REGULARITY OF QUASIGEODESICS CHARACTERISES HYPERBOLICITY

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Abstract. We characterise hyperbolic groups in terms of quasigeodesics in the Cayley graph forming regular languages. We also obtain a quantitative characterisation of hyperbolicity of geodesic metric spaces by the non-existence of certain local $(3, 0)$–quasigeodesic loops. As an application we make progress towards a question of Shapiro regarding groups admitting a uniquely geodesic Cayley graph.

1. Introduction

Hyperbolic groups were introduced by Gromov [Gro87] and revolutionised the study of finitely generated groups. Arguably, their most remarkable feature is that hyperbolicity connects several, and at a first glance independent, areas of mathematics. Confirming this, there are several different characterisations of hyperbolicity — such as the geometric thin triangle condition [Gro87], the dynamical characterisation via convergence actions [Bow98], surjectivity of the comparison map in bounded cohomology [Min01; Min02; Fra18] and vanishing of ℓ^∞-cohomology [Ger98], linear isoperimetric inequality [Gro87], all asymptotic cones being $\mathbb{R}$-trees [Gro87], and others [Gro87; Gro93; Pap95b; AG99; Gil02; CN07; Wen08a].

Another significant feature of hyperbolic groups is that they present very strong algorithmic properties. Most notably, they have solvable isomorphism problem [DG11; Sel95], they are biautomatic [ECHLPT92] and so the word problem can be solved via finite state automata, and sets of their rational quasigeodesics form a regular language [HR03].

This last property will be a central focus in the paper, and we call it rational regularity, or for short $\mathbb{Q}$**REG**.

Definition 1.1. A finitely generated group G is $\mathbb{Q}$**REG** if the (λ, ϵ)–quasigeodesics in the Cayley graph of G with respect to any finite generating set form a regular language for all rational numbers $\lambda \geqslant 1$ and $\epsilon \geqslant 0$.

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As mentioned, in [HR03] Holt and Rees prove that every word hyperbolic group is $\mathbb{Q}$**REG**. It is natural to ask if this provides a characterization of hyperbolic groups, as was conjectured in [CRSZ20, Problem 1]. The main result of the paper is the following.

Theorem 1.2. *A finitely generated group is hyperbolic if and only if it is* $\mathbb{Q}$**REG**.

We remark that it is necessary to not consider only geodesics. In [Can84], Cannon proved that for any finite generating set the geodesics in a hyperbolic group form a regular language. However, this does not characterise hyperbolicity: Neumann and Shapiro [NS95, Propositions 4.1, 4.4] prove that for any finite generating set the geodesics in an abelian group form a regular language.

The key ingredient in the proof of Theorem 1.2 is a strong quantitative characterisation of hyperbolicity. It is known that a geodesic metric space is hyperbolic if and only if local quasigeodesics are global quasigeodesics [Gro87, Proposition 7.2.E]. More precisely, the contrapositive can be stated as follows: a space is non-hyperbolic if and only if there exists a pair of constants (λ, ϵ), a sequence $L_n \to \infty$ and a sequence of L_n-locally (λ, ϵ)-quasigeodesic paths which are not global (λ', ϵ')-quasigeodesics for any uniform choice of constants (λ', ϵ'). Hence, *a priori*, to check for non-hyperbolicity one would want to consider all choices of (λ, ϵ) and all choices of locally (λ, ϵ)-quasigeodesic paths. We strengthen the above result, by proving that one needs only consider L-locally $(3, 0)$-quasigeodesic loops whose length is comparable to L.

Theorem 1.3. *A geodesic metric space X is not hyperbolic if and only if there exists a sequence $L_n \to \infty$ and a sequence of L_n-locally $(3, 0)$-quasigeodesic loops γ_n that satisfy $\ell(\gamma_n) \leqslant K L_n$, where K is some constant that does not depend on n.*

Although striking, the presence of a sharp gap in the behaviour of local-quasigeodesics is not surprising. For instance, it is known that the Dehn function of a finitely presented group has a gap. A deep theorem of S. Wenger [Wen08b], extending results of [Gro87; Ol 91; Bow95; Pap95a], shows that if the isoperimetric function satisfies $D(x) \leqslant \frac{1-\epsilon}{4\pi} x^2$, then it is in fact linear.

Our strategy in proving Theorem 1.3 relies on the study of asymptotic cones of metric spaces. If X is non-hyperbolic, then there is a cone that is not a tree [Gro93, 2.A], and it contains a simple loop. By using a series of approximations, we exploit this loop to produce a family of loops of controlled length that are locally $(3, 0)$–quasigeodesic. To prove Theorem 1.2, we use such a sequence of loops to essentially contradict a version of the pumping lemma, yielding that if a group is not hyperbolic the languages of quasigeodesics cannot be all regular.

A question of Shapiro. A natural class of graphs to consider is the one of geodetic graphs. A graph is called *geodetic* if for any pair of vertices there is exactly one geodesic connecting them. In [Sha97], M. Shapiro asked when a group admits a (locally finite) geodetic Cayley graph. He conjectures that such a group needs to be *plain*, that is, a free product where the factors are either free or finite. Surprisingly, the question is still open, although there are some algorithmic characterizations of plain groups [EP20]. More precisely, in [Pap93] Papasoglu proved that a geodetic hyperbolic group is virtually free. It is still open whether all geodetic groups are hyperbolic, and whether all geodetic, virtually free groups are plain.

We provide an answer to the first implication under an additional, language theoretic, assumption.

Theorem 1.4. *Let G be a finitely generated group with a generating set S such that $\Gamma(G,S)$ is geodetic. If there exists $\lambda > 3$ such that the language of $(\lambda, 0)$–quasigeodesics is regular, then G is hyperbolic and hence virtually free.*

Structure of the paper. In Section 2 we give the necessary background on Cayley graphs, hyperbolic metric spaces, quasigeodesics, languages, automata, and asymptotic cones. In Section 3 we prove our main technical propositions and deduce the theorems from the introduction.

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2. Background

2.1. **Cayley graphs, hyperbolicity, and quasigeodesics.** Throughout this paper, let G be a finitely generated group with generating set S. We denote by $\Gamma(G,S)$ the *Cayley graph* of G with respect to S, that is, the graph with vertices G and edges $\{g, gs\}$ where $g \in G$ and $s \in S$. We denote by $|g|$ the word-length of g with respect to S; equivalently, this is equal to $d_{\Gamma(G,S)}(e,g)$.

Let $\delta \geqslant 1$. A metric space X is δ-*hyperbolic* if every geodesic triangle in X is δ-thin. Here a geodesic triangle is δ-*thin* if every edge is contained in the δ-neighbourhood of the two other edges. We say a finitely generated group G is *hyperbolic* if the Cayley graph $\Gamma(G,S)$ is a δ-hyperbolic metric space for some finite generating set S.

Let $\lambda \geqslant 1$ and $\epsilon \geqslant 0$. Given metric spaces X and Y a (λ,ϵ)-*quasi-isometric embedding* $f \colon X \to Y$ is a (λ,ϵ)-coarsely Lipschitz function. If f is additionally ϵ-coarsely surjective, then we say f is a (λ,ϵ)-*quasi-isometry*. If there exists a quasi-isometry $X \to Y$, then we say X and Y are *quasi-isometric*.

A (λ, ϵ)-*quasigeodesic* of length $a > 0$ in X is a path $c\colon [0, a] \to X$ such that for any two points x, y in c we have

$$d_c(x, y) \leqslant \lambda d_X(x, y) + \epsilon,$$

where d_c is the usual metric on $[0, a]$ and d_X is the metric on X. We say $c\colon [0, a] \to X$ is an L-*locally* (λ, ϵ)-*quasigeodesic* or an (L, λ, ϵ)-*local-quasigeodesic* if c restricted to each subset of $[0, a]$ of length L is a (λ, ϵ)–quasigeodesic. We say c is a (L, λ, ϵ)-*quasigeodesic loop* if, in addition, $c(0) = c(a)$.

2.2. **Regular languages and automata.** The following definitions are standard and may be found in [ECHLPT92, Chapter 1]. Given a finite set A, let $A^\star$ be the free monoid generated by A, i.e. the set of finite words that can be written with letters in A. A *language* over the alphabet A is a subset $L \subseteq A^\star$. A *finite state automaton* (FSA) $\mathcal{M}$ over the alphabet A consists a finite oriented graph $\Gamma(\mathcal{M})$, together with an edge label function $\ell\colon E(\Gamma(\mathcal{M})) \to A$, a chosen vertex $q_I \in V(\Gamma(\mathcal{M}))$ called the *initial state* and subset $Q_F \subset V(\Gamma(\mathcal{M}))$ of *final states*. The vertices of $\Gamma(\mathcal{M})$ are often referred to as *states*.

Let $\mathcal{M}$ be an FSA over an alphabet A. We say a string $w \in L$ is *accepted* by A if and only if there is an oriented path γ in $\Gamma(\mathcal{M})$ starting from q_I and ending in a vertex $q \in Q_F$ such that γ is labelled by w. A language L is *regular* if and only if there exists an FSA $\mathcal{M}$ such that L coincides with the strings of $A^\star$ accepted by $\mathcal{M}$.

Let G be a group generated by a finite set A. An element $w \in A^*$ labels a path in $\Gamma(G, A)$ which starts at 1_G. We say w is a *geodesic/*(λ, ϵ)*–quasigeodesic/*(L, λ, ϵ)*-local-quasigeodesic word* if it labels a path in $\Gamma(G, A)$ with the corresponding property. We say that the set $L^{(\lambda,\epsilon)}$ of (λ, ϵ)–quasigeodesic words w over A form the (λ, ϵ)*–quasigeodesic language of* G *over* A.

2.3. **Asymptotic cones.** In this section we will give the necessary background on asymptotic cones. The idea first appeared in the proof of Gromov's Polynomial Growth Theorem [Gro81], however, it was first formalised by Wilkie and van den Dries [WD84].

An *ultrafilter* ω on $\mathbb{N}$ is a set of nonempty subsets of $\mathbb{N}$ which is closed under finite intersection, upwards-closed, and if given any subset $X \subseteq \mathbb{N}$, contains either X or $\mathbb{N}\backslash X$. We say ω is *non-principal* if ω contains no finite sets. We may equivalently view ω as a finitely additive measure on the class $2^{\mathbb{N}}$ of subsets of $\mathbb{N}$ such that each subset has measure equal to 0 or 1, and all finite sets have measure 0. If some statement $P(n)$ holds for all $n \in X$ where $X \in \omega$, then we say that $P(n)$ holds ω-*almost surely.*

Let ω be a non-principal ultrafilter on $\mathbb{N}$ and let X be a metric space. If $(x_n)_{n\in\mathbb{N}}$ is a sequence of points in X, then a point x satisfying for every $\epsilon > 0$ that $\{n \mid d(x_n, x) \leqslant \epsilon\} \in \omega$, is called an ω-*limit* of x_n and denoted by $\lim_\omega x_n$. Given a bounded sequence $x_n \in X$, there always exists a unique ultralimit $\lim_\omega x_n$.

Let ω be a non-principal ultrafilter on $\mathbb{N}$. Let $(X_n, d_n)_{n\in\mathbb{N}}$ be a sequence of metric spaces with specified base-points $p_n \in X_n$. Say a sequence $(y_n)_{n\in\mathbb{N}}$ is *admissible* if the sequence $(d_{X_n}(p_n, y_n))_{n\in\mathbb{N}}$ is bounded. Given admissible sequences $x = (x_n)$ and $y = (y_n)$, the sequence $(d_{X_n}(x_n, y_n))$ is bounded and we define $\hat{d}_\infty(x, y) := \lim_\omega d_n(x_n, y_n)$. Denote the set of admissible sequences by $\mathcal{X}$. For $x, y \in \mathcal{X}$ define an equivalence relation by $x \sim y$ if $\hat{d}_\infty(x, y) = 0$. The *ultralimit* of (X_n, p_n) with respect to ω is the metric space (X_∞, d_∞), where $X_\infty = \mathcal{X}/\sim$ and for $[x], [y] \in X_\infty$ we set $d_\infty([x], [y]) = \hat{d}_\infty(x, y)$. Given an admissible sequence of elements $x_n \in X_n$ we define their ultralimit in X_∞ to be $\lim_\omega x_n := [(x_n)]$. Given a sequence of subsets $A_n \subset X_n$ we can define their ultralimit in X_∞ to be the set $\lim_\omega(A_n) := \{[(x_n)] \mid x_n \in A_n\}$, where we only consider admissible sequences (x_n).

Let ω be a non-principal ultrafilter on $\mathbb{N}$ and let (μ_n) be a diverging, non-decreasing sequence. Let (X, d) be a metric space and consider the sequence of metric spaces $X_n = \left(X, \frac{1}{\mu_n} d\right)$ for $n \in \mathbb{N}$ with basepoints (p_n). The ω-ultralimit of the sequence (X_n, p_n) is called the *asymptotic cone* of X with respect to ω, (μ_n), and (p_n) and denoted $\mathrm{Cone}_\omega(X, (\mu_n), (p_n))$. If the sequence of basepoints is constant, then we denote the asymptotic cone by $\mathrm{Cone}_\omega(X, (\mu_n))$. In the case of a finitely generated group, we assume that the basepoint is always the identity.

The following is [DS05, Proposition 3.29(c)] which we will use in the proof of Theorem 1.2.

Proposition 2.1. *Consider a non-principal ultrafilter ω on $\mathbb{N}$ and a sequence of metric spaces (X_n, d_n) with basepoints $p_n \in X_n$. Suppose there exists a simple geodesic triangle in (X_∞, d_∞). Then there exists a (possibly different) simple geodesic triangle Δ, a constant $k \geqslant 2$, and a sequence of simple geodesic k-gons P_n in X_n such that $\lim_\omega(P_n) = \Delta$.*

3. Proofs of the results

Definition 3.1. Let X be a metric space and consider the following condition:

($\star$) There exists an increasing sequence of positive numbers $L_n \to \infty$ and a pair of constants $K, \lambda \geqslant 1$ such that for every n there exists an L_n-locally $(\lambda, 0)$–quasigeodesic loop γ_n in X with $\ell(\gamma_n) \leqslant K L_n$.

At times, it is convenient to specify the values of the constants K, λ. In that case we say that a space satisfies ($\star$) with constants (K, λ). We say that a group satisfies ($\star$) if there exists a finite generating set S such that the Cayley graph $\Gamma(G, S)$ satisfies ($\star$).

Proposition 3.2. *Suppose G is a finitely generated group that satisfies ($\star$) with constants K, λ. Then, for all $\lambda' > (2K-1)\lambda$, the set of $(\lambda', 0)$–quasigeodesics in G do not form a regular language. In particular, G is not* $\mathbb{Q}$**REG**.

Proof. To prove the proposition we will show that any automata accepting the language of $(\lambda', 0)$–quasigeodesics must have infinitely many distinct states. In particular, the language is not accepted by an FSA and so is not regular.

Fix a generating set such that the Cayley graph satisfies ($\star$) with constants K, λ. Let $\lambda' > (2K-1)\lambda$. Let κ be the positive constant

$$\kappa := \frac{1}{\lambda} - \frac{2K-1}{\lambda'}. \tag{1}$$

Let $m \in \mathbb{N}$ and let n be such that

$$L_n > \frac{1}{\kappa}\left(2KL_m + 1 + \frac{1}{\lambda}\right). \tag{2}$$

Parametrise the loops γ_m and γ_n by arclength. Evidently, we can assume that $\gamma_m(0) = \gamma_n(0) = e$. We fix the following notation:

- Let t_m and t_n be the maximal natural numbers such that $\gamma_m\big|_{[0,t_m]}$ and $\gamma_n\big|_{[0,t_n]}$ are $(\lambda, 0)$–quasigeodesics;
- let T_n be the minimal natural number such that $\gamma_n\big|_{[0,T_n]}$ is *not* a $(\lambda', 0)$–quasigeodesic;
- let $g_m := \gamma_m(t_m)$ and $g_n := \gamma_n(t_n)$;
- let $h_n := \gamma_n(t_n)^{-1}\gamma_n(T_n)$. So $\gamma_n(T_n) = g_n h_n$.

We want to show that the $(\lambda', 0)$–quasigeodesics $\gamma_m\big|_{[0,t_m]}$ and $\gamma_n\big|_{[0,t_n]}$ are in different states at time t_m and t_n respectively. This will follow from the fact that $\gamma_n\big|_{[0,t_n]}$ concatenated with the path $\gamma_n\big|_{[t_n,T_n]}$ *is not* a $(\lambda', 0)$–quasigeodesic, but $\gamma_m\big|_{[0,t_m]}$ concatenated with the same path *is* a $(\lambda', 0)$–quasigeodesic. The former statement follows from the definition of T_n. Looking for a contradiction, suppose that the latter statement is false. That is,

$$\lambda'|g_m h_n| < t_m + T_n - t_n \tag{3}$$

The following six inequalities are easily verified

$$L_m \leqslant t_m \leqslant KL_m; \tag{4}$$

$$L_n \leqslant t_n; \tag{5}$$

$$T_n \leqslant KL_n \leqslant Kt_n; \tag{6}$$

$$\frac{t_m}{\lambda} \leqslant |g_m| \leqslant \frac{t_m + 1}{\lambda} + 1; \tag{7}$$

$$\frac{t_n}{\lambda} \leqslant |g_n| \leqslant \frac{t_n + 1}{\lambda} + 1; \tag{8}$$

$$\frac{T_n - 1}{\lambda'} - 1 \leqslant |\gamma_n(T_n)| \leqslant \frac{T_n}{\lambda'}. \tag{9}$$

We have $|h_n| \geqslant |g_n| - |\gamma_n(T_n)|$, so by (8) and (9) we see that $|h_n| \geqslant \frac{t_n}{\lambda} - \frac{T_n}{\lambda'}$. It then follows from (6) that

$$|h_n| \geqslant \left(\frac{1}{\lambda} - \frac{K}{\lambda'}\right) t_n. \tag{10}$$

Now, $|g_m h_n| \geqslant |h_n| - |g_m|$, so by (10) and (7) we obtain

$$|g_m h_n| \geqslant \left(\frac{1}{\lambda} - \frac{K}{\lambda'}\right) t_n - \frac{t_m + 1}{\lambda} - 1. \tag{11}$$

Combining our assumption (3) with (6) we obtain

$$\lambda' |g_m h_n| \leqslant t_m + (K-1)t_n. \tag{12}$$

Next, combining (11) and (12) yields

$$\frac{t_m + (K-1)t_n}{\lambda'} \geqslant \left(\frac{1}{\lambda} - \frac{K}{\lambda'}\right) t_n - \frac{t_m + 1}{\lambda} - 1.$$

This rearranges to

$$\begin{aligned} 1 + \frac{1}{\lambda} &\geqslant \left(\frac{1}{\lambda} - \frac{(2K-1)}{\lambda'}\right) t_n - \left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right) t_m; \\ &\geqslant \kappa t_n - 2t_m, \end{aligned}$$

where κ is defined in (1). Now,

$$t_n \leqslant \frac{1}{\kappa}\left(2t_m + 1 + \frac{1}{\lambda}\right),$$

and so by (4) and (5) we have

$$L_n \leqslant \frac{1}{\kappa}\left(2KL_m + 1 + \frac{1}{\lambda}\right)$$

which contradicts (2). Hence, the concatenation of $\gamma_m|_{[0,t_m]}$ and $\gamma_n|_{[t_n,T_n]}$ is a $(\lambda', 0)$–quasigeodesic. In particular $\gamma_m(t_m)$ and $\gamma_n(t_n)$ are in different states as $(\lambda', 0)$–quasigeodesics.

Let $\xi : \mathbb{N} \to \mathbb{N}$ be the function

$$\xi(m) = \min\left\{ n \in \mathbb{N} : L_n > \frac{1}{\kappa}\left(2KL_m + 1 + \frac{1}{\lambda}\right)\right\}.$$

Let $(n_i)_{i \in \mathbb{N}}$ be the integer sequence defined inductively by $n_1 = 1$, $n_{i+1} = \xi(n_i)$. By the above, we know that for every $i \in \mathbb{N}$ and for every $j < i$, $\gamma_{n_i}(t_{n_i})$ is in a different state to $\gamma_{n_j}(t_{n_j})$ as a $(\lambda', 0)$-quasigeodesic. It follows that there are infinitely many different $(\lambda', 0)$-states. Hence, the $(\lambda', 0)$–quasigeodesics in G cannot form a regular language. □

Proposition 3.3. *If a metric space X satisfies $(\star)$, then X is not hyperbolic.*

Proof. If X is hyperbolic then it satisfies the local-to-global property for quasigeodesics: for every choice of λ, ϵ there exist constants $L = L(\lambda, \epsilon)$, $\lambda' = \lambda'(\lambda, \epsilon)$ and $\epsilon' = \epsilon'(\lambda, \epsilon)$ such that every L-locally (λ, ϵ)–quasigeodesic is a global (λ', ϵ')–quasigeodesic.

Suppose X satisfies the local-to-global property for quasigeodesics and X satisfies $(\star)$ with constants (K, λ). Let $L = L(\lambda, 0)$ be the constant given by the local-to-global property. Choose $n \in \mathbb{N}$ such that $L_n \geqslant L$. Then γ_n is an L-locally $(\lambda, 0)$–quasigeodesic. However, γ_n is a loop and so cannot be a (λ', ϵ')–quasigeodesic for any choice of λ', ϵ'. □

Theorem 3.4. *If X is a non-hyperbolic geodesic metric space, then X satisfies $(\star)$ with constants $(3, 0)$.*

Proof. Since X is not hyperbolic, there exists an ultrafilter ω and a non-decreasing scaling sequence μ_n such that $\mathrm{Cone}_\omega(X, (\mu_n))$ is not a real tree. In particular, there exists a simple geodesic triangle $\Delta \subseteq \mathrm{Cone}_\omega(X, (\mu_n))$. Using Proposition 2.1, up to replacing Δ with another simple geodesic triangle, we obtain that $\Delta = \lim_\omega(P_n)$, where P_n is a geodesic k-gon in X for some k. Let $z_n^1, \dots z_n^k$ be the vertices of P_n, where the labels are taken respecting the cyclic order on P_n. From now on, we always consider the indices mod k. Denote by e_n^i the geodesic segment connecting z_n^i, z_n^{i+1}, that is the appropriate restriction of P_n.

Consider the points $z_\omega^i = (z_n^i) \in \Delta$, and let $e_\omega^i = \lim_\omega(e_i)$. It is a standard argument to show that e_ω^i are geodesic segments whose endpoints are z_ω^i, see for instance [DK18, Lemma 10.48, Exercise 10.71]. Since $\lim_\omega(P_n) = \Delta$, we have $e_\omega^i \subseteq \Delta$, and in particular that the vertices of the triangle Δ belong to $\{z_\omega^1, \dots, z_\omega^k\}$. Since there are only k edges, for any $\rho > 1$, ω–almost surely we have

$$\ell(e_n^i) \leqslant \rho \mu_n \ell(e_\omega^i). \tag{13}$$

In particular, ω–almost surely we have $\ell(P_n) \leqslant \rho\mu_n \ell(\Delta)$, that is to say that the length of the polygons P_n is bounded above by a linear function of μ_n. Our goal is to find $\kappa \geqslant 1, c > 0$ and modify the polygons P_n to obtain paths that are $(c\mu_n)$–locally $(\kappa, 0)$–quasigeodesics, and whose lengths are comparable to those of the P_n.

To this end, we restrict our attention to only some edges of P_n. We say that an index $1 \leqslant i \leqslant k$ is *active* if $e_\omega^i \neq \{z_\omega^i\}$. Let $i_1 \leqslant \dots \leqslant i_d$ be the active indices. From now on, we will only consider edges with active indices, and thus we rename $e_\omega^{i_j}$ as a_ω^j and $e_n^{i_j}$ as a_n^j. Thus, $a_\omega^1, \dots, a_\omega^d$ is a subdivision of the triangle Δ into a geodesic d-gon. Since Δ is simple and its edges are compact, we have that there exists a $\delta > 0$ such that for all edges a_ω^i and points $x \in a_\omega^i$ we have

$$\max\{d(x, a_\omega^{i-1}), d(x, a_\omega^{i+1})\} \geqslant \delta.$$

For any active edge a_n^i and $x \in a_n^i$, ω–almost surely we have

$$\max\{d(x, a_n^{i-1}), d(x, a_n^{i+1})\} \geqslant \delta\rho^{-1}\mu_n. \tag{14}$$

The intuitive idea is now as follows. For infinitely many n we have a collection of geodesic segments (a_n^j) whose length keeps increasing (13) and such that we have some control on the distance between them (14). Using this, we can connect the segments to obtain loops of controlled length which are locally quasigeodesics.

More formally, fix n such that both (13), (14) are satisfied and orient the a_n^j with the orientation of P_n that agrees with the numbering. Let $L_n = \frac{1}{2}\delta\rho^{-1}\mu_n$. From now on, we will drop the subscript n and denote, for instance $L_n = L$. Let q^1 be the first point of a^1 such that $d(q^1, a^2) \leqslant L$. By the continuity of the distance function and the choice of q^1 we see that $d(q^1, a^2) = L$.

Let p^2 be a point in a^2 such that $d(q^1, p^2) = L$. Therefore, we see that $d(p^2, a^3) \geqslant \delta\rho^{-1}\mu_n > L$. Let q^2 be the first point in a^2 after p^2 such that $d(q^2, a^3) \leqslant L$. Again, we have $d(q^2, a^3) = L$, and let $p^3 \in a^3$ be a point such that $d(q^2, p^3) = L$. We iterate this procedure until we obtain a point $q^d \in a^d$ and a point $p^1 \in a^1$ such that $d(q^d, p^1) = L$.

We claim that $d(p^j, q^j) \geqslant L$. Indeed, since $d(p^j, a^{j-1}) \leqslant L$, (14) implies $d(p^j, a^{j+1}) \geqslant \delta\rho^{-1}\mu_n = 2L$, and the result follows from the triangle inequality.

From now on, we denote by $[p^j, q^j]$ the restriction of a^j between p^j, q^j, and we choose once and for all geodesic segments $[q^j, p^{j+1}]$ connecting q^j, p^{j+1}. Let $\gamma_n = \gamma$ be the concatenation

$$\gamma = [p^1, q^1] * [q^1, p^2] * \cdots * [q^d, p^1],$$

where we consider γ to be parameterized by arc length. We will show that γ is a $(L; 3, 0)$–local quasigeodesic.

Let x, y be two points of γ of parameterized distance less than L. We denote by $d_\gamma(x, y)$ the parameter distance. We will prove that $d_\gamma(x, y) \leqslant 3d(x, y)$. If a and b are contained in the same segment of γ, then the inequality clearly holds. Thus, we can assume that x and y are on two consecutive segments of γ since the length of each segment of γ is at least L.

Firstly, consider the case $x \in [p^j, q^j]$, $y \in [q^j, p^{j+1}]$. If $x = q^j$, then we would be in the previous case, so $x \neq q^j$. We claim $d(x, y) > d(q^j, y)$. If not, this would contradict the choice of q^j as the first point at distance L from a^{j+1}. Indeed, $d(x, y) \leqslant d(q^j, y)$, implies $d(x, p^{j+1}) \leqslant d(q^j, p^{j+1})$. Therefore, $d(x, y) > d(q^j, y)$. In particular:

$$d_\gamma(x, y) = d(x, q^j) + d(q^j, y) \leqslant \big(d(x, y) + d(y, q^j)\big) + d(q^j, y) \leqslant 3d(x, y).$$

Consider now the case $x \in [q^{j-1}, p^j]$ and $y \in [p^j, q^j]$. Since $d(q^{j-1}, a^j) = d(q^{j-1}, p^j)$, we have $d(x, y) \geqslant d(x, p^j)$. Hence

$$d_\gamma(x, y) = d(x, p^j) + d(p^j, y) \leqslant d(x, p^j) + \big(d(p^j, x) + d(x, y)\big) \leqslant 3d(x, y).$$

Thus, γ is a $(L; 3, 0)$–local quasigeodesic, where $L = L_n = \frac{1}{2}\delta\rho^{-1}\mu_n$. To conclude the proposition, we need to bound the length of γ linearly in terms of μ_n. However, observe that $d(q^j, p^{j+1}) = L$ for all j, and $d(p^j, q^j) \leqslant \ell(a_n^j) \leqslant \rho\mu_n\ell(a_\infty^j)$. Setting $M = \max \ell(a_\infty^j)$, we obtain

$$\ell(\gamma) \leqslant d\left(\frac{1}{2}\delta\rho^{-1}\mu_n + \rho M\mu_n\right) = d\left(\frac{1}{2}\delta\rho^{-1} + \rho M\right)\mu_n. \qquad \square$$

Corollary 3.5. *A a finitely generated group is not hyperbolic if and only if it satisfies* $(\star)$.

The proof of the three theorems in the introduction follow easily from the previous results.

Proof of Theorem 1.3. One direction is given by Theorem 3.4 and the other by Proposition 3.3. □

Proof of Theorem 1.2. One direction is the main result of [HR03]. For the other, let G be a finitely generated group in $\mathbb{Q}\mathbf{REG}$. Now, Proposition 3.2 implies that G does not have the property $(\star)$. Proposition 3.3 and Corollary 3.5 show that for a finitely generated group H the property $(\star)$ is equivalent to H not being hyperbolic. In particular, G is hyperbolic. □

Proof of Theorem 1.4. Let Γ be a graph. An *isometrically embedded circuit* (IEC) is a simplicial loop of length $2n + 1$ such that the restriction of each subsegment of length at most n is a geodesic. We claim that if Γ is geodetic and not hyperbolic then there are IEC of arbitrarily large length. Indeed, by [Pap95b, Theorem 1.4], a Cayley graph is hyperbolic if and only if all geodesic bigons are uniformly thin, i.e. any two geodesics sharing endpoints have uniformly bounded Hausdoff distance.

Since Γ is geodetic, if the geodesic endpoints are vertices, the geodesics need to coincide. It is straightforward to check that if the endpoints are both in an edge one can reduce to a case where at least one endpoint is a vertex. So, the only case left is a bigon where one endpoint is a vertex and the other is the midpoint of an edge. By [EP20, Lemma 4], such a configuration produces an IEC. Since a bound on the Hausdorff distance of the geodesics would imply hyperbolicity, we get arbitrarily large IECs.

Observe that an IEC of length $2n + 1$ is an n–local geodesic. By Proposition 3.2, we conclude that if a group is non-hyperbolic and geodetic, then for any $\lambda > 3$ the set of $(\lambda, 0)$–quasigeodesics cannot form a regular language. Thus such a group must be hyperbolic and geodetic and so, by [Pap93, Section 4], virtually free. □

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