-hand side minimum is also equal to m. □

Observe that the above result applies in particular to finite products of RFRS limit groups.

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