A NOTE ON THE RATIONAL HOMOLOGICAL DIMENSION OF LATTICES IN POSITIVE CHARACTERISTIC

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Abstract. We show via ℓ^2-homology that the rational homological dimension of a lattice in a product of simple simply connected Chevalley groups over global function fields is equal to the rational cohomological dimension and to the dimension of the associated Bruhat–Tits building.

1. Introduction

Let k be the function field of an irreducible projective smooth curve C defined over a finite field $\mathbb{F}_q$. Let S be a finite non-empty set of (closed) points of C. Let $\mathcal{O}_S$ be the ring of rational functions whose poles lie in S. For each $p \in S$ there is a discrete valuation ν_x of k such that $\nu_p(f)$ is the order of vanishing of f at p. The valuation ring $\mathcal{O}_p$ is the ring of functions that do not have a pole at p, that is

$$\mathcal{O}_S = \bigcap_{p \notin S} \mathcal{O}_p.$$

Let $\bar{k}$ denote the algebraic closure of k. Let $\mathbf{G}$ be an affine group scheme defined over $\bar{k}$ such that $\mathbf{G}(\bar{k})$ is almost simple. For each $p \in S$ there is a completion k_p of k and the group $\mathbf{G}(k_p)$ acts on the Bruhat–Tits building X_p. Thus, we may embed $\mathbf{G}(\mathcal{O}_S)$ diagonally into the product $\prod_{p\in S} \mathbf{G}(k_p)$ as an arithmetic lattice.

The *rational cohomological dimension* of a group Γ is defined to be

$$\mathrm{cd}_{\mathbb{Q}}(\Gamma) := \sup\{n\colon H^n(\Gamma; M) \neq 0,\ M \text{ a } \mathbb{Q}\Gamma\text{-module}\},$$

the *rational homological dimension* is defined completely analogously as

$$\mathrm{hd}_{\mathbb{Q}}(\Gamma) := \sup\{n\colon H_n(\Gamma; M) \neq 0,\ M \text{ a } \mathbb{Q}\Gamma\text{-module}\}.$$

In [Gan12] it is shown that $\mathrm{cd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \prod_{p\in S} \dim(X_p)$. In light of this Ian Leary asked the author what is $\mathrm{hd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S))$?

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Theorem A. *Let* $\mathbf{G}$ *be a simple simply connected Chevalley group. Let* k *and* $\mathcal{O}_S$ *be as above, then*

$$\mathrm{hd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \mathrm{cd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \prod_{p\in S}\dim(X_p).$$

More generally, we obtain the following.

Corollary B. *Let* Γ *be a lattice in a product of simple simply connected Chevalley groups over global function fields with associated Bruhat–Tits building* X*, then* $\mathrm{hd}_{\mathbb{Q}}(\Gamma) = \mathrm{cd}_{\mathbb{Q}}(\Gamma) = \dim(X)$*.*

The author expects these results are well-known, however, they do not appear in the literature so we take the opportunity to record them here.

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2. ℓ^2–HOMOLOGY AND MEASURE EQUIVALENCE

Let Γ be a group. Both Γ and the complex group algebra $\mathbb{C}\Gamma$ act by left multiplication on the Hilbert space $\ell^2\Gamma$ of square-summable sequences. The *group von Neumann algebra* $\mathcal{N}\Gamma$ is the ring of Γ-equivariant bounded operators on $\ell^2 G$. The non-zero divisors of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}\Gamma$ can be identified with the *ring of affiliated operators* $\mathcal{U}\Gamma$.

There are inclusions $\mathbb{Q}\Gamma \subseteq \mathcal{N}\Gamma \subseteq \ell^2\Gamma \subseteq \mathcal{U}\Gamma$ and it is also known that $\mathcal{U}\Gamma$ is a self-injective ring which is flat over $\mathcal{N}\Gamma$. For more details concerning these constructions we refer the reader to [Lüc02] and especially to Theorem 8.22 of Section 8.2.3 therein. The *von Neumann dimension* and the basic properties we need can be found in [Lüc02, Section 8.3].

The ℓ^2-Betti numbers of a group Γ, denoted $b_i^{(2)}(\Gamma)$, are then defined to be the von-Neumann dimensions of the homology groups $H_i(\Gamma;\mathcal{U}\Gamma)$. The following lemma is a triviality.

Lemma 2.1. *Let* Γ *be a discrete group and suppose that* $b_i^{(2)}(\Gamma) > 0$*, then the homology group* $H_i(\Gamma;\mathcal{U}\Gamma)$ *is non-trivial.*

Two countable groups Γ and Λ are said to be *measure equivalent* if there exist commuting, measure-preserving, free actions of Γ and Λ on some infinite Lebesgue measure space (Ω, m), such that the action of each of the groups Γ and Λ admits a finite measure fundamental domain. The key examples of measure equivalent groups are lattices in the same locally-compact group [Gro93]. The relevance of this for us is the following deep theorem of Gaboriau.

Theorem 2.2 (Gaboriau's Theorem [Gab02])**.** *Suppose a discrete group Γ is measure equivalent to a discrete group Λ, then $b_p(\Gamma) = 0$ if and only if $b_p(\Lambda) = 0$.*

3. Proofs

Proof of Theorem A. We first note that the group $\Gamma := \mathbf{G}(\mathcal{O}_S)$ is measure equivalent to the product $\Lambda := \prod_{p\in S} \mathbf{G}(\mathbb{F}_q[t_p])$ for some suitably chosen $t_p \in \mathcal{O}_p$. By [PST18, Theorem 1.6] (see also [Dym04; Dym06; Dav+07]) the group $\mathbf{G}(\mathbb{F}_q[t_p])$ has one non-vanishing ℓ^2-Betti number in dimension $\dim(X_p)$. Hence, by the Künneth formula Λ has one non-vanishing ℓ^2-Betti number in dimension $d = \prod_{p\in S} \dim(X_p)$ Thus, by Gaboriau's theorem, the group Γ has exactly one non-vanishing ℓ^2-Betti number in dimension d. It follows from Lemma 2.1 that $\mathrm{hd}_{\mathbb{Q}}(\Gamma) \geqslant d$. The reverse inequality follows from the fact that Γ acts properly on the d-dimensional space $\prod_{p\in S} \dim(X_p)$. ☐

Proof of Corollary B. The proof of the corollary is entirely analogous. First, we split $\mathbf{G}$ into a product of simple groups $\prod_{i=1}^{n} \mathbf{G}_i$ corresponding to the decomposition of the Bruhat–Tits building $X = \prod_{i=1}^{n} X_i$. Let Λ_i be a lattice in $\mathbf{G}_i$ and let $\Lambda = \prod_{i=1}^{n} \Lambda_i$. Each Λ_i has a non-vanishing ℓ^2-Betti Number in dimension $\dim(X_i)$. In particular, Λ has a non-vanishing ℓ^2-Betti Number in dimension $\dim(X) = \prod_{i=1}^{n} \dim(X_i)$. By Gaboriau's Theorem Γ also has non-vanishing ℓ^2-Betti Number in dimension $\dim(X)$. It follows from Lemma 2.1 that $\mathrm{hd}_{\mathbb{Q}}(\Gamma) \geqslant d$. The reverse inequality follows from the fact that Γ acts properly on the d-dimensional space $\prod_{p\in S} X_p$. ☐

Remark 3.1. A similar argument can be applied to lattices in products of simple simply-connected algebraic groups over locally compact p-adic fields. One obtains the analogous result for such a lattice Γ that $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = \mathrm{hd}_{\mathbb{Q}}(\Gamma) = \dim(X)$, where X is the associated Bruhat–Tits building.

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