HYPERBOLICALLY EMBEDDED SUBGROUPS AND QUASI-ISOMETRIES OF PAIRS

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ABSTRACT. We give technical conditions for a quasi-isometry of pairs to preserve a subgroup being hyperbolically embedded. We consider applications to the quasi-isometry and commensurability invariance of acylindrical hyperbolicity of finitely generated groups.

1. INTRODUCTION

A group G is *acylindrically hyperbolic* if it admits a non-elementary, acylindrical action on a hyperbolic space. An alternative characterisation is that G is acylindrically hyperbolic if and only if G contains a *hyperbolically embedded subgroup* H, denoted $H \hookrightarrow_h G$, we will give a characterisation from [MR21] in Proposition 3.5.

The class of acylindrically hyperbolic groups generalises the classes of non-elementary hyperbolic and relatively hyperbolic groups whilst sharing many similar properties [Osi16]. In spite of this there are still foundational questions that remain open, for instance, it is known that a group being hyperbolic or relatively hyperbolic is invariant under quasi-isometry [Gro87] [Dru09], but the corresponding question for acylindrical hyperbolicity is still open.

Question 1.1. [Osi18, Question 2.20(a)] Is the class of finitely generated acylindrically hyperbolic groups closed under quasi-isometry?

Some partial results are known, for instance acylindrical hyperbolicity passes to finite-index subgroups and is preserved by quotienting out a finite normal subgroup [MO15]. If the group is $\mathcal{AH}$-accessible then acylindrical hyperbolicity can be passed to finite extensions [MO19]. The property of being $\mathcal{AH}$-accessible also passes to finite-index overgroups [Bal20]. However, not every finitely presented acylindrically hyperbolic group is $\mathcal{AH}$-accessible [ABO19, Theorem 2.18]. Some experts in the field do not expect a complete positive answer to Question 1.1.

This article relies on the notion of quasi-isometry of pairs, and our results provide technical conditions to ensure a quasi-isometry of pairs carries the property of being a hyperbolically embedded subgroup.

Definition 1.2 (Quasi-isometry of pairs)**.** Consider two pairs $(G, \mathcal{P})$ and $(H, \mathcal{Q})$ where G and H are finitely generated groups with chosen word metrics dist_G and dist_H. Denote the Hausdorff

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distance between subsets of H by hdist_H. An (L,C)-quasi-isometry $q\colon G \to H$ is an (L,C,M)-*quasi-isometry of pairs* $q\colon (G,\mathcal{P}) \to (H,\mathcal{Q})$ if the relation

$$\dot{q} = \{(A,B) \in G/\mathcal{P} \times H/\mathcal{Q}\colon\ \mathsf{hdist}_H(q(A),B) < M\}$$

satisfies that the projections into $G/\mathcal{P}$ and $H/\mathcal{Q}$ are surjective.

Example 1.3 (Quasi-isometry of pairs and finite extensions)**.** Let H be a finite index normal subgroup of finitely generated group G, and let $\mathcal{Q}$ be a finite collection of subgroups of H. Then the inclusion $(H,\mathcal{Q}) \hookrightarrow (G,\mathcal{Q})$ is a quasi-isometry of pairs if the collection $\{hQh^{-1}\colon h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by G, see Proposition 4.1.

Recall that the *commensurator* of a subgroup P of a group G is the subgroup

$$Comm_G(P) = \{g \in G\colon P \cap gPg^{-1} \text{ is a finite index subgroup of } P \text{ and } gPg^{-1}\}.$$

Definition 1.4 (Refinements)**.** Let $\mathcal{P}$ be a collection of subgroups of group G. A *refinement* $\mathcal{P}^*$ of $\mathcal{P}$ is a set of representatives of conjugacy classes of the collection of subgroups

$$\{\mathrm{Comm}_G(gPg^{-1})\colon P \in \mathcal{P} \text{ and } g \in G\}.$$

Example 1.5 (Refinements and qi of pairs)**.** Let $\mathcal{Q}$ be a finite collection of subgroups of a finitely generated group H and let $\mathcal{Q}^*$ be a refinement. If each $Q \in \mathcal{Q}$ is finite index in $\mathrm{Comm}_H(Q)$ then the identity map on G is a quasi-isometry of pairs $(H,\mathcal{Q}) \to (H,\mathcal{Q}^*)$.

Example 1.6 (Refinements and finite extensions)**.** Let A be a group, let $\mathcal{H}$ be an almost malnormal collection of infinite subgroups, and let $F \leqslant \mathsf{Aut}(A)$ be a finite subgroup. If F acts freely on $\mathcal{H}$ and $\mathcal{H}_F$ is a collection of representatives of F-orbits in $\mathcal{H}$, then a refinement of $\mathcal{H}$ in $A \rtimes F$ is $\mathcal{H}_F$.

Definition 1.7 (Reduced collections)**.** A collection of subgroups $\mathcal{P}$ of a group G is *reduced* if for any $P,Q \in \mathcal{P}$ and $g \in G$, if P and gQg^{-1} are commensurable then $P = Q$ and $g \in P$.

Our first result, Theorem A, describes a strategy to obtain positive results to Question 1.1. For a group G with a generating set S, let $\Gamma(G,S)$ denote the corresponding *Cayley graph*, see Definition 2.4.

Theorem A (Theorem 3.11)**.** *Let $q\colon G \to H$ be a quasi-isometry of finitely generated groups, let $\mathcal{P}$ and $\mathcal{Q}$ be finite collections of subgroups of G and H respectively, and let S and T be (not necessarily finite) generating sets of G and H respectively. Suppose*

(1) $q\colon (G,\mathcal{P}) \to (H,\mathcal{Q})$ is a quasi-isometry of pairs, and
(2) $q\colon \Gamma(G,S) \to \Gamma(H,T)$ is a quasi-isometry.

The following statements hold:

(1) If $\mathcal{P}$ and $\mathcal{Q}$ are reduced collections in G and H respectively; then $\mathcal{P} \hookrightarrow_h (G,S)$ if and only if $\mathcal{Q} \hookrightarrow_h (G,T)$.
(2) If $\mathcal{Q}$ contains only infinite subgroups and $\mathcal{Q} \hookrightarrow_h (G,T)$ then $\mathcal{P}^* \hookrightarrow_h (G,S)$.

Qi-characteristic collections. The first numbered hypothesis of Theorem A raises the following problem: Given a finite collection of subgroups $\mathcal{Q}$ of a group H and a quasi-isometry

$q\colon G \to H$ of finitely generated groups, is there a collection $\mathcal{P}$ of subgroups of G such that $q\colon (G,\mathcal{P}) \to (H,\mathcal{Q})$ is a quasi-isometry of pairs?

This problem was studied in [MS21] where the notion of qi-characteristic collection is introduced and it is proved that if the collection $\mathcal{Q}$ is qi-characteristic in H, then any quasi-isometry of finitely generated groups induces a collection $\mathcal{P}$.

Definition 1.8 (Qi-characteristic)**.** [MS21] A collection of subgroups $\mathcal{P}$ of a finitely generated group G is *quasi-isometrically characteristic* (or shorter *qi-characteristic*) if $\mathcal{P}$ is finite; each $P \in \mathcal{P}$ has finite index in its commensurator; and for every $L \geqslant 1$ and $C \geqslant 0$ there is $M = M(G,\mathcal{P},L,C) \geqslant 0$ such that every (L,C)-quasi-isometry $q\colon G \to G$ is an (L,C,M)-quasi-isometry of pairs $q\colon (G,\mathcal{P}) \to (G,\mathcal{P})$.

Example 1.9. The argument by Behrstock, Druţu and Mosher proving quasi-isometric rigidity of relative hyperbolicity with respect to non-relatively hyperbolic groups (NRH groups) shows that if H is hyperbolic group relative to a collection $\mathcal{Q}$ of NRH subgroups, then $\mathcal{Q}$ is qi-characteristic [BDM09, Theorems 4.1 and 4.8]. Another example is provided by mapping class groups. Ruling out a few surfaces of low complexity, any self quasi-isometry of the mapping class group is at uniform distance from left multiplication by an element of the group, see the work of Behrstock, Kleiner, Minsky and Mosher [Beh+12, Theorem 1.1]. As a consequence, the hyperbolically embedded (virtually cyclic) subgroup generated by a pseudo-Anosov is qi-characteristic.

Corollary B. *Let G and H be finitely generated groups, let T be a generating set of H, let $\mathcal{Q}$ be a finite collection of subgroups of H such that $\mathcal{Q} \hookrightarrow_h (H,T)$, and let $q\colon G \to H$ be a quasi-isometry. If*

(1) $\mathcal{Q}$ is a qi-characteristic collection of subgroups of H, and
(2) there is a generating set $S \subset G$ such that $q\colon \Gamma(G,S) \to \Gamma(H,T)$ is a quasi-isometry;

then there is a finite collection $\mathcal{P}$ of subgroups of G such that $\mathcal{P} \hookrightarrow_h (G,S)$ and $q\colon (G,\mathcal{P}) \to (H,\mathcal{Q})$ is a quasi-isometry of pairs.

Proof. Without loss of generality, assume that all subgroups in $\mathcal{Q}$ are proper infinite subgroups. Note that removing finite subgroups from $\mathcal{Q}$ preserves being qi-characteristic and that $\mathcal{Q} \hookrightarrow_h (H,T)$. On the other hand, if $\mathcal{Q}$ contains H, then the theorem is trivial by taking $\mathcal{P}$ the collection that contains only G and S any finite generating set of G. Since $\mathcal{Q}$ is qi-characteristic, the quasi-isometry $q\colon G \to H$ induces a finite collection $\mathcal{P}$ such that $q\colon (G,\mathcal{P}) \to (H,\mathcal{Q})$ is a quasi-isometry of pairs, this is precisely [MS21, Theorem 1.1]. Then the second statement of Theorem A and $\mathcal{Q} \hookrightarrow_h (H,T)$ imply that $\mathcal{P}^* \hookrightarrow_h (G,S)$. □

Uniform Quasi-actions. The second numbered hypothesis of Theorem A raises the problem: Given a group H with a generating set T and a quasi-isometry $q\colon G \to H$ of finitely generated groups, is there a generating set $S \subset G$ such that $q\colon \Gamma(G,S) \to \Gamma(H,T)$ is a quasi-isometry of Cayley graphs?

We show that a positive answer to this question is equivalent to asking that the quasi-action of G on H induced by q is T-uniform in the following sense, see Proposition C.

Definition 1.10 (Uniform induced quasi-action)**.** Let G and H be finitely generated groups and let $q\colon G \to H$ be a quasi-isometry with quasi-inverse $\bar{q}$. Let $T \subset H$ be a generating set (possibly infinite). We say that the quasi-action of G on H induced by q is *uniform with respect to* T if there are constants $L \geqslant 1$, $C \geqslant 0$ such that for each $g \in G$ the function $q_g\colon H \to H$ given by $q_g(h) = q(g \cdot \bar{q}(h))$ is an (L, C)-quasi-isometry $q_g\colon \Gamma(H, T) \to \Gamma(H, T)$.

Example 1.11 (Uniform quasi-action and finite extensions)**.** Let H be a finite index normal subgroup of finitely generated group G and let T be a generating set of H invariant under conjugation by G. The G-action by conjugation on H preserves the word metric induced by T. On the other hand, any transversal R of H in G induces a quasi-isometry $q\colon G \to H$ given by $q(hg) = h$ for $h \in H$ and $g \in R$. In this case the quasi-action of G on H induced by q is uniform with respect to T, see Lemma 2.8.

Proposition C (Proposition 2.5)**.** *Let G and H be groups with finite generating sets S_0 and T_0, and let $q\colon \Gamma(G, S_0) \to \Gamma(H, T_0)$ be a quasi-isometry. Let $T \subset H$ containing T_0. The following statements are equivalent:*

(1) The quasi-action of G on H induced by q is uniform with respect to T.
(2) There is $S \subset G$ containing S_0 such that $q\colon \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry.

Corollary D. *Let G and H be finitely generated groups with finite collections of infinite subgroups $\mathcal{P}$ and $\mathcal{Q}$ respectively. Suppose $q\colon (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs inducing a T-uniform quasi-action of G on H. If $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P}^* \hookrightarrow_h G$.*

Proof. Since the quasi-action of G on H induced by q is T-uniform, Proposition C implies that there is a generating set S of G such that $q\colon \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry. Then the second statement of Theorem A and $\mathcal{Q} \hookrightarrow_h (H, T)$ imply that $\mathcal{P}^* \hookrightarrow_h (G, S)$. □

Let us remark that for this last corollary, in the case that T is finite, then there is a finite $S \subset G$ such that $\mathcal{P} \hookrightarrow_h (G, S)$; this case is implied by the results on quasi-isometric rigidity of relative hyperbolicity in [BDM09].

Finite Extensions. The following application is a particular instance of Theorem 4.3 in the main body of the article.

Theorem E (Theorem 4.3)**.** *Let H be a finite index normal subgroup of a finitely generated group G, and let $\mathcal{Q}$ be a finite collection of infinite subgroups of H such that $\mathcal{Q} \hookrightarrow_h (H, T)$. Suppose:*

(1) The set T is invariant under conjugation by G.
(2) The collection $\{hQh^{-1} : h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by G.

If $\mathcal{Q}^*$ is a refinement of $\mathcal{Q}$ in G, then $\mathcal{Q}^* \hookrightarrow_h G$.

Example 1.12. Let $G = \langle a, b, t \colon tat^{-1} = b,\ t^2 = 1 \rangle \cong F_2 \rtimes \mathbb{Z}_2$, let $H = \langle a, b \rangle$, and let $\mathcal{Q} = \{\langle a \rangle, \langle b \rangle\}$. Note that $\mathcal{Q} \hookrightarrow_h H$, and, for instance one can take $\mathcal{Q}^* = \{\langle a \rangle\}$ and observe that $\mathcal{Q}^* \hookrightarrow_h G$. In contrast, for $\mathcal{Q}_0 = \{\langle a \rangle\} \hookrightarrow_h H$ the theorem does not apply since the conjugates of $\langle a \rangle$ in H are not invariant under conjugation by elements of G.

The next result illustrates concrete examples were Theorem E applies.

Theorem F (Theorem 5.2)**.** *Let A be a finitely generated group with a (not necessarily finite) generating set T, and let $\mathcal{H}$ be a finite collection of infinite subgroups such that $\mathcal{H} \hookrightarrow_h (A,T)$. If $F \leqslant \mathsf{Aut}(A)$ is finite, T and $\mathcal{H}$ are F-invariant, and the F-action on $\mathcal{H}$ is free, then $\mathcal{H}_F \hookrightarrow_h (A \rtimes F, T \cup F)$ where $\mathcal{H}_F$ is collection of representatives of F-orbits in $\mathcal{H}$.*

Example 1.13. Let $A = \ast_{i=1}^n B_i$ with each B_i isomorphic to a fixed finitely generated group B. Let $F = \mathbb{Z}_n$ act on A by cyclically permuting the copies of B. Consider the generating set of A given by $T = \bigcup_{i=1}^n B_i \backslash \{1\}$, then T is F-invariant. Now, the collection $\mathcal{H} = \{B_1, \dots, B_n\}$ is hyperbolically embedded into (A,T) and F acts freely by conjugation on $\mathcal{H}$. All of the hypotheses of the previous theorem have been verified so we conclude that $B_1 \hookrightarrow_h (A \rtimes F, T \cup F)$.

Organization. The rest of the article is divided into four sections. Section 2 is on quasi-actions, it contains the proof of Proposition C as well as some corollaries. The proof of Theorem A is the content of Section 3. Then Sections 4 and 5 contain the proofs of Theorem E and Theorem F respectively.

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2. Uniform quasi-actions

Definition 2.1 (Uniform quasi-action)**.** Let G be a group and let X be a metric space. Let $\mathsf{QI}(\mathsf{X})$ denote the set of quasi-isometries $X \to X$. A function $G \to \mathsf{QI}(\mathsf{X})$, $g \mapsto f_g$, is a *quasi-action* if there is $K \geqslant 0$ such that for any $g_1, g_2 \in G$

1. the map $f_{g_1 g_2}$ is at distance at most K from the map $f_{g_1} \circ f_{g_2}$ in the L_∞-distance, and
2. the map $f_{g_1} \circ f_{g_1^{-1}}$ is at distance at most K from the identity.

The quasi-action $G \to \mathsf{QI}(X)$ is *uniform* if there are constants $L \geqslant 1$ and $C \geqslant 0$ such that for any $g \in G$ the map f_g is an (L,C)-quasi-isometry.

It is well known that a quasi-isometry $q \colon G \to H$ of finitely generated groups induces a uniform quasi-action of G on H:

Definition 2.2 (Uniform quasi-action induced by a quasi-isometry)**.** Let G be a group with a word metric induced by a finite generating set, let X be a metric space, let $q \colon G \to X$ and $\bar{q} \colon X \to G$ be (L_0, C_0)-quasi-isometries such that $q \circ \bar{q}$ and $\bar{q} \circ q$ are at distance less than C_0 from the identity maps on X and G respectively. For $g \in G$, let

$$L_g \colon G \to G, \qquad x \mapsto gx;$$

and let

$$q_g \colon X \to X \qquad q_g = q \circ \mathbf{g} \circ \bar{q}.$$

It is an exercise to verify that there are constants $L \geqslant 1$ and $C \geqslant 0$ such that:

- For $g \in G$, $q_g \colon X \to X$ is an (L, C)-quasi-isometry.
- (G quasi-acts on X) For $g_1, g_2 \in G$, the map $q_{g_1 g_2}$ is at distance at most C from the map $q_{g_1} \circ q_{g_2}$; and the map $q_{g_1} \circ q_{g_1^{-1}}$ is at distance at most C from the identity.
- (G acts C_0-transitively on X) For every $x, y \in X$ there is $g \in G$ such that $\mathsf{dist}_G(x, q_g(y)) \leqslant C$.

The map $G \to \mathsf{QI}(X)$ given by $g \mapsto q_g$ is called the *uniform quasi-action of G on X induced by q and $\bar{q}$.*

Remark 2.3 (Equivalence of Definitions 2.2 and 1.10)**.** In the context of Definition 1.10, if the induced quasi-action of G on H is uniform with respect to T, then $G \to \mathsf{QI}(\Gamma(H, T))$ given by $g \mapsto q_q$ is a uniform quasi-action in the sense of Definition 2.2. Indeed, since T contains a finite generating set of H there is $M > 0$ such that $\mathsf{dist}_{(H,T)} \leqslant M \,\mathsf{dist}_{(H,T_0)}$. Hence if two functions $H \to H$ are at finite L_∞-distance with respect to $\mathsf{dist}_{(H,T_0)}$, then the same holds for $\mathsf{dist}_{(H,T)}$.

Definition 2.4 (Cayley Graph)**.** Let G be a group with a generating set S. The *Cayley graph* $\Gamma(G, S)$ *of G with respect to S* is the G-graph with vertex set G and edge set $\{\{g, gs\} \colon g \in G,\ s \in S\}$.

Proposition 2.5 (Proposition C)**.** *Let G and H be groups with finite generating sets S_0 and T_0, and let $q \colon \Gamma(G, S_0) \to \Gamma(H, T_0)$ be a quasi-isometry. Let $T \subset H$ containing T_0. The following statements are equivalent:*

(1) The quasi-action of G on H induced by q is uniform with respect to T.
(2) There is $S \subset G$ containing S_0 such that $q \colon \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry.

Proof. That the second statement implies the first one is immediate. Conversely, suppose that q and $\bar{q}$ are (L_0, C_0)-quasi-isometries $\Gamma(G, S_0) \to \Gamma(H, T_0)$ and $\Gamma(H, T_0) \to \Gamma(G, S_0)$ respectively. Without loss of generality assume that $q(e) = e$ and $\bar{q}(e) = e$ where e denotes the identity in each corresponding group.

Let $K_0 = L_0 + C_0 + 1$ and define

$$S = \{f^{-1}g \in G \colon \text{ there are } h \in H \text{ and } t \in T \text{ such that } \mathsf{dist}_{(H,T_0)}(q(f), h) \leqslant K_0 \text{ and } \mathsf{dist}_{(H,T_0)}(q(g), ht) \leqslant K_0\}.$$

Note that $S_0 \subset S$ since $q \colon \Gamma(G, S_0) \to \Gamma(H, T_0)$ is an (L_0, C_0)-quasi-isometry. In particular, S is a generating set of G.

Let $L_1 \geqslant 1$ and $C_1 \geqslant 0$ be such that the G-action on H induced by q is (L_1, C_1)-uniform with respect to T. In particular, for every $g \in G$ the function $q_g \colon H \to H$ is an (L_1, C_1)-quasi-isometry $\Gamma(H, T) \to \Gamma(H, T)$.

Now, we prove that if the induced quasi-action of G on H is uniform with respect to T, then $q \colon \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry. Observe that every vertex of $\Gamma(H, T)$ is at distance at most C_0 from $q(G)$ with respect to $\mathsf{dist}_{(H,T_0)}$ and hence with respect to $\mathsf{dist}_{(H,T)}$. Below we prove inequalities (1) and (2) which will conclude proof.

Claim: *There is constant $\bar{L}$ such that*

$$\mathsf{dist}_{(H,T)}(q(a), q(b)) \leqslant \bar{L} \,\mathsf{dist}_{(G,S)}(a, b). \tag{1}$$

for any $a, b \in G$.

Proof of claim: Let $s \in S$. Then there are $f, g \in G$, $h \in H$ and $t \in T$ such that $s = f^{-1}g$ and

$$\mathsf{dist}_{(H,T_0)}(q(f), h) \leqslant K_0, \quad \mathsf{dist}_{(H,T_0)}(q(g), ht) \leqslant K_0.$$

It follows that

$$\mathsf{dist}_{(H,T)}(q_f(e), q_g(e)) = \mathsf{dist}_{(H,T)}(q(f), q(g)) \leqslant 2K_0 + 1.$$

Since the quasi-action of G on $\Gamma(H,T)$ is (L_1, C_1)-uniform, the previous inequality implies that

$$\begin{aligned}
\mathsf{dist}_{(H,T)}(e, q(s)) &= \mathsf{dist}_{(H,T)}(q_e(e), q_{f^{-1}g}(e)) \\
&\leqslant L_1 \,\mathsf{dist}_{(H,T)}(q_f \circ q_e(e), q_f \circ q_{f^{-1}g}(e)) + C_1 \\
&\leqslant L_1 \,\mathsf{dist}_{(H,T)}(q_f(e), q_g(e)) + 3C_1 \\
&\leqslant L_1(2K_0 + 1) + 3C_1 =: \bar{L}_0.
\end{aligned}$$

For any $g \in G$ and $s \in S$, we have that

$$\begin{aligned}
\mathsf{dist}_{(H,T)}(q(g), q(gs)) &= \mathsf{dist}_{(H,T)}(q_g(e), q_{gs}(e)) \\
&\leqslant L_1 \,\mathsf{dist}_{(H,T)}(q_{g^{-1}} \circ q_g(e), q_{g^{-1}} \circ q_{gs}(e)) + C_1 \\
&\leqslant L_1 \,\mathsf{dist}_{(H,T)}(e, q_{g^{-1}gs}(e)) + 3C_1 \\
&\leqslant L_1 \,\mathsf{dist}_{(H,T)}(q(e), q(s)) + 3C_1.
\end{aligned}$$

and hence

$$\mathsf{dist}_{(H,T)}(q(g), q(gs)) \leqslant \mathsf{dist}_{(H,T_0)}(q(g), q(gs)) \leqslant \bar{L}$$

where $\bar{L} = L_1(\bar{L}_0) + 3C_1$. If $a, b \in G$ and $[u_0, \ldots, u_\ell]$ is a geodesic in $\Gamma(G,S)$ from a to b, then the triangle inequality implies inequality (1). ◆

Claim: *For any $a, b \in G$ we have*

$$\mathsf{dist}_{(G,S)}(a, b) \leqslant \mathsf{dist}_{(H,T)}(q(a), q(b)). \tag{2}$$

Proof of claim: Suppose that $[h_0, \ldots, h_\ell]$ is a geodesic in $\Gamma(H,T)$ from $q(a)$ to $q(b)$. Since $q\colon \Gamma(G,S_0) \to \Gamma(H,T_0)$ is a (L_0,C_0)-quasi-isometry, for each i, there is $g_i \in G$ such that $\mathsf{dist}_{(H,T_0)}(q(g_i), h_i) \leqslant C_0$. Let $g_0 = a$ and $g_\ell = b$. Observe that $g_i^{-1}g_{i+1} \in S$ for $0 \leqslant i < \ell$, and hence $\mathsf{dist}_{(G,S)}(g_i, g_{i+1}) \leqslant 1$. Now, $[g_0, \ldots, g_\ell]$ is a path in $\Gamma(G,S)$ from a to b and therefore $\mathsf{dist}_{(G,S)}(a, b) \leqslant \mathsf{dist}_{(H,T)}(q(a), q(b))$ proving inequality (2). ◆ □

Corollary 2.6. *Let G and H be groups with finite generating sets S_0 and T_0. Let $q\colon G \to H$ be a group homomorphism which is also an (L_0,C_0)-quasi-isometry $q\colon \Gamma(G,S_0) \to \Gamma(H,T_0)$. If $T \subset H$ contains T_0, then there is $S \subset G$ containing S_0 such that $q\colon \Gamma(G,S) \to \Gamma(H,T)$ is a quasi-isometry.*

Proof. Let $\bar{q}\colon H \to G$ be a quasi-inverse of q and, by increasing L_0 and C_0 if necessary, assume that $\bar{q}\colon \Gamma(H,T_0) \to \Gamma(G,S_0)$ is a (L_0,C_0)-quasi-isometry. Moreover, suppose $q \circ \bar{q}$ and $\bar{q} \circ q$ are at distance at most C_0 from the corresponding identity maps with respect to $\mathsf{dist}_{(H,T_0)}$ and $\mathsf{dist}_{(G,S_0)}$. Note that for any $g \in G$,

$$q_g(h) = q(g \cdot \bar{q}(h)) = q(g) \cdot q(\bar{q}(h)).$$

Hence q_g is an $(1, C_0)$-quasi-isometry since it is the composition of $q \circ \bar{q}$ followed by the isometry given by multiplication on the left by $q(g)$. Then the proof concludes by invoking Proposition 2.5. □

The following result is the particular case of Corollary 2.6 in which H is a finite index subgroup of G. In this case, one can give a more algebraic description of the generating set S. The proof follows the same lines as the previous argument modulo Lemma 2.8.

Proposition 2.7. *Let H be a finite index normal subgroup of a finitely generated group G. Let T be a generating set of H, let R be a right transversal of H in G, and let $S = T \cup R$. If the G-action by conjugation on H is a uniform quasi-action on $\Gamma(H,T)$, then the inclusion $\Gamma(H,T) \hookrightarrow \Gamma(G,S)$ is a quasi-isometry.*

We divert the proof of the proposition after the following lemma.

Lemma 2.8. *Let H be a finite index normal subgroup of a finitely generated group G. Let T be a generating set of H containing a finite generating set T_0, let R be transversal of H in G, let S_0 be a finite generating set of G, and let $q\colon \Gamma(G,S_0) \to \Gamma(H,T_0)$ be the quasi-isometry defined by $q(hg) = h$ for $h \in H$ and $g \in R$. The following statements are equivalent:*

(1) The G-action by conjugation on H is a uniform quasi-action on $\Gamma(H,T)$.
(2) The quasi-action of G on H induced by q is uniform with respect to T.

Proof. Take as the quasi-inverse of q the inclusion $H \hookrightarrow G$. For $h \in H$, let $L_h\colon H \to H$ be given by $L_h(x) = hx$, i.e. multiplication on the left. Note that $L_h\colon \Gamma(H,T) \to \Gamma(H,T)$ is an isometry for every $h \in H$.

Let $g \in G$ and suppose that $g = h_* g_*$ where $h_* \in H$ and $g_* \in R$. Then

$$q_g(h) = q(gh) = q(ghg^{-1}h_*g_*) = ghg^{-1}h_* = h_*g_*hg_*^{-1}h_*^{-1}h_* = h_*g_*hg_*^{-1}$$

and hence

$$q_g = L_{h_*} \circ \mathrm{Ad}(g_*),$$

where $\mathrm{Ad}(g_*)$ is conjugation by g_*. It follows $q_g\colon \Gamma(H,T) \to \Gamma(H,T)$ is an (L,C)-quasi-isometry for all $g \in G$ if and only if $\mathrm{Ad}(g_*)\colon \Gamma(H,T) \to \Gamma(H,T)$ is an (L,C)-quasi-isometry for all $g_* \in R$. In particular, the first statement implies the second by Remark 2.3, and the second statement implies the first since the constants L and C hold for all conjugations. □

Proof of Proposition 2.7. Let $T_0 \subset T$ be a finite generating set of H, let $S_0 = T_0 \cup R$. Note that S_0 is a finite generating set of G. Then $q\colon \Gamma(G,S_0) \to \Gamma(H,T_0)$ is a (L_0, C_0) quasi-isometry for some $L_0 \geqslant 1$ and $C_0 \geqslant 0$, and the quasi-inverse $\bar{q}$ can be taken as the inclusion $\Gamma(H,T_0) \hookrightarrow \Gamma(G,S_0)$.

Observe that in $\Gamma(G,S)$ the vertices $g = hr$ and $q(g) = h$ are adjacent since $r \in S$. Therefore, if $[v_0, \ldots, v_\ell]$ is a geodesic path in $\Gamma(H,T)$ from $q(a)$ to $q(b)$, then $[a, v_0, \ldots, v_\ell, b]$ is a path in $\Gamma(G,S)$ from a to b, and hence

$$\mathsf{dist}_{(G,S)}(a,b) \leqslant \mathsf{dist}_{(H,T)}(q(a), q(b)) + 2.$$

We now prove the other inequality. Since the G-action on H by conjugation is a uniform quasi-action on $\Gamma(H,T)$, Lemma 2.8 implies that the quasi-action of G on H induced by q is (L_1, C_1)-uniform with respect to T, for some $L_1 \geqslant 1$ and $C_1 \geqslant 0$.

Let $K_0 = L_0 + C_0 + 1$. Observe that

$$S \subseteq \{f^{-1}g \in G\colon \text{ there are } h \in H \text{ and } t \in T \text{ such that } \mathsf{dist}_{(H,T_0)}(q(f), h) \leqslant K_0 \text{ and } \mathsf{dist}_{(H,T_0)}(q(g), ht) \leqslant K_0\}.$$

Indeed, let $s \in S = T \cup R$, there are two cases. First, if $s \in T$ let $f = h = e$ and $g = t = s$; and second if $s \in R$ let $f = h = e$, $g = s$ and t any element of T_0. Then, exactly as in the first claim in the proof of Proposition 2.5, one defines a constant $\bar{L} = \bar{L}(L_1, C_1, K_0)$ and deduces the inequality

$$\mathsf{dist}_{(H,T)}(q(a), q(b)) \leqslant \bar{L}\,\mathsf{dist}_{(G,S)}(a, b). \tag{3}$$

It remains to show

$$\mathsf{dist}_{(G,S)}(a, b) \leqslant \mathsf{dist}_{(H,T)}(q(a), q(b)) + 2. \tag{4}$$

for any $a, b \in G$, concluding the proof. This is clear since $\Gamma(H,T)$ is a subgraph of $\Gamma(G,T)$ and $\mathsf{dist}_{G,S}(g, q(g)) \leqslant 1$ for any $g \in G$. □

The following example by Minasyan and Osin illustrates the need for the hypothesis relating to the conjugation action in Corollary 2.7.

Example 2.9. [MO19] Let $H = \langle a, b\rangle$ be the free group of rank two, let $G = \langle a, b, t\colon tat^{-1} = b,\quad t^2 = e\rangle$, let $T = \{b, a, a^{-1}, a^2, a^{-2}, \dots\}$ and $S = T \cup \{t\}$. The inclusion $\Gamma(H,T) \to \Gamma(G,S)$ is not a quasi-isometry. Indeed, in G we have $ta^nt^{-1} = b^n$ and hence $\mathsf{dist}_{(G,S)}(e, b^n) = 3$ but $\mathsf{dist}_{(H,T)}(e, b^n) = n$ for every n. In particular, the map $\Gamma(H,T) \to \Gamma(H,T)$ given by $h \mapsto tht^{-1}$ is not a quasi-isometry, and hence the G-action on H by conjugation is not an action by quasi-isometries.

3. Quasi-isometries and Hyperbolically embedded subgroups

In this section, we will prove Theorem A. The theorem is obtained by putting together a simple characterization of hyperbolically embedded subgroups in terms of coned-off Cayley graphs which appeared in work of Rashid and the second author, see [MR21, Propositions 1.5 and 5.8]; some results about quasi-isometries of pairs from [HMS21], and some basic facts about hyperbolically embedded subgroups from [DGO17]. Below we state these results and then we discuss the proof of Theorem 3.11.

Definition 3.1 (Reduced collections)**.** A collection of subgroups $\mathcal{Q}$ of a group H is *reduced* if for any $P, Q \in \mathcal{Q}$ and $g \in H$, if P and gQg^{-1} are commensurable subgroups then $P = Q$ and $g \in P$.

Remark 3.2. An almost malnormal collection is reduced.

Definition 3.3 (Fine)**.** Let Γ be a graph and let v be a vertex of Γ. Let

$$T_v\Gamma = \{w \in V(\Gamma) \mid \{v, w\} \in E(\Gamma)\}.$$

denote the set of the vertices adjacent to v. For $x, y \in T_v\Gamma$, the *angle metric* $\angle_v(x, y)$ is the length of the shortest path in the graph $\Gamma\backslash\{v\}$ between x and y, with $\angle_v(x, y) = \infty$ if there is no such path. The graph Γ is *fine at v* if $(T_v\Gamma, \angle_v)$ is a locally finite metric space. The graph Γ is *fine at $C \subseteq V(\Gamma)$* if Γ is fine at v for all $v \in C$.

Definition 3.4 (Coned-off Cayley graph)**.** Let G be a group, let $\mathcal{P}$ be an arbitrary collection of subgroups of G, and let S be a subset of G. Denote by $G/\mathcal{P}$ the set of all cosets gP with $g \in G$ and $P \in \mathcal{P}$. The *coned-off Cayley graph of G with respect to $\mathcal{P}$* is the graph $\hat{\Gamma}(G, \mathcal{P}, S)$ with vertex set $G \cup G/\mathcal{P}$ and edges are of the following type

- $\{g, gs\}$ for $s \in S$,
- $\{x, gP\}$ for $g \in G$, $P \in \mathcal{P}$ and $x \in gP$.

We call vertices of the form gP *cone points.*

Proposition 3.5. [MR21] *Let $\mathcal{P}$ be a collection of infinite subgroups of G and let S be a subset of G. Then $\mathcal{P} \hookrightarrow_h (G, S)$ if and only if the Coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{P}, S)$ is a connected hyperbolic graph which is fine at every cone vertex.*

Proposition 3.6. [HMS21, Proposition 5.6] *Let G and H be groups, let $S \subset G$ and $T \subset H$, and let $S_0 \subset S$ and $T_0 \subset T$ be finite generating sets of G and H respectively. Consider collections $\mathcal{P}$ and $\mathcal{Q}$ of subgroups of G and H respectively. Let $q\colon G \to H$ be a function.*

Suppose q is a quasi-isometry $\Gamma(G, S) \to \Gamma(H, T)$, is a quasi-isometry of pairs $(G, \mathcal{P}, S_0) \to (H, \mathcal{Q}, T_0)$, and $\dot{q}$ is a bijection $G/\mathcal{P} \to H/\mathcal{Q}$.

(1) Let $\hat{q} = q \cup \dot{q}$, then $\hat{q}$ is a quasi-isometry $\hat{\Gamma}(G, \mathcal{P}, S) \to \hat{\Gamma}(H, \mathcal{Q}, T)$.
(2) If $\hat{\Gamma}(H, \mathcal{Q}, T)$ is fine at cone vertices, then $\hat{\Gamma}(G, \mathcal{P}, S)$ is fine at cone vertices.
(3) If $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P} \hookrightarrow_h (G, S)$.

Items (1) and (2) of Proposition 3.6 are taken from [HMS21, Proposition 5.6], and the last item is a direct consequence of Proposition 3.5.

Proposition 3.7. [HMS21, Proposition 5.12] *Let $q\colon (G, \mathcal{P}) \to (H, \mathcal{Q})$ be a (L, C, M)-quasi-isometry of pairs. Then*

(1) $\dot{q}$ is a surjective function $G/\mathcal{P} \to H/\mathcal{Q}$ if $\mathcal{Q}$ is reduced.
(2) $\dot{q}$ is a bijection $G/\mathcal{P} \to H/\mathcal{Q}$ if $\mathcal{P}$ and $\mathcal{Q}$ are reduced.

Proposition 3.8. [HMS21, Proposition 6.2] *Let $\mathcal{P}^*$ be a refinement of a finite collection of subgroups $\mathcal{P}$ of a finitely generated group G. If P is a finite index subgroup of $\mathrm{Comm}_G(P)$ for every $P \in \mathcal{P}$, then $(G, \mathcal{P})$ and $(G, \mathcal{P}^*)$ are quasi-isometric pairs via the identity map on G.*

Proposition 3.9. [HMS21, Proposition 6.7] *Let $q\colon (G, \mathcal{P}) \to (H, \mathcal{Q})$ be a quasi-isometry of pairs. If $\mathcal{Q}$ is an almost malnormal finite collection of infinite subgroups and $\mathcal{P}$ is a finite collection, then any refinement $\mathcal{P}^*$ of $\mathcal{P}$ is almost malnormal.*

Proposition 3.10. [DGO17, Proposition 4.33] *Let $\mathcal{P}$ be a collection of subgroups of a group G. If $\mathcal{P} \hookrightarrow_h G$ then $\mathcal{P}$ is an almost malnormal collection.*

We are now ready to prove Theorem A.

Theorem 3.11 (Theorem A)**.** *Let $q\colon G \to H$ be a quasi-isometry of finitely generated groups, let $\mathcal{P}$ and $\mathcal{Q}$ be finite collections of subgroups of G and H respectively, and let S and T be (not necessarily finite) generating sets of G and H respectively. Suppose*

(1) $q\colon (G, \mathcal{P}) \to (H, \mathcal{Q})$ is a quasi-isometry of pairs, and
(2) $q\colon \Gamma(G, S) \to \Gamma(H, T)$ is a quasi-isometry.

The following statements hold:

1. *If $\mathcal{P}$ and $\mathcal{Q}$ are reduced collections in G and H respectively; then $\mathcal{P} \hookrightarrow_h (G,S)$ if and only if $\mathcal{Q} \hookrightarrow_h (G,T)$.*
2. *If $\mathcal{Q}$ contains only infinite subgroups and $\mathcal{Q} \hookrightarrow_h (G,T)$ then $\mathcal{P}^* \hookrightarrow_h (G,S)$.*

Proof. For the first statement, since $\mathcal{P}$ and $\mathcal{Q}$ are reduced, Proposition 3.7 implies that $\dot{q}\colon G/\mathcal{P} \to H/\mathcal{Q}$ is a bijection. Then Proposition 3.6 implies that $\hat{\Gamma}(G,\mathcal{P},S)$ is hyperbolic and fine at cone vertices if and only if $\hat{\Gamma}(H,\mathcal{Q},T)$ is hyperbolic and fine at cone vertices. Then Proposition 3.5 concludes the proof of the first statement.

The second statement is a consequence of the first statement as follows. That $\mathcal{Q} \hookrightarrow_h H$ implies that $\mathcal{Q}$ is an almost malnormal collection of subgroups in H, see Proposition 3.10. It follows that $\mathcal{Q}$ is reduced in H. Then, since $\mathcal{Q}$ contains only infinite subgroups, Proposition 3.9 implies that $\mathcal{P}^*$ is reduced. By Proposition 3.8, $q\colon (G,\mathcal{P}^*) \to (H,\mathcal{Q})$ is a quasi-isometry of pairs. Then $\mathcal{Q} \hookrightarrow_h H$ and the first statement of the proposition imply that $\mathcal{P}^* \hookrightarrow (G,S)$. □

4. Hyperbolically embedded subgroups and commensurability

In this section we prove Theorem E. The argument uses the following proposition which is a strengthening of [MS21, Proposition 2.15]. It essentially follows from the proof in the cited article; but we have included the proof for the convenience of the reader.

Proposition 4.1. *Let H be a finite index subgroup of a finitely generated group G, and let $\mathcal{Q}$ be a finite collection of subgroups of H. The following statements are equivalent:*

1. *The inclusion $H \hookrightarrow G$ is a quasi-isometry of pairs $(H,\mathcal{Q}) \hookrightarrow (G,\mathcal{Q})$.*
2. *For any $Q \in \mathcal{Q}$ and $g \in G$, there is $Q' \in \mathcal{Q}$ and $h \in H$ such that $\mathsf{hdist}_G(gQ, hQ') < \infty$.*

Proof. That (1) implies (2) is trivial. Assume statement (2). Since H is a finite index subgroup of the finitely generated group G, assume $H \hookrightarrow G$ is an (L,C) quasi-isometry. Since H is finite index in G, and $\mathcal{Q}$ is a finite collection, the H-action on $G/\mathcal{Q}$ has finitely many orbits. For $gQ \in G/\mathcal{Q}$, let

$$\mathsf{hdist}_G(gQ, H/\mathcal{Q}) := \min\left\{\mathsf{hdist}_G(gQ, hQ')\colon hQ' \in H/\mathcal{Q}\right\}.$$

Let $\mathcal{R}$ be a finite collection of orbit representatives of the H-action on $G/\mathcal{Q}$. By hypothesis, for $gQ \in \mathcal{R}$ there is $hQ' \in H/\mathcal{Q}$ such that $\mathsf{hdist}(gQ, hQ') < \infty$ and therefore

$$M = \max\{\mathsf{hdist}_G(gQ, H/\mathcal{Q})\colon gQ \in \mathcal{R}\} < \infty$$

is a well defined integer since $\mathcal{R}$ is a finite set. Since the subset $H/\mathcal{Q}$ of $G/\mathcal{Q}$ is H-invariant,

$$\mathsf{hdist}_G(gQ, H/\mathcal{Q}) = \mathsf{hdist}_G(hgQ, H/\mathcal{Q})$$

for every $gQ \in \mathcal{R}$ and $h \in H$. Since $\mathcal{R}$ is a collection of representatives of orbits of $G/\mathcal{Q}$,

$$\mathsf{hdist}_G(gQ, H/\mathcal{Q}) \leqslant M$$

for every $gQ \in G/\mathcal{Q}$. Hence $(H,\mathcal{Q}) \hookrightarrow (G,\mathcal{Q})$ is an (L,C,M) quasi-isometry of pairs . □

Remark 4.2. Let G be a group and let T and S generating sets with finite symmetric difference. Then the identity map on G is a quasi-isometry $\Gamma(G,T) \to \Gamma(G,S)$.

Theorem 4.3 (Theorem E). *Let H be a finite index normal subgroup of a finitely generated group G, and let $\mathcal{Q}$ be a finite collection of infinite subgroups of H such that $\mathcal{Q} \hookrightarrow_h (H,T)$. Suppose:*

(1) The G-action by conjugation on H is a uniform quasi-action on $\Gamma(H,T)$.
(2) The collection $\{hQh^{-1}: h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by G.

If $\mathcal{Q}^*$ is a refinement of $\mathcal{Q}$ in G and R is a transversal of H in G, then $\mathcal{Q}^* \hookrightarrow_h (G, T \cup R)$.

Proof. Since H is finitely generated, by adding a finitely many elements we can assume that T generates H. Note that this preserves $\mathcal{Q} \hookrightarrow_h (H,T)$ by [DGO17, Cor. 4.27], and the quasi-isometry type of $\Gamma(H,T)$ by Remark 4.2. Under this assumption, the conclusion will follow from the second statement of Theorem 3.11 applied to the quasi-isometry of finitely generated groups given by the inclusion $H \hookrightarrow G$.

Since $\mathcal{Q} \hookrightarrow_h (H,T)$, $\mathcal{Q}$ is an almost malnormal collection, see Proposition 3.10. The assumption that $\mathcal{Q}$ consist only of infinite subgroups implies that for any $Q \in \mathcal{Q}$,

$$Q = \mathrm{Comm}_H(Q) = \mathrm{Comm}_G(Q) \cap H.$$

Since H is finite index in G, we have that Q is finite index in $\mathrm{Comm}_G(Q)$. Then, Proposition 3.8 implies that the identity map on G is a quasi-isometry of pairs $(G,\mathcal{Q}) \to (G,\mathcal{Q}^*)$. On the other hand, since the collection $\{hQh^{-1}: h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by elements of G, we have for any $g \in G$ and $Q \in \mathcal{Q}$ there is $h \in H$ such that $gQg^{-1} = hQ'h^{-1}$ and hence

$$\mathsf{hdist}_G(gQ, hQ') \leqslant \mathsf{hdist}_G(gQ, Q^g) + \mathsf{hdist}_G(Q^g, (Q')^h) + \mathsf{hdist}((Q')^h, hQ') < \infty.$$

Proposition 4.1 implies that $H \hookrightarrow G$ is a quasi-isometry of pairs $(H,\mathcal{Q}) \to (G,\mathcal{Q})$. It follows that $H \hookrightarrow G$ is a quasi-isometry of pairs $(H,\mathcal{Q}) \to (G,\mathcal{Q}^*)$ as it is the composition $(H,\mathcal{Q}) \hookrightarrow (G,\mathcal{Q}) \to (G,\mathcal{Q}^*)$. Let R be a transversal of H in G and let $S = T \cup R$. Since the G-action by conjugation on H is uniform on $\Gamma(H,T)$, Proposition 2.7 implies that $H \hookrightarrow G$ is a quasi-isometry $\Gamma(H,T) \to (G,S)$. The hypothesis of Theorem 3.11 has been verified and therefore, $\mathcal{Q} \hookrightarrow_h (H,T)$ implies $\mathcal{Q}^* \hookrightarrow_h (G,S)$. □

5. Semi-direct products and Hyperbolically embedded subgroups

In this section we will prove Theorem F about semi-direct products. The hypothesis of the following proposition and theorem reflects the issues posed by the example of Minasyan and Osin (Example 2.9).

Proposition 5.1. *Let A be a group with (not necessarily finite) generating set T, let $\mathcal{H}$ be a collection of subgroups, and let $F \leqslant \mathsf{Aut}(A)$ be a finite subgroup. Suppose that T and $\mathcal{H}$ are F-invariant, and the F-action on $\mathcal{H}$ is free. Let $\mathcal{H}_F$ be a collection of representatives of F-orbits in $\mathcal{H}$. Then the inclusion $A \hookrightarrow A \rtimes F$ induces:*

(1) a quasi-isometry $\Gamma(A,T) \to \Gamma(A \rtimes F, T \cup F)$;
(2) and, if A is finitely generated, a quasi-isometry of pairs $(A,\mathcal{H}) \to (A \rtimes F, \mathcal{H}_F)$.

Proof. To prove the first statement, let $S = T \cup F$ and let dist_T and dist_S be the word metrics on A and $A \rtimes F$ induced by T, and S respectively. Let $q\colon A \hookrightarrow A \rtimes F$ be the inclusion, and let $\bar{q}\colon A \rtimes F \to A$ such that for $a \in A$ and $f \in F$, $\bar{q}(af) = a$. Note that $\bar{q}$ is a well defined

A-equivariant map since each element of $A \rtimes F$ can be expressed as a product af in a unique way. Observe that $\bar{q} \circ q$ is the identity on A, and $q \circ \bar{q}$ is at distance one from the identity map on $A \rtimes F$ with respect to dist_S. Since the Cayley graph $\Gamma(A, T)$ is a subgraph of $\Gamma(A \rtimes F, T \cup F)$, it is immediate that for any $u, v \in A$, $\mathsf{dist}_S(q(u), q(v)) \leqslant \mathsf{dist}_T(u, v)$. To conclude the proof the statement, we show that for any $u, v \in A \rtimes F$, $\mathsf{dist}_T(\bar{q}(u), \bar{q}(v)) \leqslant \mathsf{dist}_S(u, v)$. Note that it is enough to consider the case that $\mathsf{dist}_S(u, v) = 1$. Let $w_1, w_2 \in A \rtimes F$ such that $\mathsf{dist}_S(w_1, w_2) = 1$. Then $w_1 = a_1 f_1$ and $w_2 = a_2 f_2$ and $\bar{q}(w_i) = a_i$. It follows that $g = (a_1 f_1)^{-1} a_2 f_2 \in T \cup F$. Observe that

$$g = f_1^{-1} a_1^{-1} a_2 f_2 = (a_1^{-1})^{f_1^{-1}} f_1^{-1} a_2 f_2 = (a_1^{-1})^{f_1^{-1}} a_2^{f_1^{-1}} f_1^{-1} f_2 = (a_1^{-1} a_2)^{f_1^{-1}} f_1^{-1} f_2 \in T \cup F.$$

There are two cases, either $g \in T$ or $g \in F$, since $T \cap F = \varnothing$. We regard $T \cup F$ and F as a subset and a subgroup of $A \rtimes F$ respectively. If $g \in T$, then $f_1 = f_2$ and hence $(a_1^{-1} a_2)^{f_1^{-1}} \in T$; since T is F-invariant, a_1 and a_2 are adjacent in Γ, and hence $\mathsf{dist}_T(\bar{q}(w_1), \bar{q}(w_2)) = 1$. If $g \in F$, then $a_1 = a_2$ and hence $\mathsf{dist}_T(\bar{q}(w_1), \bar{q}(w_2)) = 0$.

For the second statement, suppose that A is finitely generated and let dist denote word metric on $A \rtimes F$ induced by finite generating set, and let $\mathsf{hdist}_{A \rtimes F}$ be the induced Hausdorff distance. Let $M = \max_{f \in F} \mathsf{dist}(1, f)$. Since the inclusion $A \hookrightarrow A \rtimes F$ is a quasi-isometry of finitely generated groups and $\mathcal{H}_F \subset \mathcal{H}$, it is enough to prove that for any $H \in \mathcal{H}$ there is a left coset $(A \rtimes F)/\mathcal{H}_F$ at Haudorff distance at most M in $A \rtimes F$. Let $H \in \mathcal{H}$. Since the F-action on $\mathcal{H}$ by conjugation is free, there is a unique $f \in F$ and a unique $K \in \mathcal{H}_F$ such that $H = fKf^{-1}$. Observe that

$$\mathsf{hdist}(H, fK) = \mathsf{hdist}(fKf^{-1}, fK) \leqslant \mathsf{dist}(1, f^{-1}) \leqslant M,$$

and this completes the proof. □

Theorem 5.2 (Theorem F). *Let A be a finitely generated group with (not necessarily finite) generating set T, and let $\mathcal{H}$ be a finite collection of infinite subgroups such that $\mathcal{H} \hookrightarrow_h (A, T)$. If $F \leqslant \mathsf{Aut}(A)$ is finite, T and $\mathcal{H}$ are F-invariant and the F-action on $\mathcal{H}$ is free, then $\mathcal{H}_F \hookrightarrow_h (A \rtimes F, T \cup F)$ where $\mathcal{H}_F$ is collection of representatives of F-orbits in $\mathcal{H}$.*

Proof. By Proposition 5.1, the inclusion $A \hookrightarrow A \rtimes F$ induces a quasi-isometry $\Gamma(A, T) \to \Gamma(A \rtimes F, T \cup F)$, and a quasi-isometry of pairs $(A, \mathcal{H}) \to (A \rtimes F, \mathcal{H}_F)$. Since $\mathcal{H} \hookrightarrow_h A$, the collection $\mathcal{H}$ is almost malnormal in A; then the assumption that F acts freely on $\mathcal{H}$ implies that a refinement of $\mathcal{H}$ in $A \rtimes F$ is $\mathcal{H}_F$, this was observed in Example 1.6. Since $\mathcal{H}$ contains only infinite subgroups and $\mathcal{H} \hookrightarrow_h A$, Theorem 3.11 implies that $\mathcal{H}_F \hookrightarrow_h (A \rtimes F, T \cup F)$. □

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