AUTOMORPHISMS OF RELATIVELY HYPERBOLIC GROUPS AND THE FARRELL–JONES CONJECTURE

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Abstract. We prove the fibred Farrell–Jones Conjecture (FJC) in A-, K-, and L-theory for a large class of suspensions of relatively hyperbolic groups, as well as for all suspensions of one-ended hyperbolic groups. We deduce two applications:

(1) FJC for the automorphism group of a one-ended group hyperbolic relative to virtually polycyclic subgroups;

(2) FJC is closed under extensions of FJC groups with kernel in a large class of relatively hyperbolic groups.

Along the way we prove a number of results about JSJ decompositions of relatively hyperbolic groups which may be of independent interest.

1. Introduction

Let G be a group. The Farrell–Jones Conjecture (FJC) is one of the most prominent open conjectures in algebraic and differential topology. In its simplest form the K-theoretic conjecture predicts that a certain assembly map

$$H_n^G(\mathrm{pr})\colon H_n^G(\underline{\underline{E}}G;\mathbf{K}_R) \to K_n(RG)$$

is an isomorphism. Here $\underline{\underline{E}}G$ is the classifying space for the family of virtually cyclic subgroups, $\mathbf{K}_R$ is the algebraic K-theory spectrum for the ring R, and $K_n(RG)$ is the algebraic K-theory of the group ring RG. There are variants of the conjecture for Waldhausen's A-theory and for L-theory. The conjecture for L-theory, as well as a detailed account of the Farrell–Jones Conjecture, and the objects involved can be found in W. Lück's book project [Lüc]. For recent progress on A-theory the reader should consult [ELP$^+$18].

Computing the algebraic K-theory of a group ring RG is a very difficult problem. In principle, knowing that FJC holds for G gives a method of computing $K_n(RG)$ using equivariant algebraic topology. It also has a number of other applications, for example, to the Borel Conjecture [BL12] and to computing the Whitehead group $\mathrm{Wh}(G)$. Knowledge of $\mathrm{Wh}(G)$ is

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a fundamental step in classifications of higher dimensional manifolds with fundamental group G.

From this point onward, by the Farrell–Jones Conjecture for G, we mean the most general setting, that is, the *fibred Farrell–Jones Conjecture with respect to the family of virtually cyclic subgroups* $\mathcal{VC}$. See for example [ELP$^+$18] and [Lüc] for a discussion of these terms. We will denote the classes of group satisfying the FJC for X-theory by $\mathbf{FJC_X}$ where X is A-, K-, or L-theory.

The class $\mathbf{FJC_K}$ of groups known to satisfy FJC for algebraic K-theory is large: containing hyperbolic groups [BLR08a], many relatively hyperbolic groups [Bar17], CAT(0) groups [Weg12] (see also [BL12] and [KR17]), soluble groups [Weg15], $\mathrm{GL}_n(\mathbb{Z})$ [BLRR14] and more generally lattices in connected Lie groups [BFL14] and S-arithmetic groups [Rüp16], as well as mapping class groups [BB19], normally poly-free groups [BKW21], and suspensions of virtually torsion free hyperbolic groups [BFW23]. The class enjoys many closure properties: it passes to arbitrary subgroups, finite index overgroups, and directed colimits. For more information the reader is referred to the surveys [BLR08b, LR05, Lüc10, Bar16].

One property that is not known is whether $\mathbf{FJC_X}$ is closed under extensions $1 \to N \to \Gamma \to Q \to 1$. One direction of interest is to put conditions on N so that Γ is in $\mathbf{FJC_X}$ whenever Q is. By [BFL14, Theorem 2.7] and [ELP$^+$18, Theorem 1.1(ii)] this reduces to understanding cyclic extensions of G; which is to say the suspensions $N_\Phi = N \rtimes_\Phi \mathbb{Z}$, where Φ is some automorphism of N defining this *suspension*.

Intuitively, a group G is *hyperbolic relative to* $\mathcal{P}$ if its geometry is hyperbolic "away from the subgroups $P \in \mathcal{P}$." One (of many: see [Hru10] for the definitions as well as proofs of their equivalence) way to formalise this uses the notion of coning off a Cayley graph: take a vertex for every coset gP of each element of P, and add an edge from each element of gP to the new vertex. The group G is hyperbolic relative to $\mathcal{P}$ if the resulting graph is δ-hyperbolic in the sense of Gromov, and *fine*: every edge is contained in finitely many cycles of a given finite length. An automorphism of G lies in the subgroup $\mathrm{Aut}(G, \mathcal{P})$ if it preserves the conjugacy classes of every subgroup $P \in \mathcal{P}$. For more information on $\mathrm{Aut}(G, \mathcal{P})$ see [MO12] and [GL15].

Recently, Bestvina, Fujiwara and Wigglesworth [BFW23] proved the suspension of a virtually torsion free hyperbolic group satisfies the Farrell–Jones conjecture. We extend this result to a large class of relatively hyperbolic groups.

Theorem A. *Let $(G, \mathcal{P})$ be a virtually torsion-free or one-ended relatively hyperbolic group with $\mathcal{P}$ finite and let $\Phi \in \mathrm{Aut}(G, \mathcal{P})$. If for every $[P] \in \mathcal{P}$ we have $P_\Phi \in \mathbf{FJC_X}$, then $G_\Phi \in \mathbf{FJC_X}$.*

These hypotheses include, for instance, all suspensions of toral relatively hyperbolic groups and more generally one-ended or virtually torsion-free groups that are hyperbolic relative to virtually polycyclic or soluble subgroups. Note that this removes the assumption of virtual torsion-freeness in [BFW23] for one-ended hyperbolic groups. This is pertinent since it is a well known question of Gromov whether every hyperbolic group is residually finite (and hence virtually torsion-free).

With infinitely ended groups more care is needed, we discuss this further in Section 1.B.

1.A. **Applications.** Our first application is a result on extensions with relatively hyperbolic kernel. A group is *non-relatively hyperbolic* or NRH if it is not hyperbolic relative to a collection of proper subgroups.

Corollary B. *Let $(N, \mathcal{P})$ be a virtually torsion-free or one-ended relatively hyperbolic group such that $\mathcal{P}$ consists of finitely many conjugacy classes of groups which are NRH and whose suspensions $P \rtimes_\Psi \mathbb{Z}$ are in $\mathbf{FJC_X}$ for all automorphisms Ψ of P. Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence. If Q is in $\mathbf{FJC_X}$, then Γ is in $\mathbf{FJC_X}$.*

The assumption that peripheral subgroups are NRH is needed for Corollary B, since it requires Theorem A to hold for arbitrary automorphisms. The key point being that $\mathrm{Aut}(N; \mathcal{P})$ has finite index in $\mathrm{Aut}(N)$ under this extra hypothesis.

It is a major open problem whether $\mathrm{Out}(F_N)$ satisfies $\mathbf{FJC_X}$. Whilst we do not solve this, using Theorem A we are able to show automorphism groups of one-ended groups hyperbolic relative to virtually polycyclic groups satisfy FJC. In particular, $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ for G a one-ended hyperbolic group satisfy $\mathbf{FJC_X}$.

Theorem C. *If G is a one ended group hyperbolic relative to finitely many conjugacy classes of virtually polycyclic groups, then $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ are in $\mathbf{FJC_X}$.*

1.B. **Remarks on the proofs.** As is usual for (relatively) hyperbolic groups, there are two main flavours to our arguments, depending on the number of ends of G. In both cases we apply a result of Knopf [Kno19] allowing us to deduce that a group acting acylindrically on a tree satisfies the Farrell–Jones conjecture if and only if its vertex groups do, though the source of the trees is different in each case.

For one-ended relatively hyperbolic groups, we have access to the powerful machinery of JSJ decompositions developed (in this generality) by Guirardel and Levitt [GL17]. We consider three related trees: the canonical JSJ decomposition T^{can} relative to the peripheral subgroups $\mathcal{P}$, a refinement T^ϕ of T^{can} which better suits the study of an outer automorphism ϕ and another tree that we call T^{Per}. This tree is the canonical JSJ tree relative to the (non-elementary) periodic subgroups of the *outer* automorphism ϕ. That is, we require that the periodic subgroups of every representative $\Phi\,\mathrm{ad}_g$, are elliptic. Our main structural result about this tree is Theorem 5.19: even without assuming that the periodic subgroups are finitely generated, they agree exactly with the rigid vertices of T^{Per}. We prove that this ensures that the induced action of $G \rtimes_\Phi \mathbb{Z}$ is acylindrical, and then analyse vertex groups that can appear in this new action.

We remark that the strong uniqueness properties of the JSJ decomposition imply that for a one-ended, torsion free hyperbolic group, the rigid vertex groups of the tree considered in [BFW23] agree with those in our T^{Per}.

We consider the case when G is infinitely ended and has a finite index subgroup which is a free product of one-ended groups. Being virtually torsion free is sufficient but not necessary for this to occur, and we provide necessary and sufficient conditions in Proposition 8.1. Once we have a free product splitting we have the following combination-type theorem.

Theorem D. *Let $G = G_1 * \ldots * G_k * F_N$ be a free product of finitely generated groups, let $\mathcal{F}' = \{[G_1], \ldots, [G_k]\}$ and let $\Phi \in \mathrm{Aut}(G, \mathcal{F}')$. For every $i \in \{1, \ldots, k\}$, denote by Φ_i an element of the outer class of Φ preserving G_i. If for every $i \in \{1, \ldots, k\}$ the group $G_i \rtimes_{\Phi_i} \mathbb{Z}$ is in $\mathbf{FJC_X}$, then $G \rtimes_\Phi \mathbb{Z}$ is in $\mathbf{FJC_X}$.*

Theorem D is proved by induction on the Grushko rank $k+N$. There are two kinds of induction step, depending on whether the maximal periodic free factor system is *sporadic* or not. A free factor system $(G, \mathcal{F})$ is sporadic if $G \cong G_1 * G_2$ or $G \cong G_1 * \mathbb{Z}$. This division might seem unusual to experts; a more standard division (for instance, in [BFW23] as well as throughout the $\mathrm{Out}(F_n)$ literature) depends instead on if the automorphism is polynomially or exponentially growing. Polynomially growing automorphisms are always sporadic in this sense, but so are some exponentially growing automorphisms. The non-sporadic case uses Dahmani and Li's work on relative hyperbolicity for suspensions of free factors [DL22], whereas in the sporadic case we use the fact that these splittings are *rigid*.

These rigidity arguments hold equally well for sporadic Stallings–Dunwoody decompositions, and so we are still able to obtain some results without first passing to a finite index free product: see Proposition 8.3.

1.C. **Fixed and Periodic Subgroups, and the classes $\mathcal{AC}(\mathbf{VNil})$ versus $\mathbf{FJC_X}$.** Some previous results of this flavour have concluded the stronger property that the suspension is in the class $\mathcal{AC}(\mathbf{VNil})$. Every group in this class satisfies the Farrell–Jones conjecture [BB19]. This class has similar closure properties to the class of groups satisfying the Farrell–Jones conjecture, except that $\mathbf{FJC_X}$ is closed under directed colimits while $\mathcal{AC}(\mathbf{VNil})$ is not known to be. However, for the majority of the paper we work directly with the class $\mathbf{FJC_X}$.

The reason for this is that we have to understand the periodic subgroups of certain automorphisms as an ascending union of fixed subgroups, and consider the action of the automorphism on this subgroup. In general our hypotheses do not guarantee that this union stabilises — we do not have a *virtual neatness* property to rely on.

However, there are hypotheses that ensure virtual neatness, and if we assume these then again the suspensions will be in $\mathcal{AC}(\mathbf{VNil})$. One set of sufficient conditions is,

Theorem 5.22. *Let G be a hyperbolic group relative to a collection $\mathcal{P}$ of slender groups and let $\Phi \in \mathrm{Aut}(G)$. There exists $N \in \mathbb{N}^*$ such that $\mathrm{Per}(\Phi) = \mathrm{Fix}(\Phi^N)$ and $\mathrm{Per}(\Phi)$ is finitely generated.*

If we add these hypotheses to our main theorem, we can prove the suspensions lie in $\mathcal{AC}(\mathbf{VNil})$.

Theorem E. *Suppose $(G, \mathcal{P})$ is one-ended or virtually torsion free, and hyperbolic relative to finitely many conjugacy classes of slender subgroups. Then for every automorphism Φ of G, $\Gamma := G \rtimes_\Phi \mathbb{Z}$ is in $\mathcal{AC}(\mathbf{VNil})$.*

Except for replacing each periodic subgroup with the fixed subgroup of a power, the proof of this theorem is identical to the proof of Theorem A. We discuss this in a little more detail after completing that proof.

1.D. **Structure of the paper.** Section 2 introduces the relevant background results on the Farrell–Jones conjecture.

Section 3 contains definitions and results on free products and their automorphisms, needed for Section 6.

Section 4 collects results on JSJ decompositions of one ended relatively hyperbolic groups, and provides a lemma on acylindricity when passing to the action of a suspension.

The one-ended case of Theorem A is proved in Section 5 by careful analysis of a certain JSJ tree. From this analysis, we also deduce Theorem 5.22.

In Section 6 we prove Theorem D and the infinitely-ended case of Theorem A. Using these results we prove Theorem A and Theorem E.

In Section 7, we deduce Theorem C from Theorem A.

Finally, in Section 8 we extend Theorem A as far as possible with our current techniques to groups which are infinitely ended but do not split as free products. The arguments of the last three sections are almost independent of Sections 4 and 5, apart from requiring the background information on trees of cylinders from Section 4.A.

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2. Background on the Farrell–Jones conjecture

For full context and background on the Farrell–Jones conjecture, see for instance Lück's book project [Lüc]. In this section, we recall some properties of the class $\mathbf{FJC_X}$ of groups which satisfy the Farrell–Jones conjecture for X-theory where X is A, K, or L.

Theorem 2.1. *The class* $\mathbf{FJC_X}$ *is closed under the following operations:*

(1) taking subgroups;
(2) taking finite index overgroups;
(3) finite direct products;
(4) finite free products;
(5) directed colimits.

Proof. The cases of K- and L-theory are given in [GMR15, Theorem 2.1]. The case of A-theory is [ELP+18, Theorem 1.1(ii)]. □

While $\mathbf{FJC_X}$ is not known to be closed under extensions, there is a partial result which we make use of.

Theorem 2.2. *Let* $1 \to N \to \Gamma \to Q \to 1$ *be a short exact sequence with* $N \in \mathbf{FJC_X}$*. If for every infinite cyclic subgroup* C *of* Q*, the preimage of* C *in* Γ *belongs to* $\mathbf{FJC_X}$*, then* Γ *belongs to* $\mathbf{FJC_X}$*.*

Proof. The cases of K- and L-theory are given in [BFL14, Theorem 1.7]. The case of A-theory is given in [ELP$^+$18, Theorem 1.1(ii)]. □

Here is an easy, mild strengthening of commensurability towards virtual isomorphism.

Lemma 2.3. *Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence with N finite. Then if Q is in $\mathbf{FJC_X}$ then so is Γ, and if Γ is residually finite and in $\mathbf{FJC_X}$ then so is Q.*

Proof. For the first statement apply Theorem 2.2 to the short exact sequence, noting that both finite groups and virtually cyclic groups are in $\mathbf{FJC_X}$. For the second, observe that if Γ is residually finite then there is a finite index subgroup Γ_0 of Γ whose intersection with N is trivial, and then $\Gamma_0 \cong Q_0$ for some finite index subgroup Q_0 of Q. The result follows from commensurability. □

We refer for instance to the work of Bowditch [Bow12] for the definition of a relatively hyperbolic group.

Theorem 2.4 (Bartels)**.** *Let G be a group hyperbolic relative to a collection $\{[P_1], \dots, [P_n]\}$ of conjugacy classes of subgroups. If, for every $i \in \{1, \dots, n\}$, we have $P_i \in \mathbf{FJC_X}$, then $G \in \mathbf{FJC_X}$.*

Proof. This result is due to Bartels. The cases of K- and L-theory are [Bar17, Corollary 4.6]. The case of A-theory is also ostensibly due to Bartels combined with some recent developments on the A-theoretic FJC. We sketch the relevant details. The key here is that Bartels' space Δ for a relatively hyperbolic group pair $(G, \mathcal{P})$ is finitely $\mathcal{P}$-amenable (see [Bar17, Theorem 3.1]). By [Kno19, Proof of Theorem 1.8(a)] this implies that G is strongly transfer reducible over $\mathcal{F}$. The result now follows from [ELP$^+$18, Theorem 6.19]. □

Let G be a group acting by isometries on a tree T. Recall that the action is *acylindrical* if there exists $K \geqslant 0$ such that the stabiliser of any geodesic path of length at least K is finite.

Theorem 2.5 (Knopf)**.** *Let G be a group acting acylindrically by isometries on a tree T. If every vertex stabiliser belongs to $\mathbf{FJC_X}$, then G belongs to $\mathbf{FJC_X}$.*

Proof. The result is due to S. Knopf. For K-theory we refer to [Kno19, Corollary 4.2]. The result for L-theory is [Kno19, Corollary 4.3], note that here one has the additional hypothesis that index 2 overgroups of the stabilisers in G must satisfy $\mathbf{FJC_L}$. But this follows from Theorem 2.1. For A-theory, as in Bartels' result, one combines finite $\mathcal{F}$-amenability [Kno19, Proposition 4.1] with the recent developments for A-theory [Kno19, Proof of Theorem 1.8(a)] and [ELP$^+$18, Theorem 6.19]. □

Let G be a group and let $\Phi \in \mathrm{Aut}(G)$. Let $\mathrm{Per}(\Phi) = \langle \mathrm{Fix}(\Phi^n) \rangle_{n \in \mathbb{N}}$ be the periodic subgroup of Φ. At several points in our arguments we will need the following lemma.

Lemma 2.6. *Let G be a group belonging to $\mathbf{FJC_X}$ and let $\Phi \in \mathrm{Aut}(G)$. The group $\mathrm{Per}(\Phi) \rtimes_\Phi \mathbb{Z}$ belongs to $\mathbf{FJC_X}$.*

Proof. Note that

$$\mathrm{Per}(\Phi) \rtimes_\Phi \mathbb{Z} = \langle \mathrm{Fix}(\Phi^n) \rangle_{n\in\mathbb{N}} \rtimes_\Phi \mathbb{Z} = \bigcup_{n\in\mathbb{N}} \left(\mathrm{Fix}(\Phi^{n!}) \rtimes_\Phi \mathbb{Z} \right),$$

which is an increasing union of subgroups. Therefore, by Theorem 2.1 (5), it suffices to prove that, for every $n \in \mathbb{N}$, the group $\mathrm{Fix}(\Phi^{n!}) \rtimes_\Phi \mathbb{Z}$ belongs to $\mathbf{FJC_X}$.

Let $n \in \mathbb{N}$. Note that $\mathrm{Fix}(\Phi^{n!}) \rtimes_{\Phi^{n!}} \mathbb{Z}$ is a finite index subgroup of $\mathrm{Fix}(\Phi^{n!}) \rtimes_\Phi \mathbb{Z}$. By Theorem 2.1 (2), the group $\mathrm{Fix}(\Phi^{n!}) \rtimes_\Phi \mathbb{Z}$ belongs to $\mathbf{FJC_X}$ if and only if the group $\mathrm{Fix}(\Phi^{n!}) \rtimes_{\Phi^{n!}} \mathbb{Z}$ belongs to $\mathbf{FJC_X}$.

The group $\mathrm{Fix}(\Phi^{n!}) \rtimes_{\Phi^{n!}} \mathbb{Z}$ is isomorphic to $\mathrm{Fix}(\Phi^{n!}) \times \mathbb{Z}$. By Theorem 2.1, the group $\mathbb{Z}$ belongs to $\mathbf{FJC_X}$. Since $G \in \mathbf{FJC_X}$ and since $\mathbf{FJC_X}$ is closed under taking subgroups by Theorem 2.1 (1), the group $\mathrm{Fix}(\Phi^{n!})$ belongs to $\mathbf{FJC_X}$. Since $\mathbf{FJC_X}$ is closed under taking direct products by Theorem 2.1 (3), the group $\mathrm{Fix}(\Phi^{n!}) \times \mathbb{Z}$ and hence the group $\mathrm{Per}(\Phi) \rtimes_\Phi \mathbb{Z}$ belongs to $\mathbf{FJC_X}$. □

3. Free Products of Groups and their Automorphisms

3.A. **Free products of groups.** Let $N \in \mathbb{N}$, let $G_1, \ldots, G_k$ be countable groups and let $G = G_1 * \ldots * G_k * F_N$. Let $\mathcal{F} = \{[G_1], \ldots, [G_k]\}$ be the set consisting of the conjugacy classes of the G_i. We refer to $(G, \mathcal{F})$ as a *free product*.

An element $g \in G$ is *peripheral* if there exists $[A] \in \mathcal{F}$ with $g \in A$. Otherwise, g is *nonperipheral*. A subgroup P of G is *peripheral* if every element of P is peripheral, and is *nonperipheral* otherwise.

A *free factor system of* $(G, \mathcal{F})$ is a set $\mathcal{F}' = \{[A_1], \ldots, [A_\ell]\}$ of conjugacy classes of proper subgroups of G such that:

(1) for every $i \in \{1, \ldots, k\}$, there exists $[A] \in \mathcal{F}'$ such that $G_i \subseteq A$;
(2) there exists a subgroup B of G such that $G = A_1 * \ldots * A_\ell * B$.

The set of free factor systems of G is equipped with a partial order where $\mathcal{F}_1 \leqslant \mathcal{F}_2$ if, for every $[A_1] \in \mathcal{F}_1$, there exists $[A_2] \in \mathcal{F}_2$ with $A_1 \subseteq A_2$. A free factor system $\mathcal{F}'$ is *sporadic* if either $\mathcal{F}' = \{[A_1], [A_2]\}$ and $G = A_1 * A_2$ or $\mathcal{F}' = \{[A_1]\}$ and $G = A_1 * \mathbb{Z}$. Otherwise, the free factor system $\mathcal{F}'$ is *nonsporadic*. The free product $(G, \mathcal{F})$ is *sporadic* (resp. *nonsporadic*) if $\mathcal{F}$ is.

We denote by $\mathrm{Aut}(G, \mathcal{F})$ the subgroup of automorphisms of G preserving $\mathcal{F}$ and by $\mathrm{Out}(G, \mathcal{F})$ the subgroup of outer automorphisms of G preserving $\mathcal{F}$. An automorphism $\Phi \in \mathrm{Aut}(G, \mathcal{F})$ is *fully irreducible* if no power of Φ fixes a free factor system of $(G, \mathcal{F})$.

A $(G, \mathcal{F})$*-tree* is a tree equipped with an action of G without inversion such that, for every $[A] \in \mathcal{F}$, the group A is elliptic in T. A *Grushko* $(G, \mathcal{F})$*-tree* is a $(G, \mathcal{F})$-tree T with trivial edge stabilisers and such that, for every $v \in VT$, the conjugacy class of the stabiliser G_v of v is trivial or contained in $\mathcal{F}$.

Let $\mathcal{F}'$ be a sporadic free factor system of $(G, \mathcal{F})$. There is a unique, up to unique G-equivariant homeomorphism, reduced $(G, \mathcal{F}')$-tree $T_{\mathcal{F}'}$, which we call the *Bass-Serre tree* of $(G, \mathcal{F}')$. The tree $T_{\mathcal{F}'}$ has a unique orbit of edges. The tree $T_{\mathcal{F}'}$ is canonical in the sense that every element $\Phi \in \mathrm{Aut}(G, \mathcal{F}')$

induces a G-equivariant homeomorphism of $T_{\mathcal{F}'}$. Therefore, for every $\Phi \in \mathrm{Aut}(G, \mathcal{F}')$, the group $G \rtimes_\Phi \mathbb{Z}$ acts by homeomorphisms on $T_{\mathcal{F}'}$.

3.B. **Growth under an automorphism of a free product.** Let $(G, \mathcal{F})$ be a free product and let T be a Grushko $(G, \mathcal{F})$-tree. We turn T into a metric graph by assigning length 1 to every edge of T.

Let $g \in G$. The *translation length* of g in T is $\|g\|_T = \inf_{x \in T} d(x, gx)$. The translation length of g only depends on the conjugacy class of g.

Let $\Phi \in \mathrm{Aut}(G, \mathcal{F})$. An element $g \in G$ has *$\|.\|_T$-polynomial growth under iteration of* Φ if there exists $P \in \mathbb{Z}[X]$ such that, for every $n \in \mathbb{N}$:

$$\|\Phi^n(g)\|_T \leqslant P(n).$$

Note that any elliptic element of G in T has $\|.\|_T$-polynomial growth under iteration of Φ.

A subgroup P of G is a *$\|.\|_T$-polynomial subgroup of* Φ if there exists an automorphism $\Psi \in \mathrm{Aut}(G, \mathcal{F})$ contained in the outer class of some power of Φ such that $\Psi(P) = P$ and every element of P has $\|.\|_T$-polynomial growth under iteration of Ψ.

Let $\mathcal{P}_T(\Phi)$ be the set of conjugacy classes of maximal $\|.\|_T$-polynomial subgroups of Φ. When Φ is fully irreducible, the set $\mathcal{P}_T(\Phi)$ satisfies some additional properties. Recall that a subgroup A of G is *malnormal* if, for every $g \in G - A$, we have $A \cap gAg^{-1} = \{e\}$.

Proposition 3.1. [DL22, Proposition 1.13] *Let $(G, \mathcal{F})$ be a nonsporadic free product and let $\Phi \in \mathrm{Aut}(G, \mathcal{F})$ be fully irreducible. Let T be a Grushko $(G, \mathcal{F})$-tree.*

(1) The set $\mathcal{P}_T(\Phi)$ is finite.
(2) For every $[A] \in \mathcal{P}_T(\Phi)$, the subgroup A is malnormal in G.

Let $\mathcal{P} = \{[P_1], \dots, [P_\ell]\}$ be a finite set of conjugacy classes of malnormal subgroups of G. Let $\Phi \in \mathrm{Aut}(G)$ be an automorphism such that, for every $i \in \{1, \dots, \ell\}$, there exists $g_i \in G$ such that $\mathrm{ad}_{g_i} \circ \Phi(P_i) = P_i$. The *suspension of* $\mathcal{P}$ is the set $\{[P_i \rtimes_{\mathrm{ad}_{g_i} \circ \Phi} \mathbb{Z}]\}$ considered as a set of conjugacy classes of subgroups of $G \rtimes_\Phi \mathbb{Z}$.

The following result is due to Dahmani–Li [DL22].

Theorem 3.2. [DL22, Corollary 2.3] *Let $(G, \mathcal{F})$ be a nonsporadic free product and let $\Phi \in \mathrm{Aut}(G, \mathcal{F})$ be fully irreducible. Let T be a Grushko $(G, \mathcal{F})$-tree and let $\mathcal{P}_T(\Phi)$ be the set of conjugacy classes of maximal $\|.\|_T$-polynomial subgroups of Φ. There exists $n \in \mathbb{N}$ such that the group $G \rtimes_{\Phi^n} \mathbb{Z}$ is hyperbolic relative to the suspension of $\mathcal{P}_T(\Phi)$.*

Note that Theorem 3.2 implies that the set $\mathcal{P}_T(\Phi)$ does not depend on T when Φ is fully irreducible. In the rest of the section, we give a precise description of the set $\mathcal{P}_T(\Phi)$ for a fully irreducible automorphism.

We first need a result, which can be found for instance in the work of Francaviglia–Martino–Syrigos [FMS21] concerning the existence of a limiting tree of a fully irreducible automorphism.

Lemma 3.3. [FMS21, Lemma 2.14.1] *Let $(G, \mathcal{F})$ be a nonsporadic free product and let $\Phi \in \mathrm{Aut}(G, \mathcal{F})$ be a fully irreducible automorphism. There exist* *a Grushko $(G, \mathcal{F})$-tree S, an $\mathbb{R}$-tree T equipped with an isometric action of G and a constant $\lambda > 1$ such that, for every $g \in G$, we have*

$$\lim_{n\to\infty} \frac{1}{\lambda^n} \|\Phi^n g\|_S = \|g\|_T. \tag{1}$$

Let $(G, \mathcal{F})$ be a free product. Note that, for any subgroup A of G, the free factor system $\mathcal{F}$ induces a free factor system $\mathcal{F}|_A$ of A. Following the terminology of for instance Guirardel-Horbez [GH22, Definition 3.2], an $\mathbb{R}$-tree equipped with an isometric action of G is an *arational* $(G, \mathcal{F})$-tree if the following holds:

(1) the tree T is not a Grushko $(G, \mathcal{F})$-tree;
(2) for every $[A] \in \mathcal{F}$, the group A is elliptic in T;
(3) for every free factor system $\mathcal{F} < \mathcal{F}'$ and every $[A] \in \mathcal{F}'$ such that A is nonperipheral, the action of A on its minimal tree in T is a Grushko $(A, \mathcal{F}|_A)$-tree.

Proposition 3.4. *Let $(G, \mathcal{F})$ be a nonsporadic free product, let $\Phi \in \mathrm{Aut}(G, \mathcal{F})$ be a fully irreducible automorphism and let S be a Grushko $(G, \mathcal{F})$-tree given by Lemma 3.3. For every $[P] \in \mathcal{P}_S(\Phi)$, either $[P] \in \mathcal{F}$ or P is nonperipheral and infinite cyclic.*

Proof. Let T be the $\mathbb{R}$-tree associated with S given by Lemma 3.3. Note that, by Equation (1), for every $\|.\|_S$-polynomially growing element $g \in G$, we have $\|g\|_T = 0$.

By [GH22, Theorems 3.4,4.1] the $\mathbb{R}$-tree T is an *arational* $(G, \mathcal{F})$-tree. It has trivial arc stabilisers (because it is mixing [Hor14, Lemma 4.9]). Thus, for every $[P] \in \mathcal{P}_S(\Phi)$, the group P fixes a point in T.

By [Hor14, Lemma 4.6], using the fact that T is arational, for every point $x \in T$, the stabiliser G_x of x is either peripheral or nonperipheral and infinite cyclic. Thus, for every $[P] \in \mathcal{P}_S(\Phi)$, the elliptic subgroup P is either peripheral or nonperipheral and infinite cyclic. By maximality of P, either $[P] \in \mathcal{F}$ or P is nonperipheral and infinite cyclic. □

4. Actions on trees and JSJ decompositions

4.A. **Tree of cylinders.** Let G be a group acting on a tree T. In order to construct an acylindrical action of G on a tree, we will modify the tree T using the technology of *tree of cylinders* introduced by Guirardel and Levitt [GL11].

Let $\mathcal{E}$ be a class of subgroups of G, stable under conjugation. An $\mathcal{E}$*-tree* is a tree T equipped with an action of G without edge inversion and such that the stabiliser of any edge is contained in $\mathcal{E}$. An equivalence relation $\sim$ on $\mathcal{E}$ is *admissible* if, for any $A, B \in \mathcal{E}$, the following holds:

(1) for any $g \in G$, if $A \sim B$, then $gAg^{-1} \sim gBg^{-1}$;
(2) if $A \subseteq B$, then $A \sim B$;
(3) for every $\mathcal{E}$-tree T, if $A \sim B$ and A and B are elliptic in T, then $\langle A, B\rangle$ is elliptic in T.

Inclusion is an admissible relation for every class of groups $\mathcal{E}$. If $\mathcal{E}$ is the class of virtually infinite cyclic groups, then commensurability is an admissible equivalence relation, where two groups $A, B \in \mathcal{E}$ are commensurable if $A \cap B$ has finite index in both A and B.

Let T be an $\mathcal{E}$-tree and let $\sim$ be an admissible equivalence relation on $\mathcal{E}$. If e is an edge of T, we denote by G_e its stabiliser in T. We define an equivalence relation $\sim_T$ on the set of edges of T by setting, for all edges $e, e' \in ET$, $e \sim_T e'$ if and only if $G_e \sim G_{e'}$. A *cylinder* Y of T is a $\sim_T$-equivalence class, seen as a subforest of T. A cylinder is in fact a subtree of T (see [GL11, Lemma 4.2]).

Definition 4.1. Let T be an $\mathcal{E}$-tree. The *tree of cylinders of* T is the bipartite tree T_c whose vertex set $VT_c = V_0T_c \coprod V_1T_c$ is defined as follows:

(1) V_0T_c is the set of vertices ot T belonging to at least two distinct cylinders;
(2) V_1T_c is the set of cylinders ot T;
(3) there is an edge between $v_0 \in V_0T$ and $v_1 \in V_1T$ if the vertex in T corresponding to v_0 belongs to the cylinder corresponding to v_1.

The tree of cylinders of T is a tree equipped with an action of G without edge inversion.

4.B. **JSJ decompositions of one-ended relatively hyperbolic groups.** We now let G be a one-ended hyperbolic group relative to a family $\mathcal{P} = \{[P_1], \ldots, [P_n]\}$ of conjugacy classes of groups and let $\Phi \in \mathrm{Aut}(G, \mathcal{P})$. Let G_Φ be the suspension $G \rtimes_\Phi \mathbb{Z}$. In time, we will also assume the suspensions of the P_i belong to $\mathbf{FJC_X}$, and want to apply Theorem 2.5 to G_Φ in order to prove that $G_\Phi \in \mathbf{FJC_X}$. That is, we will construct a simplicial tree T on which G_Φ acts acylindrically. The construction of the tree T uses the theory of *JSJ decomposition of groups*, which we now discuss, following the work of Guirardel-Levitt [GL11, GL15, GL17].

A subgroup of G is *elementary* if it is virtually cyclic or conjugate into some P_i with $i \in \{1, \ldots, n\}$. Let $\mathcal{A}$ be the family of all elementary subgroups of G.

Let $\sim_{\mathcal{A}}$ be the equivalence relation on $\mathcal{A}$ given by $A \sim_{\mathcal{A}} B$ if $\langle A, B\rangle$ is elementary. The equivalence relation $\sim_{\mathcal{A}}$ defines an admissible equivalence relation called *coelementarity*.

Let $\mathcal{H}$ be any set of conjugacy classes of subgroups of G. Recall that an $(\mathcal{A}, \mathcal{P} \cup \mathcal{H})$*-tree* is an $\mathcal{A}$-tree T such that, for every $[A] \in \mathcal{P} \cup \mathcal{H}$, the group A is elliptic in T. We denote by $\mathrm{Out}(G, \mathcal{P} \cup \mathcal{H}^{(t)})$ the subgroup of $\mathrm{Out}(G, \mathcal{P} \cup \mathcal{H})$ consisting of every $\psi \in \mathrm{Out}(G, \mathcal{P} \cup \mathcal{H})$ such that, for every $[A] \in \mathcal{H}$, there exists $\Psi \in \psi$ with $\Psi(A) = A$ and $\Psi|_A = \mathrm{id}_A$.

Let T be a tree equipped with an action of G by isometries with a finite number of orbits of edges. If H is a subgroup of $\mathrm{Out}(G, \mathcal{P} \cup \mathcal{H})$ preserving the G-equivariant homeomorphism class of a tree T, we denote by H^0 the finite index subgroup of H acting trivially on $G\backslash T$. Note that, for every $v \in VT$, we have a homomorphism $H^0 \to \mathrm{Out}(G_v)$.

Using [GL17, Theorem 9.18], [GL15, Theorem 3.9] and [GL11, Proposition 6.1], we have the following theorem.

Theorem 4.2. [GL11, GL15, GL17] *Let G be a one-ended hyperbolic group relative to a family $\mathcal{P}$ of non virtually cyclic groups. Let $\mathcal{H}$ be any family of conjugacy classes of subgroups of G. There exists a tree of cylinders $T_{\mathcal{H}}$ for coelementarity equipped with an isometric action of G such that:*

(1) the group $\mathrm{Aut}(G,\mathcal{P}\cup\mathcal{H})$ *preserves the* G*-equivariant homeomorphism class of* $T_{\mathcal{H}}$*;*
(2) the action of G *on* $T_{\mathcal{H}}$ *is* 2*-acylindrical;*
(3) the number of orbits of edges is finite;
(4) edge stabilisers are infinite elementary;
(5) the tree $T_{\mathcal{H}}$ *is an* $(\mathcal{A},\mathcal{P}\cup\mathcal{H})$*-tree;*
(6) vertex stabilisers corresponding to cylinders are elementary subgroups;
(7) vertex stabilisers G_v *corresponding to a vertex* v *of the original tree satisfy one of the following:*
- G_v *is nonelementary and* Quadratically Hanging (QH) with finite fibre*; (see* [GL17, Definition 5.13]*);*
- *the vertex* v *is nonelementary and* rigid*: the stabiliser of* v *is elliptic in every* $(\mathcal{A},\mathcal{P}\cup\mathcal{H})$*-tree (see* [GL17, Definition 2.14]*).*

(8) if e_1, e_2 *are two distinct edges adjacent to the same nonelementary vertex, then* $G_{e_1}\cap G_{e_2}$ *is finite and* $\langle G_{e_1}, G_{e_2}\rangle$ *is not elementary;*
(9) if $[H]\in\mathcal{H}$ *is not elementary, then* H *stabilises a unique rigid vertex. (This follows from* [GL17, Definition 5.13(3)]*, and the fact that QH vertices with finite fibre have virtually cyclic extended boundary subgroups).*

Moreover, if $\mathcal{H}=\{[H_1],\dots,[H_k]\}$ *with every* H_i *finitely generated:*

(10) [GL15, Theorem 3.9] *for every rigid vertex* $v\in VT_{\mathcal{H}}$*, the homomorphism* $\mathrm{Out}^0(G,\mathcal{P}\cup\mathcal{H}^{(t)})\to\mathrm{Out}(G_v)$ *is finite;*
(11) for every edge $e\in ET_{\mathcal{H}}$*, the homomorphism* $\mathrm{Out}^0(G,\mathcal{P}\cup\mathcal{H}^{(t)})\to\mathrm{Out}(G_e)$ *is finite.*

When the family $\mathcal{H}$ is trivial, we will refer to $T_{\mathcal{H}}$ as T^{can}. (Our superscript convention here is certainly not standard: we use it because we will shortly need to discuss minimal invariant trees for subgroups coming from multiple underlying actions. This choice lets us write T^{can}_H, for instance, keeping both the tree and the subgroup conveniently in the notation.)

We now prove a general lemma in order to deduce acylindrical actions of G_Φ with $\Phi\in\mathrm{Aut}(G)$ on trees out of acylindrical actions of G. If G_Φ acts on a tree T, we denote by F_Φ the G-equivariant isometry of T induced by Φ.

Lemma 4.3. *Let* $K\geqslant 1$*, and let* $\phi=[\Phi]\in\mathrm{Out}(G)$*. Suppose that* G_Φ *acts on a tree* T *with finitely many orbits of edges and that the action of* G *on* T *is* K*-acylindrical. Suppose that for every geodesic path* γ *of length* 3 *and every automorphism* $\Psi\in\phi$ *such that* F_Ψ *preserves* γ*, there exist a vertex* v *of* γ *and* $g\in G_v$ *of infinite order fixed by a power of* Ψ*.*

(1) Let $n\in\mathbb{N}^*$ *and let* $\Psi\in\phi^n$*. Suppose that* F_Ψ *fixes pointwise a geodesic edge path of length at least equal to* $2K+7$*. There exists* $N\in\mathbb{N}^*$ *such that* Ψ^N *fixes elementwise a nonabelian free group* $L\subseteq G$ *consisting of loxodromic elements of* T*. Moreover, for every* $g\in L$*, the isometry* F_{Ψ^N} *fixes elementwise the axis of* g*.*
(2) Let $n\in\mathbb{N}^*$ *and let* $\Psi\in\phi^n$*. Suppose that there exists* $g\in\mathrm{Fix}(\Psi)$ *which acts loxodromically on* T*. There exist* $m\in\mathbb{Z}$*,* $n\in\mathbb{Z}^*$ *such that* $g^mF_{\Psi^n}$ *fixes pointwise the axis of* g*.*

(3) The action of G_Φ on T is acylindrical if and only if for every $n \in \mathbb{N}$ and every Ψ in the outer class of Φ^n, the group $\mathrm{Fix}(\Psi)$ *is elliptic in T.*

Proof. (1) Suppose that F_Ψ fixes pointwise a geodesic edge path γ of length $2K+7$. Thus, the path γ is not reduced to an edge and the isometry F_Ψ fixes the initial and the terminal paths γ_1, γ_2 of γ length 3. Since the action of G on T is K-acylindrical, for all vertices v_1 of γ_1 and v_2 of γ_2, the intersection $G_{v_1} \cap G_{v_2}$ is finite.

Let $i \in \{1,2\}$. Note that F_Ψ preserves γ_i. By hypothesis, there exist $N_i \in \mathbb{N}^*$, a vertex $v_i \in \gamma_i$ and an infinite order element $g_i \in G_{v_i}$ which is fixed by Ψ^{N_i}.

Let $N = N_1 N_2$ and let $L = \langle g_1, g_2 \rangle \subseteq \mathrm{Fix}(\Psi^N)$. Since $G_{v_1} \cap G_{v_2}$ is finite, we have $\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2) = \varnothing$. By standard ping-pong arguments, the group L is a non-abelian free group which contains a (non-abelian) subgroup consisting of loxodromic elements of T. This proves the first part of Assertion (1). **Comment 1 (by Naomi)**: the group L certainly isn't purely loxodromic, since its generators are elliptic. I think to cook up the loxodromic guy weaker hypotheses would do: so long as g_1, g_2 are non-trivial and at least one has order at least 3, and the fixed point sets don't intersect (maybe that's harder to achieve in our setup) you can begin to play ping pong

Let $g \in L$ be loxodromic. Since $\Psi^N(g) = g$, the isometry F_{Ψ^N} commutes with g. In particular, the axis of g is contained in the characteristic set of F_{Ψ^N}. Since F_{Ψ^N} is elliptic, this implies that F_{Ψ^N} fixes pointwise the axis of g. This concludes the proof of Assertion (1).

(2) Let $g \in \mathrm{Fix}(\Psi)$ be a loxodromic element. As in the proof of Assertion (1), the axis of g is preserved by F_Ψ and is contained in the characteristic set of F_Ψ. Thus, we have a homomorphism $\Lambda \colon \langle g, F_\Psi \rangle \to \mathbb{R}$ given by the translation length on the axis of g. Since T is a simplicial tree, the image of Λ is a discrete subset of $\mathbb{R}$. Thus, the image of Λ is cyclic. Therefore, the kernel of Λ is nontrivial: there exist $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$ such that $g^m F_{\Psi^n}$ fixes pointwise the axis of g. This proves Assertion (2).

(3) Suppose that there exist $n \in \mathbb{N}^*$, $\Psi \in \phi^n$ and $g \in \mathrm{Fix}(\Psi)$ such that the action of g on T is loxodromic. By Assertion (2), there exist $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^*$ such that $g^m T_{\Psi^n}$ fixes pointwise the axis of g. Hence the action of G_Φ is not acylindrical.

Conversely, suppose that the action of $G_\Phi = \langle G, t \rangle$ on T is not acylindrical. Since the action of G on T is K-acylindrical, there exist $g \in G$ and $k \in \mathbb{N}^*$ such that the element gt^k fixes an edge path of length $2K+7$. Let $\Psi \in \phi^k$ be the automorphism corresponding to gt^k. By Assertion (1), some power of Ψ fixes a loxodromic element of G. This proves Assertion (3) and concludes the proof. ∎

5. The periodic JSJ decomposition

We now specialise to the tree we will use to prove that suspensions of one-ended relatively hyperbolic groups (under reasonable assumptions on the parabolic subgroups) satisfy the Farrell–Jones conjecture. Let G be a one-ended relatively hyperbolic group and let $\Phi \in \mathrm{Aut}(G)$. We explain in the following section the construction of G-trees which are naturally associated with Φ.

5.A. **Trees associated with an automorphism of a one-ended relatively hyperbolic group.** Recall that, if $\Phi \in \mathrm{Aut}(G)$, we denote by $\mathrm{Per}(\Phi)$ the subgroup of G consisting of all $g \in G$ such that there exists $n \in \mathbb{N}^*$ with $\Phi^n(g) = g$. Let $\phi = [\Phi] \in \mathrm{Out}(G)$. We denote by $\mathrm{NP}(\phi)$ the set of all representatives $\Phi \in \phi$ such that $\mathrm{Per}(\Phi)$ is not an elementary subgroup. If $\phi \in \mathrm{Out}(G)$, we set $\mathrm{Per}_{\mathrm{NP}}(\phi) = \{[\mathrm{Per}(\Phi)]\}_{\Phi \in \mathrm{NP}(\phi)}$.

We work with three ϕ-invariant trees for G. The first is the canonical JSJ tree T^{can}, the second is the tree T^{Per} obtained by applying Theorem 4.2 with $\mathcal{H} = \bigcup_{n \in \mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$. The third one is obtained from T^{can} by blowing-up JSJ trees at QH with fibre vertices. The following lemmas motivate the construction.

Lemma 5.1. *Let G be a hyperbolic group relative to $\mathcal{P}$ and let $\phi = [\Phi] \in \mathrm{Out}(G, \mathcal{P})$. Let $n \in \mathbb{N}^*$ and let $\Psi, \Theta \in \phi^n$ be such that Ψ and Θ fix elementwise the same nonelementary subgroup H of G.*

There exists $N \in \mathbb{N}^*$ such that, $\Psi^N = \Theta^N$.

Proof. Since Ψ and Θ fix H elementwise, Ψ and Θ differ by an inner automorphism in the centraliser of H. Since H is nonelementary, its centraliser is finite (see for instance [Osi06, Theorem 4.19]). Thus, up to taking powers of Ψ and Θ fixing elementwise the centraliser of H, we have $\Psi = \mathrm{ad}_g \circ \Theta$ where $g \in C_G(H)$ and $g \in \mathrm{Fix}(\Theta)$. Thus, for every $m \geqslant 1$, we have $\Psi^m = \mathrm{ad}_{g^m} \circ \Theta^m$. As g is finite order, there exists $N \in \mathbb{N}^*$ such that $\Psi^N = \Theta^N$. □

Lemma 5.2. *Suppose G is the vertex group of a QH with fibre vertex of T^{can} and let $\phi = [\Phi] \in \mathrm{Out}(G)$.*

1. *The group $\mathrm{Per}(\Phi)$ is finitely generated, and there is some $k \in \mathbb{N}$ so that $\mathrm{Per}(\Phi) = \mathrm{Fix}(\Phi^k)$.*
2. *There exists $k \in \mathbb{N}^*$ so that if $[g]$ is a periodic conjugacy class of ϕ then $[g]$ is fixed by ϕ^k.*
3. *As Φ varies over the outer classes ϕ^n with $n \in \mathbb{N}^*$, there are only finitely many conjugacy classes of periodic subgroups $\mathrm{Per}(\Phi)$.*
4. *There exists $k \in \mathbb{N}^*$ so that if $[K]$ is a conjugacy class of periodic subgroups of ϕ, then $[K]$ is fixed by ϕ^k.*

This result does not seem surprising, and in fact the same statement is true for all hyperbolic groups (see Theorem 5.22). However, this special case is necessary to begin the arguments on JSJ decompositions we use throughout this section, including to prove the general statement.

Proof. We first prove Lemma 5.2 when G is a hyperbolic 2-orbifold. Recall that hyperbolic 2-orbifolds are good, and let H be a characteristic finite index subgroup of G corresponding to an orientable surface cover of the orbifold. (This can be obtained by taking the characteristic core of the subgroup corresponding to any such cover, since G is finitely generated.) In particular, Φ preserves H. We now consider two periodic subgroups: $\mathrm{Per}_G(\Phi) \leqslant G$ and its subgroup $\mathrm{Per}_H(\Phi|_H) \leqslant H$. If g_1 and g_2 are elements of

$\mathrm{Per}_G(\Phi)$ representing the same coset of H in G, then in fact they represent the same coset of $\mathrm{Per}_H(\Phi|_H)$ in $\mathrm{Per}_G(\Phi)$, so this is a finite index subgroup. The restriction $\Phi|_H$ can be represented by an element of the mapping class group of the surface, and it follows from [Iva92] that periodic subgroups here are finitely generated. Finite generation is a commensurability invariant, so the same is true of $\mathrm{Per}_G(\Phi)$. Then taking a sufficiently high power to fix every element of a finite generating set shows that $\mathrm{Per}(\Phi) = \mathrm{Fix}(\Phi^k)$.

Let $[g]$ be a ϕ-periodic conjugacy class. If g has finite order, then, as there exists finitely many conjugacy classes of finite order elements in G, some power of ϕ fixes $[g]$. Suppose now that g has infinite order, and let $t \in \mathbb{N}^*$ be such that $g^t \in H$. Since G is hyperbolic, g^t has finitely many t-th roots in G, the number of such roots depending only on the finite numbers of orders of the finite subgroups of G. Thus, if $\ell \in \mathbb{N}^*$ is such that $\phi^\ell \in \mathrm{Out}(H)$ fixes the conjugacy class of g^t, then a power of ϕ^ℓ fixes the conjugacy class of g and this power does not depend on g. If H is a free group, then the existence (and uniformity) of ℓ follows from the work of Handel–Mosher [HM20, Theorem II.4.1]. If H is the fundamental group of a closed orientable surface, this follows from the work of Ivanov [Iva92].

Since the third statement is true for free and surface groups (by Ivanov [Iva92] for the surface case, Bestvina–Handel [BH92] for the free case with noncyclic periodic subgroups, and for instance Guirardel–Levitt [GL16] for the general case), it will suffice to bound the number of subgroups $\mathrm{Per}_G(\Phi)$ containing (with finite index) a given restriction $\mathrm{Per}_H(\Phi|_H)$. If this is non-elementary, then it follows from Lemma 5.1 that any two automorphisms in ϕ fixing it have a common power, and hence the same periodic subgroups. (First replace $\Phi^k \,\mathrm{ad}(g_1)$ and $\Phi^\ell \,\mathrm{ad}(g_2)$ with their ℓ-th and k-th powers respectively, so they represent the same outer automorphism, then another power so as to fix the common non-elementary subgroup $\mathrm{Per}_H(\Phi|_H)$, then apply the lemma as written.)

Now assume $\mathrm{Per}_H(\Phi|_H)$ is elementary, and we want to control the periodic subgroups of Φ in G restricting to it. Recall that in a hyperbolic group, every virtually cyclic subgroup is contained in a unique maximal one, and let M be the maximal virtually cyclic subgroup containing $\mathrm{Per}_H(\Phi|_H)$. Since Φ preserves $\mathrm{Per}_H(\Phi|_H)$, it must also preserve M, and we consider the induced automorphism of M. By for instance [MO10, Lemma 6.6], $\mathrm{Out}(M)$ is finite, and so passing to a power Φ^k the induced automorphism is inner. Composing with an inner automorphism coming from M, some representative Ψ of Φ^k fixes M; in particular M is itself a periodic subgroup.

Note that this inclusion between the original periodic subgroup and the one for Ψ can stay proper at all powers: the infinite order inner automorphism of D_∞ gives an example.

Finally, as there exist only finitely many conjugacy classes of finite subgroups in G, there exist only finitely many conjugacy classes of finite periodic subgroups for any power of ϕ.

The final property follows from taking a high enough power to fix (up to composing with appropriate inner automorphisms) the generating sets of a representative of each conjugacy class.

Let G be the vertex group of a QH with finite fibre vertex. Let F be a finite normal subgroup of G_v such that G_v/F is isomorphic to the fundamental group $\pi_1(\Sigma_v)$ of a 2-orbifold Σ_v. Since G_v is a hyperbolic group, it contains finitely many conjugacy classes of finite subgroups. Thus, for every $\phi \in \mathrm{Out}(G_v)$, there exists $M \geqslant 1$ such that ϕ^M induces an element $\phi^M|_{\Sigma_v} \in \mathrm{Out}(\pi_1(\Sigma_v))$.

Let $\Phi \in \mathrm{Aut}(G)$. Up to taking a power of Φ, we may suppose that Φ induces an element of $\mathrm{Aut}(\pi_1(\Sigma_v))$. Let $\ell \in \mathbb{N}^*$ be the integer associated with $\Phi|_{\pi_1(\Sigma_v)}$ which satisfies both Assertions (1) and (2). Let $g \in \mathrm{Per}(\Phi)$. Then the image of g in $\pi_1(\Sigma_v)$ is fixed by $\Phi^\ell|_{\pi_1(\Sigma_v)}$. Thus, Φ^ℓ preserves the left coset gF. As F is finite, the automorphism $\Phi^{\ell|\mathrm{Aut}(F)|}$ acts trivially on F. Thus, we see that $\Phi^{\ell|F||\mathrm{Aut}(F)|}$ fixes g. This proves Assertion (1). Similarly, suppose that $[g]$ is a periodic conjugacy class. Then $\Phi^\ell|_{\pi_1(\Sigma_v)}$ preserves the conjugacy class in $\pi_1(\Sigma_v)$ induced by g. Thus, Φ^ℓ sends g to $hk_1gk_2h^{-1}$ with $h \in G$ and $k_1, k_2 \in F$. Let $\Psi^\ell = \mathrm{ad}_{h^{-1}} \circ \Phi^\ell$. Then Ψ^ℓ sends g to k_1gk_2. Note that $\Psi^{\ell|\mathrm{Aut}(F)|}$ acts trivially on F and sends g to $k_1'gk_2'$ with $k_1', k_2' \in F$. Thus, $\Psi^{\ell|\mathrm{Aut}(F)||F|}$ fixes g and $\Phi^{\ell|\mathrm{Aut}(F)||F|}$ fixes the conjugacy class of g. This proves Assertion (2).

For the third assertion, notice that the periodic subgroups of the induced action on $\pi_1(\Sigma_v)$ contain (by passing to a finite index surface subgroup, and if necessary then an infinite index one) a preserved (and periodic) free group. This splits back to G_v, and since the arguments given earlier used only Lemma 5.1 and properties of virtually cyclic subgroups of hyperbolic groups, they apply equally well here (note that it is enough to take *any* non-elementary periodic subgroup to apply Lemma 5.1). Again, the final assertion follows by taking a high enough power to fix (up to composing with appropriate inner automorphisms) the generating sets of a representative of each conjugacy class. □

Let v be a QH with fibre vertex of T^{can}, let $e_1, \ldots, e_k$ be representatives of the G_v-orbits of edges in T^{can} adjacent to v and let T_v be the JSJ tree of G_v relative to $\bigcup_{n \in \mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}([\phi^n|_{G_v}]) \cup \{[G_{e_1}], \ldots, [G_{e_k}]\}$. The idea is to blow up, at every such vertex v the tree T_v. But, we want the resulting tree to be *compatible* with T^{Per}, so that we need to be careful when attaching the edges of T^{can} to vertices in T_v.

Two $(\mathcal{A}, \mathcal{P})$-trees T and T' are *compatible* if there exists an $(\mathcal{A}, \mathcal{P})$-tree U such that both T and T' are obtained from U by collapsing some orbits of edges. By [GL17, Proposition A.26], there exists a unique such minimal tree U which refines T and T'. The tree U satisfies the following properties: a subgroup H of G stabilises a point in U if and only if H stabilises a point in both T and T'. Moreover, for every edge $e \in EU$, the image of e in either T or T' is not reduced to a point.

By universality of T^{can} (see [GL17, Corollary 9.18(3)]), the trees T^{can} and T^{Per} are compatible. We denote by T_0^{ref} their minimal refinement. By minimality, since ϕ preserves both T^{can} and T^{Per}, the tree T_0^{ref} is preserved by ϕ. The tree T_0^{ref} is obtained from T^{can} by blowing up, for every vertex $v \in T^{\mathrm{can}}$, a tree S_v. The tree S_v is the minimal G_v-tree in T_0^{ref}.

Suppose that v is a QH vertex of T^{can} with finite fibre. Let H be a subgroup of G such that $[H] \in \bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}([\phi^n|_{G_v}]) \cup \{[G_{e_1}], \ldots, [G_{e_k}]\}$. Then H is elliptic in both T^{can} and T^{Per}, so that H is elliptic in T_0^{ref}. Since S_v is the minimal G_v-tree in T_0^{ref}, the group H is also elliptic in S_v. Thus, by universality of T_v, the trees T_v and S_v are compatible. Let U_v be their common minimal refinement. The tree U_v is invariant by the outer automorphism of G_v induced by ϕ. Moreover, since both S_v and T_v are 2-acylindrical, the minimality of U_v implies that U_v is 4-acylindrical.

Let T_1^{ref} be the tree obtained from T_0^{ref} by blowing up the trees U_v at every tree S_v of T_0^{ref} and attaching an edge e of T_0^{ref} to the centre of the fixed point set of $\mathrm{Stab}(e)$ in U_v. The centre exists since for every $i \in \{1, \ldots, k\}$, the group G_{e_i} is elliptic in U_v and U_v is 4-acylindrical. It is the (necessarily common) midpoint of the longest paths in the fixed point set of $\mathrm{Stab}(e)$. The centre is not necessarily a vertex of U_v, so this construction might require to subdivide some edges of U_v.

Note that the tree T_1^{ref} is preserved by ϕ as it is obtained from T_0^{ref} by blowing-up in a canonical way trees preserved by ϕ.

Let T_0^{ϕ} be the tree obtained from T_1^{ref} by the following operations. We have a natural collapse map $q\colon T_1^{\mathrm{ref}} \to T^{\mathrm{can}}$. Let $v \in VT^{\mathrm{can}}$. If v is not QH with fibre, then collapse $q^{-1}(v)$ to a point. If v is QH with fibre, then $q^{-1}(v) = U_v$ and we collapse U_v to the tree T_v. The resulting tree is our desired T_0^{ϕ}. Note that, since T_0^{ϕ} is obtained from T^{can} by blowing up at each vertex $v \in VT^{\mathrm{can}}$, trees which are invariant by ϕ (namely, the trees T_v), the tree T_0^{ϕ} is also invariant by ϕ.

Notice that T_1^{ref} is a common refinement of T_0^{ϕ} and T^{Per}, so that T_0^{ϕ} and T^{Per} are compatible. Since T^{can} and all the trees T_v are acylindrical, so is T_0^{ϕ}.

Finally, let T^{ϕ} be the tree obtained from T_0^{ϕ} by collapsing all the edges whose endpoints are both elementary. Note that the resulting new vertices are elementary since every infinite elementary subgroup is contained in a unique maximal elementary one (see [GL15, Lemma 3.1]).

The tree T^{ϕ} is preserved by ϕ, it is compatible with T^{Per} and the action of G on T^{ϕ} is acylindrical. Additionally, the tree T^{ϕ} is a bipartite tree: every edge has an endpoint which is elementary and an endpoint which is either rigid or QH with fibre. (Here, rigid vertices could correspond to rigid vertices either in T^{can} or in some T_v, while QH vertices are QH in a T_v, though T_v could be a point.) Note that, unlike T^{can}, the tree T^{ϕ} is not necessarily preserved by every element of $\mathrm{Out}(G, \mathcal{P})$. However, it is preserved by ϕ, which is sufficient for our considerations. Moreover, the tree T^{ϕ} is not necessarily compatible with every $(\mathcal{A}, \mathcal{P})$-tree, but we will only need the fact that it is compatible with T^{Per}. The fact that we replace T^{can} by T^{ϕ} is due to the following lemmas.

Lemma 5.3. *Let $v \in VT^{\phi}$ be either rigid or QH with fibre and let $e_1, e_2 \in ET^{\phi}$ be two distinct edges adjacent to v. Then $G_{e_1} \cap G_{e_2}$ is finite and $\langle G_{e_1}, G_{e_2}\rangle$ is not elementary. In particular, the tree T^{ϕ} is 2-acylindrical.*

Proof. Note that, since v is rigid or QH with fibre, the preimages of e_1 and e_2 in T_0^ϕ are unique edges since T^ϕ is obtained from T_0^ϕ by collapsing edges with elementary endpoints. Thus, it suffices to prove the result for T_0^ϕ.

Let $v \in VT_0^\phi$ be either rigid or QH with fibre and let $e_1, e_2 \in ET_0^\phi$ be two distinct edges adjacent to v. Suppose first that both e_1 and e_2 are edges in either T^{can} or T_w for some $w \in VT^{\text{can}}$ which is QH with fibre. Then $G_{e_1} \cap G_{e_2}$ is finite and $\langle G_{e_1}, G_{e_2}\rangle$ is not elementary by Theorem 4.2 (8).

Thus, we may suppose, up to reordering, that e_1 is an edge coming from T^{can} and e_2 is an edge coming from some T_w with $w \in VT^{\text{can}}$ QH with fibre. In particular, since w is QH with fibre, the group G_{e_1} is virtually cyclic.

Note that e_1 is attached to the centre of the fixed point set of G_{e_1} in T_w. Moreover, by Theorem 4.2 (8), this centre is not elementary if and only if the fixed point set is reduced to a point. Thus, the fixed point set of G_{e_1} in T_w is reduced to v (seen as a point in T_w).

Suppose towards a contradiction that $G_{e_1} \cap G_{e_2}$ is infinite and let v' be the endpoint of e_2 distinct from v. As explained above, the point v' is not fixed by G_{e_1}. Let $g \in G_{e_1}$ be such that $gv' \neq v'$. Then one of the endpoints of ge_2 is v since G_{e_1} fixes v. Moreover, since G_{e_1} is virtually cyclic, the intersection $G_{e_2} \cap gG_{e_2}g^{-1}$ is infinite (it contains the normal core in G_{e_1} of the intersection $G_{e_1} \cap G_{e_2}$). Therefore, e_2 and ge_2 are two distinct edges of T_w adjacent to the nonelementary vertex v such that $G_{e_2} \cap gG_{e_2}g^{-1}$ is infinite. This contradicts Theorem 4.2 (8). Thus, the intersection $G_{e_1} \cap G_{e_2}$ is finite.

We now prove that $\langle G_{e_1}, G_{e_2}\rangle$ is not elementary. Since w is a QH with fibre vertex of T^{can}, the group G_w is Gromov hyperbolic. Therefore, the only elementary subgroups of G contained in G_w are virtually cyclic. Since $G_{e_1} \cap G_{e_2}$ is finite, the group $\langle G_{e_1}, G_{e_2}\rangle$ is not virtually cyclic, hence is not elementary.

The fact that T^ϕ is 2-acylindrical follows from the fact that any edge of T^ϕ has an endpoint which is nonelementary. □

Lemma 5.4. *Let $v \in T^\phi$ be rigid. The map $\langle\phi\rangle^0 \to \mathrm{Out}(G_v)$ has finite image.*

Proof. Since v is rigid, the group G_v is the stabiliser of a rigid vertex of either T^{can} or some T_w where $w \in VT^{\text{can}}$ is QH with finite fibre.

If G_v is the stabiliser of a rigid vertex of T^{can}, then, by Theorem 4.2 (10), then map $\langle\phi\rangle^0 \to \mathrm{Out}(G_v)$ has finite image.

Suppose now that G_v is the stabiliser of a rigid vertex of some T_w where $w \in VT^{\text{can}}$ is QH with finite fibre. By Lemma 5.2, the set $\mathrm{Per}_{\mathrm{NP}}([\phi|_{G_w}]) \cup \{[G_{e_1}], \dots, [G_{e_k}]\}$ is a finite set of conjugacy classes of finitely generated subgroups of T_w. Thus, Theorem 4.2 (10) applies (to T_w) and the image of $\langle\phi\rangle^0 \to \mathrm{Out}(G_v)$ is finite. □

Lemma 5.5. *Let $v \in T^\phi$ and let $[H] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$.*

1. *Suppose that v is QH with fibre. Then $H \cap G_v$ is elementary.*
2. *Suppose that v is rigid. If $H \cap G_v$ is nonelementary, then $G_v \subseteq H$.*

Proof. Suppose first that v is QH with fibre. By construction of T^ϕ, the group G_v is the stabiliser of a QH with fibre vertex of some JSJ tree T_w, where w is a QH with fibre vertex of T^{can}. But the intersection of G_v with every subgroup H' such that $[H'] \in \mathrm{Per}_{\mathrm{NP}}([\phi|_{G_v}]) \cup \{[G_{e_1}], \dots, [G_{e_k}]\}$ is elementary by the definition of QH with fibre vertices. Thus, the intersection of H with G_v is elementary. This proves Assertion (1).

Suppose now that v is rigid.

Let $\Psi \in \phi$ be such that $\mathrm{Per}(\Psi) = H$. Suppose that the intersection $H \cap G_v$ is nonelementary. Then Ψ preserves G_v since v is the unique vertex in T^ϕ fixed by $H \cap G_v$. By Lemma 5.4, there exists $k \in \mathbb{N}^*$ such that Ψ^k acts as a global conjugation on G_v by an element $g \in G_v$. Taking a larger k if necessary, we may also assume that Ψ^k also act trivially on a nonelementary subgroup $H' \subseteq H \cap G_v$.

Note that $\Theta = \mathrm{ad}_{g^{-1}} \circ \Psi^k$ acts trivially on G_v. Thus, Ψ^k and Θ acts trivially on the same nonelementary subgroup $H' \subseteq G_v$. By Lemma 5.1, there exists $N \in \mathbb{N}$ such that $\Psi^{kN} = \Theta^N$. In particular, Ψ has a power which fixes G_v elementwise. Thus, we have $G_v \subseteq H$. □

Remark 5.6. Note that, for every $\Phi \in \mathrm{NP}(\phi)$, the isometry F_Φ of T^ϕ is elliptic. Indeed, if F_Φ is loxodromic, then Φ can only fix an element $g \in G$ which is loxodromic and whose axis is the same as the one of F_Φ. Since the action of G on T^ϕ is acylindrical, the element g is contained in a unique maximal virtually cyclic subgroup. In particular, $\mathrm{Per}(\Phi)$ is a virtually cyclic group and $\Phi \notin \mathrm{NP}(\phi)$.

Let G be a one-ended hyperbolic group relative to $\mathcal{P}$ and let $\phi = [\Phi] \in \mathrm{Out}(G, \mathcal{P})$. The rest of the section is dedicated to the proof of some properties of the set $\mathrm{Per}_{\mathrm{NP}}(\phi)$ and of the action of G_Φ on T^ϕ. To this end we prove that $\mathrm{Per}_{\mathrm{NP}}(\phi)$ is finite (see Lemma 5.11). We need the following lemmas regarding the intersection of characteristic sets of isometries in T^ϕ.

Lemma 5.7. *Let G be a one-ended hyperbolic group relative to $\mathcal{P}$, let $v \in VT^\phi$ be non-elementary and let e_1 and e_2 be distinct edges adjacent to v. Let $\phi \in \mathrm{Out}(G, \mathcal{P})$. If there is a representative $\Phi \in \phi$ such that $\Phi(G_{e_i}) = G_{e_i}$ for each $i = 1, 2$, then v is rigid and G_v is fixed elementwise by some power of Φ.*

Proof. We first prove that v is rigid. Indeed, suppose towards a contradiction that v is QH with finite fibre. Then G_{e_1} and G_{e_2} are virtually cyclic. Thus, for every $i \in \{1, 2\}$, the automorphism Φ has a power Φ^k fixing an infinite order element $g_i \in G_{e_i}$. By Lemma 5.3 and as the groups G_{e_1} and G_{e_2} are virtually cyclic, the group $\langle g_1, g_2 \rangle$ is a non-elementary subgroup. As $\langle g_1, g_2 \rangle \subseteq \mathrm{Per}(\Phi) \cap G_v$, this contradicts Lemma 5.5. Thus, the vertex v is rigid.

By Lemma 5.4, after taking a power Φ^ℓ, it acts on G_v as global conjugation by an element $g \in G_v$.

We claim that, after taking a further power of Φ, the element g is trivial. Indeed, note that, by Lemma 5.3 the stabiliser of an edge adjacent to v is almost malnormal in G_v: for every edge e' of T^ϕ adjacent to v and every $g' \in G_v$, if $g' G_e g'^{-1} \cap G_e$ is infinite then $g' \in G_e$. Moreover, if e and e' are two distinct edges adjacent to v, then $G_e \cap G_{e'}$ is finite. Since Φ preserves G_{e_1} and G_{e_2}, the power Φ^ℓ must act by conjugating by an element of the finite intersection $G_{e_1} \cap G_{e_2}$. This becomes trivial on taking a further power of Φ, which concludes the proof of the claim.

Thus we have shown that Φ has a power fixing G_v elementwise, which concludes the proof. □

Corollary 5.8. *Let G be a one-ended hyperbolic group relative to $\mathcal{P}$ and let $\phi = [\Phi] \in \mathrm{Out}(G, \mathcal{P})$. Let $[H] = [\mathrm{Per}(\Phi)] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$ and let T^{ϕ}_H be the minimal H-invariant subtree of T^{ϕ}.*

If T^{ϕ}_H contains an edge e, then the endpoints of e are respectively elementary and rigid. If v is the rigid endpoint of e, then $G_e \subseteq G_v \subseteq H$ and G_v is fixed elementwise by some power of Φ.

Proof. Note that, since T^{ϕ} is bipartite, the endpoints of e are respectively elementary and rigid or QH with fibre. It suffices to prove that an endpoint v of e cannot be QH with fibre.

Since T^{ϕ}_H contains an edge, it follows that H is not elliptic in T^{ϕ}. Thus, T^{ϕ}_H is the union of the axes of elements of H. Then, e is contained in the axis of an element g of H. Recalling the bipartite structure of T^{ϕ}, let v be the rigid or QH vertex adjacent to e. Then there exists an edge $e' \neq e$ adjacent to v and contained in the axis of g. Let $\Phi \in \phi$ be such that $\mathrm{Per}(\Phi) = H$ and let $N \in \mathbb{N}^*$ be such that $\Phi^N(g) = g$. Since Φ^N is elliptic in T^{ϕ} by Remark 5.6, it fixes pointwise the axis of g. In particular, it fixes e and e'. By Lemma 5.7, we see that v is rigid and that G_v is fixed elementwise by a power of Φ. □

Corollary 5.9. *Let G be a one-ended hyperbolic group relative to $\mathcal{P}$ and let $\phi = [\Phi] \in \mathrm{Out}(G, \mathcal{P})$. Let $[H] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$ and let T^{ϕ}_H be the minimal H-invariant subtree of T^{ϕ}.*

The tree T^{ϕ}_H does not contain a QH with fibre vertex.

Proof. Suppose first that T^{ϕ}_H is reduced to a point v. Then $H \subseteq G_v$ and v is not QH with fibre by Lemma 5.5. Suppose now that T^{ϕ}_H is not reduced to a point. Then any vertex v of T^{ϕ}_H is adjacent to an edge and the result follows from Corollary 5.8. □

Lemma 5.10. *Let G be a one-ended hyperbolic group relative to $\mathcal{P}$ and let $\phi = [\Phi] \in \mathrm{Out}(G, \mathcal{P})$. Let $n \in \mathbb{N}^*$ and let $\Psi, \Theta \in \phi^n$ be such that F_Ψ and F_Θ are elliptic isometries of T^{ϕ}. Suppose that there exist $g, h \in G$ loxodromic in T^{ϕ}, such that $\Psi(g) = g$, $\Theta(h) = h$ and $\mathrm{Ax}(g) \cap \mathrm{Ax}(h)$ contains an edge e.*

There exists $N \in \mathbb{N}^*$ such that $\Psi^N = \Theta^N$.

Proof. First note that F_Ψ (resp. F_Θ) fixes pointwise the axis of g (resp. h). Therefore, both Ψ and Θ preserves the stabilisers of the endpoints of e. By construction of T^{ϕ}, one of the endpoints v of e is either a rigid or a QH with fibre vertex. Moreover, both Ψ and Θ preserve the subgroup associated with an edge adjacent to v distinct from e.

Therefore, we can apply Lemma 5.7: there exists $N \in \mathbb{N}^*$ such that both Ψ^N and Θ^N act as the identity on the nonelementary subgroup G_v. By Lemma 5.1, up to taking powers of Ψ and Θ, we have $\Psi^N = \Theta^N$. □

Lemma 5.11. *Let G be a one-ended relatively hyperbolic group and let $\phi = [\Phi] \in \mathrm{Out}(G)$. The set $\bigcup_{n\in\mathbb{N}} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$ is finite. Consequently, there exists $N \in \mathbb{N}^*$ such that any subgroup whose conjugacy class is in $\bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$ belongs to a subgroup whose conjugacy class is in $\mathrm{Per}_{\mathrm{NP}}(\phi^N)$.*

Proof. Note that, for every $N \in \mathbb{N}^*$ and every $\Psi \in \phi$, we have $\mathrm{Per}(\Psi^N) = \mathrm{Per}(\Psi)$. Thus, we will generally take a power of the considered automorphisms if needed.

Let T^ϕ be the above described tree associated with ϕ. Up to taking a power of ϕ, we may suppose that ϕ acts trivially on $G\backslash T^\phi$, that the homomorphism $\langle\phi\rangle \to \mathrm{Out}(G_e)$ is trivial for every $e \in ET^\phi$ and that the homomorphism $\langle\phi\rangle \to \mathrm{Out}(G_v)$ is trivial for every rigid vertex v.

Counting elliptic subgroups: Let $[H] \in \bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$. Suppose first that every element of H is elliptic in T^ϕ. We claim that H is elliptic in T^ϕ. Indeed, otherwise, by acylindricity of the action, there would exist two distinct $g, h \in H$ with $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) = \varnothing$. Therefore, the element $gh \in H$ would be loxodromic by [CM87, Proposition 1.5], a contradiction. Therefore the group H is elliptic in T^ϕ.

Since $[H] \in \bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$, the group H is not contained in the stabiliser of a vertex which is elementary. By Corollary 5.8, it is also not contained in the stabiliser of a QH with finite fibre vertex. Therefore, the group H is contained in the stabiliser of a vertex v which is rigid. By Lemma 5.5, we have in fact $H = G_v$.

In particular, the stabiliser of any vertex of T^ϕ contains at most one conjugacy class of elliptic subgroups in $\bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$. Since the action of G on T^ϕ has finitely many orbits of vertices, the set $\bigcup_{n\in\mathbb{N}} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$ contains only finitely many conjugacy classes of elliptic subgroups of T^ϕ.

Counting subgroups containing loxodromic elements: Suppose now that H contains a loxodromic element h. Let $n \in \mathbb{N}^*$, let $\Psi \in \phi^n$ be such that $\mathrm{Per}(\Psi) = H$ and let $N \in \mathbb{N}^*$ be such that $\Psi^N(h) = h$. By Remark 5.6, the isometry F_{Ψ^N} is elliptic. Let D be a finite fundamental domain for the action of G on T^ϕ. Up to taking a conjugate of H, we may suppose that D contains an edge e of the axis of h.

Let $\Theta \in \phi^n$ with $[\mathrm{Per}(\Theta)] \in \mathrm{Per}_{\mathrm{NP}}(\phi^n)$ and let $h' \in G$ loxodromic be such that $\Theta^N(h') = h'$ and that $\mathrm{Ax}(h')$ contains e. As above the isometry F_{Θ^N} is elliptic in T^ϕ. By Lemma 5.10, there exists $m \in \mathbb{N}^*$ such that $\Psi^m = \Theta^m$. Therefore, we see that $\mathrm{Per}(\Psi) = \mathrm{Per}(\Theta)$. Hence the conjugacy class of H in $\mathrm{Per}_{\mathrm{NP}}(\phi^n)$ is entirely determined by the edges of the fundamental domain D contained in translates of axes of elements of H. Since D is finite, and since the natural map $\mathrm{Per}_{\mathrm{NP}}(\phi^n) \to \mathrm{Per}_{\mathrm{NP}}(\phi^{(n+1)!})$ is injective, there exist only finitely many $[H] \in \bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$ containing the conjugacy class of a loxodromic element. (In fact, their number is bounded above by the number of edges in the fundamental domain.)

As we have ruled out every case, we see that the set $\bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$ is finite.

The second assertion follows from the first since the set $\bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^{n!})$ is a nondecreasing sequence of sets exhausting $\bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$. □

Definition 5.12. Let $\phi \in \mathrm{Out}(G, \mathcal{P})$ and let $N \in \mathbb{N}^*$ be the integer given by Lemma 5.11. If $N = 1$, we say that ϕ is *almost rotationless*.

Note that Lemma 5.11 implies that every element of $\mathrm{Out}(G, \mathcal{P})$ has an almost rotationless power.

5.B. **The Periodic JSJ tree.** We now need to understand vertex stabilisers of a JSJ tree given by Theorem 4.2 and its acylindricity in order to apply Theorem 2.5. We set $\mathcal{H} = \bigcup_{n \in \mathbb{N}^*} \mathrm{Per}_{\mathrm{NP}}(\phi^n)$, and refer to the JSJ tree relative to $\mathcal{P} \cup \mathcal{H}$ as T^{Per}. (Again, while this notation is neither standard nor entirely unambiguous, we use it consistently through our proofs.)

Remark 5.13. Note that, by Lemma 5.11, there exists an almost rotationless power ϕ^N of ϕ such that every subgroup of G whose conjugacy class is in $\mathcal{H}$ is contained in a subgroup whose conjugacy class is in $\mathrm{Per}_{\mathrm{NP}}(\Phi^N)$. Thus, every $(\mathcal{A}, \mathcal{P} \cup \mathcal{H})$-tree is an $(\mathcal{A}, \mathcal{P} \cup \mathrm{Per}_{\mathrm{NP}}(\Phi^N))$-tree, and conversely. Thus the JSJ tree of G relative to $\mathcal{P} \cup \mathcal{H}$ is also the JSJ tree relative to $\mathcal{P} \cup \mathrm{Per}_{\mathrm{NP}}(\Phi^N)$ (see [GL17, Definition 2.12]). Thus, we only need to work with almost rotationless automorphisms and we will still get results regarding the periodic JSJ tree associated with an arbitrary automorphism.

Let ϕ^N be an almost rotationless power of ϕ. Let $\mathrm{Per}_{\mathrm{NP}}(\phi^N) = \{[H_1], \ldots, [H_k]\}$ where, for every $i \in \{1, \ldots, k\}$, the group H_i is not elementary and there exist $\Phi_i \in \phi^N$ such that $H_i = \mathrm{Per}(\Phi_i)$. Note that, for every $i \in \{1, \ldots, k\}$, Theorem 4.2(9) gives that the group H_i fixes a unique rigid vertex v_i in T^{Per} since H_i is nonelementary.

Recall the construction of T^ϕ at the beginning of Section 5.A. Note that the trees T^ϕ and T^{Per} are compatible. Let T^{ref} be the unique minimal tree which refines T^ϕ and T^{Per}. We denote by $p_\phi \colon T^{\mathrm{ref}} \to T^\phi$ and $p_{\mathrm{Per}} \colon T^{\mathrm{ref}} \to T^{\mathrm{Per}}$ the associated G-equivariant projections. The tree T^{ref} satisfies the following properties: a subgroup H of G stabilises a point in T^{ref} if and only if H stabilises a point in both T^ϕ and T^{Per}. Moreover, for every edge $e \in ET^{\mathrm{ref}}$, at least one of the images $p_\phi(e)$ or $p_{\mathrm{Per}}(e)$ is not reduced to a point.

Note that, since the actions of G on T^ϕ and T^{Per} are acylindrical, by minimality of T^{ref}, the action of G on T^{ref} is also acylindrical. (Any sufficiently long path in T^{ref} will project to a path of length at least 3 in at least one of T^ϕ and T^{ref}; edge stabilisers are not changed by the projection map, and so the stabiliser of the path must have been finite to begin with.) Moreover, by uniqueness of T^{ref}, and since the outer automorphism ϕ preserves T^ϕ and T^{Per}, we see that ϕ also preserves T^{ref}.

For $i \in \{1, \ldots, k\}$, let F_{Φ_i} be the isometry of T^{ref} induced by Φ_i. As in Remark 5.6, one can show that F_{Φ_i} is elliptic in T^{ref} (this uses the acylindricity of the action of G on T^{ref}). For every $i \in \{1, \ldots, k\}$, let $T^{\mathrm{ref}}_{H_i}$ be the minimal tree of H_i in T^{ref}. It might be that every element of some H_i is elliptic in T^{ref}: in this case the acylindricity of T^{ref} implies that the whole subgroup H_i is also elliptic. Then H_i stabilises a vertex in both T^{Per} and T^ϕ; since H_i is non-elementary these vertices are unique. But since every edge of T^{ref} survives in the projection to at least one of T^ϕ and T^{Per}, there cannot be an edge of T^{ref} stabilised by H_i, and we may take the unique fixed vertex as the minimal invariant tree in this case.

Lemma 5.14. *Suppose that ϕ is almost rotationless. Let $H = \mathrm{Per}(\Phi)$, v a vertex in its minimal invariant tree T^{ref}_H, and F_Φ the isometry of T^{ref} induced by Φ. There exists $n \in \mathbb{N}$ such that F_{Φ^n} fixes v.*

Proof. If T^{ref}_H consists of a single vertex v, then this is the unique vertex stabilised by H. Since Φ preserves H, F_Φ must also fix v. Otherwise, the vertex v is contained in the axis of some $g \in H$. In particular, since g is periodic, there exists $n \in \mathbb{N}^*$ with $\Phi^n(g) = g$, and hence F_{Φ^n} preserves this axis. Since F_Φ is elliptic in T^{ref}, the isometry F_{Φ^n} fixes elementwise the axis of g. In particular, F_{Φ^n} fixes the vertex v. □

Proposition 5.15. *Let G be a one-ended hyperbolic group relative to $\mathcal{P}$ and let Φ and Ψ be two representatives of an almost rotationless $\phi \in \mathrm{Out}(G, \mathcal{P})$. Let $H = \mathrm{Per}(\Phi)$ and $K = \mathrm{Per}(\Psi)$ be two non-elementary periodic subgroups of ϕ, perhaps conjugate. Then their minimal invariant trees T^{ref}_H and T^{ref}_K have non-empty intersection if and only if $H = K$.*

Proof. If $H = K$ then the minimal invariant trees T^{ref}_H and T^{ref}_K are equal, so only one direction needs proof. Consider the minimal invariant trees of H and K in T^ϕ. If T^ϕ_H and T^ϕ_K do not intersect, then neither do T^{ref}_H and T^{ref}_K, so for the remainder of the proof we assume there is an intersection here. If the intersection contains an edge, Lemma 5.10 implies that $H = K$, so from now on assume the intersection is a single vertex $v = T^\phi_H \cap T^\phi_K$.

Each of H and K stabilise a unique rigid vertex in T^{Per}, and whenever $H \neq K$ we will construct an $(\mathcal{A}, \mathcal{P} \cup \mathrm{Per}_{\mathrm{NP}}(\phi))$-tree where H and K stabilise different vertices. This prevents $\langle H, K\rangle$ being contained in a rigid vertex group, by Theorem 4.2(9). But then the rigid vertices stabilised by H and K are distinct, and the trees in T^{ref} (containing T^{ref}_H and T^{ref}_K as subtrees) collapsing to them must be disjoint.

We distinguish two cases, according to the nature of v. Note that v is not QH with fibre by Corollary 5.9.

Case 1: v is a rigid vertex. By Lemma 5.5 if H or K is elliptic and Corollary 5.8 otherwise, there exists $N \in \mathbb{N}^*$ such that $\mathrm{Stab}(v) \subseteq \mathrm{Fix}(\Phi^N)$ and $\mathrm{Stab}(v) \subseteq \mathrm{Fix}(\Psi^N)$. Hence both Ψ^N and Φ^N fix elementwise the same nonelementary subgroup. By Lemma 5.1, we see that $H = K$.

Case 2: v is elementary. Let $\mathcal{T}_v$ be the set of minimal trees T^ϕ_K with $[K] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$ which contain v. Since, for every $[K] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$, the group K is nonelementary, no tree $S \in \mathcal{T}_v$ is reduced to a point. By Lemma 5.10, for all distinct $S, S' \in \mathcal{T}_v$, the intersection $S \cap S'$ is reduced to v. Let $E(v)$ be the set of edges in S adjacent to v. We have a partition

$$E(v) = E \coprod (E_S)_{S \in \mathcal{T}_v}$$

where for every $e \in E$, the edge e is not contained in any $S \in \mathcal{T}_v$ and, for every $S \in \mathcal{T}_v$ and every $e \in E_S$, the edge e is contained in S.

Let S_v be the tree with one central vertex v_0 adjacent to all the other vertices and that the leaves v_S are indexed by the trees $S \in \mathcal{T}_v$. We suppose that the stabiliser of v_0 is equal to $\mathrm{Stab}(v)$ and, for every $S \in \mathcal{T}_v$, that the stabiliser of v_S is equal to $\mathrm{Stab}(E_S)$. Let T' be the tree obtained from T by blowing up S_v at v and attaching for every $e \in E$, the edge e to v_0 and for every $S \in \mathcal{T}_v$ and every $e \in E_S$, by attaching the edge e to v_S.

Note that the tree T' obtained is an $(\mathcal{A}, \mathcal{P})$-tree. Moreover, for any $[K_1], [K_2] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$ with $T^{\phi}_{K_1} \neq T^{\phi}_{K_2} \in \mathcal{T}_v$ the minimal trees of T'_{K_1} and T'_{K_2} of K_1 and K_2 in T' are disjoint.

Let U be the $(\mathcal{A}, \mathcal{P} \cup \mathcal{H})$-tree obtained from T' by collapsing the minimal tree of every H with $[H] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$. Then K and H fix distinct points in U. Thus, U is an $(\mathcal{A}, \mathcal{P} \cup \mathcal{H})$-tree where H and K fix distinct points. By Theorem 4.2 (7), the groups H and K fix distinct rigid vertices in T^{Per}. Therefore, the minimal trees T^{ϕ}_H and T^{ϕ}_K of H and K in T^{ref} are disjoint. ☐

Lemma 5.16. *Suppose ϕ is almost rotationless. For every $H \in \mathrm{Per}_{\mathrm{NP}}(\phi)$, the stabiliser of the vertex v_H of T^{Per} is equal to the global stabiliser $G_{T^{\mathrm{ref}}_H}$ of the minimal tree T^{ref}_H of H in T^{ref}.*

Proof. First note that, since the projection $p_{\mathrm{Per}} \colon T^{\mathrm{ref}} \to T^{\mathrm{Per}}$ is equivariant, the tree T^{ref}_H collapses onto v_H. Thus, we have $G_{T^{\mathrm{ref}}_H} \subseteq \mathrm{Stab}(v_H)$.

Conversely, let U be the $(\mathcal{A}, \mathcal{P} \cup \mathcal{H})$-tree obtained from T^{ref} by collapsing, for every $H \in \mathrm{Per}_{\mathrm{NP}}(\phi)$, the tree T^{ref}_H. For every $H \in \mathrm{Per}_{\mathrm{NP}}(\phi)$, let w_H be the vertex of U fixed by H. Note that, by Lemma 5.15, for any $H, K \in \mathrm{Per}_{\mathrm{NP}}(\phi)$ and every $g \in G$ such that $gHg^{-1} \neq K$, the trees gT^{ref}_H and T^{ref}_K are disjoint. Thus, the stabiliser of w_H is equal to $G_{T^{\mathrm{ref}}_H}$. Since the vertex v_H of T^{Per} is rigid in every $(\mathcal{A}, \mathcal{P} \cup \mathcal{H})$-tree (see Theorem 4.2 (7)), the group $\mathrm{Stab}(v_H)$ is elliptic in U and contains H. As H fixes a unique point in U, which is w_H, we see that $\mathrm{Stab}(v_H) \subseteq G_{T^{\mathrm{ref}}_H}$. ☐

Lemma 5.17. *Suppose that ϕ is almost rotationless. Let $H \in \mathrm{Per}_{\mathrm{NP}}(\phi)$ and let $v \in VT^{\mathrm{ref}}_H$. Then $\mathrm{Stab}(v) \subseteq H$ or the intersection $\mathrm{Stab}(v) \cap H$ is elementary. In this case $\mathrm{Stab}(v)$ admits a (non-reduced) splitting $\mathrm{Stab}(v) = \mathrm{Stab}(v) *_{H \cap \mathrm{Stab}(v)} (H \cap \mathrm{Stab}(v))$ with elementary edge stabilisers.*

Proof. By Lemma 5.14, up to taking a power of Φ_H, we may suppose that F_{Φ_H} fixes v. Let $p_\phi(v)$ be the projection of v in T^ϕ. We distinguish between several cases, according to the nature of $p_\phi(v)$. Note that, by Corollary 5.9, the vertex $p_\phi(v)$ is not QH with fibre.

Suppose that $p_\phi(v)$ is elementary. Then $\mathrm{Stab}(v)$ is elementary. Thus, $\mathrm{Stab}(v)$ splits naturally as $\mathrm{Stab}(v) = \mathrm{Stab}(v) *_{H \cap \mathrm{Stab}(v)} (H \cap \mathrm{Stab}(v))$ and $H \cap \mathrm{Stab}(v)$ is elementary.

Suppose now that $p_\phi(v)$ is rigid. By Lemma 5.4 applied to $p_\phi(v)$, up to taking a power of Φ_H, we may suppose that Φ_H acts on $\mathrm{Stab}(p_\phi(v))$ as a global conjugation by an element $g \in \mathrm{Stab}(p_\phi(v))$.

If g is finite, up to taking a power of Φ_H, we see that Φ_H acts as the identity on $\mathrm{Stab}(p_0(v))$, so that $\mathrm{Stab}(v) \subseteq \mathrm{Stab}(p_0(v)) \subseteq H$.

So we may suppose that g is infinite. Then $H \cap \mathrm{Stab}(p_\phi(v))$ consists of all elements $h \in \mathrm{Stab}(p_\phi(v))$ which commute with a power of g. But the commensurator of $\langle g \rangle$ in G is elementary (because maximal elementary subgroups are almost malnormal, see [GL15, Corollary 3.2]). Thus, the group $H \cap \mathrm{Stab}(p_\phi(v))$ is elementary. Hence the group $H \cap \mathrm{Stab}(v)$ is elementary and the splitting $\mathrm{Stab}(v) = \mathrm{Stab}(v) *_{H_j \cap \mathrm{Stab}(v)} (H_j \cap \mathrm{Stab}(v))$ is over an elementary subgroup.

As we have ruled out every case, this concludes the proof. ☐

Lemma 5.18. *Suppose that ϕ is almost rotationless. For every $H \in \mathrm{Per}_{\mathrm{NP}}(\phi)$, and the unique vertex $v_H \in T^{\mathrm{Per}}$ it stabilises, we have $\mathrm{Stab}(v_H) = H$.*

Proof. Let $H \in \mathrm{Per}_{\mathrm{NP}}(\phi)$. By Lemma 5.16, it suffices to show that the stabiliser $G_{T_H^{\mathrm{ref}}}$ of T_H^{ref} is equal to H.

Claim 1. The stabiliser of any edge of T_H^{ref} is contained in H.

Proof. Let $e \in ET_H^{\mathrm{ref}}$. Note that e is contained in the axis of an element $g \in H$. Since H is elliptic in T^{Per}, the image $p_{\mathrm{Per}}(e)$ of e in T^{Per} is collapsed to a point.

By minimality of T^{ref}, the image $p_\phi(e)$ is an edge, and $G_e = G_{p_\phi(e)}$. By Corollary 5.8, any edge contained in T_H^{ϕ} is adjacent to a rigid vertex v, and $G_{p_\phi(e)} \leqslant G_v \leqslant H$. ∎

Claim 2. Let $v \in VT_H^{\mathrm{ref}}$. The stabiliser $\mathrm{Stab}(v)$ of v in G is contained in H.

Proof. Suppose towards a contradiction that $\mathrm{Stab}(v)$ is not contained in H. By Lemma 5.17, there exists of splitting S_v of $\mathrm{Stab}(v)$ with elementary edge stabilisers such that $H \cap \mathrm{Stab}(v)$ fixes a point w in S_v distinct from any point fixed by $\mathrm{Stab}(v)$. This splitting induces a refinement T' of T^{ref} by blowing up S_v at the vertex v and attaching the adjacent edges accordingly. We make the additional assumption that if e is an edge in T^{ref} adjacent to v such that $\mathrm{Stab}(e) \subseteq \mathrm{Stab}(w)$, then we attach e to w. The tree T' is an $(\mathcal{A}, \mathcal{P})$-tree since any edge of T^{ref} or S_v is elementary and since $\mathrm{Stab}(v)$ fixes a point in S_v if v is elementary by Lemma 5.17.

By Claim 1, every edge of T^{ref} is attached to w when blowing up the splitting S_v. Therefore, the minimal tree T'_H of H in T' does not contain any vertex stabilised by $\mathrm{Stab}(v)$. Collasping, for every $K \in \mathrm{Per}_{\mathrm{NP}}(\phi)$, the tree T'_K gives an $(\mathcal{A}, \mathcal{P} \cup \mathrm{Per}_{\mathrm{NP}}(\phi))$-tree U such that $\mathrm{Stab}(v_H)$ does not fix a point in it. This contradicts the fact that $\mathrm{Stab}(v_H)$ is elliptic in every $(\mathcal{A}, \mathcal{P} \cup \mathrm{Per}_{\mathrm{NP}}(\phi))$-tree (see Theorem 4.2 (7)). Hence, we have $\mathrm{Stab}(v) \subseteq H$. ∎

Claim 3. Suppose that $g \in G_{T_H^{\mathrm{ref}}}$. Then $g \in H$.

Proof. If g is elliptic, then since g preserves T_H^{ref}, one of its fixed point is contained in T_H^{ref}. By Claim 2, we see that $g \in H$.

So we may suppose that g is loxodromic in T^{ref}. We claim that there exists $v \in VT_H^{\mathrm{ref}}$ with infinite nonelementary stabiliser. Indeed, first note that T_H^{ref} is not reduced to a point as g is loxodromic. In particular, H does not fix a point of T^{ref}. Thus, by minimality of T^{ref}, the projection $p_\phi(T_H^{\mathrm{ref}})$ is nontrivial. By Corollary 5.9, the tree $p_\phi(T_H^{\mathrm{ref}})$ contains a rigid vertex and the preimage in T^{ref} of such a rigid vertex contains a vertex v with infinite nonelementary stabiliser.

Let $h \in \mathrm{Stab}(v)$ be of infinite order. Since the action of G on T is acylindrical, up to taking a vertex v far enough from the axis of g, we may suppose that h and ghg^{-1} do not have a common fixed point in T. In particular, they generate a nonabelian free group by standard ping pong arguments. Moreover, this nonabelian free group contains a finitely generated nonabelian free subgroup L consisting of loxodromic elements, hence consisting of nonperipheral elements.

Note that $h, ghg^{-1}, g^2hg^{-2} \in H$ since g preserves T_H^{ref} and since any vertex stabiliser of T_H^{ref} is contained in H by Claim 2. Thus, up to taking a power of Φ, we may suppose that $\langle L, gLg^{-1}\rangle \leqslant \langle h, ghg^{-1}, g^2hg^{-2}\rangle \leqslant \mathrm{Fix}(\Phi)$. Let $\mathcal{K} = \{[\langle L, gLg^{-1}\rangle]\}$ and let $T_{\mathcal{K}}$ be the canonical JSJ tree associated with $\mathcal{K}$ given by Theorem 4.2.

Since $\langle L, gLg^{-1}\rangle$ is nonelementary, it fixes a unique rigid point $w \in VT_{\mathcal{K}}$ by Theorem 4.2 (9). We claim that g fixes w. Indeed, otherwise the path between w and gw would be fixed by gLg^{-1} which is a nonelementary subgroup. This would contradict Theorem 4.2 (2).

Hence we have $g \in \mathrm{Stab}(w)$. Since w is rigid and since $\langle L, gLg^{-1}\rangle$ is finitely generated, by Theorem 4.2 (10), the automorphism Φ acts as a global conjugation on $\mathrm{Stab}(w)$. Since Φ fixes L which is nonelementary, it must act as a periodic automorphism of $\mathrm{Stab}(w)$. In particular, we have $g \in H$, which concludes the proof of the claim. ■

Combining Lemma 5.16 and by Claim 3, we see that the stabiliser of v_H is equal to H, which concludes the proof. □

Theorem 5.19. *Let G be a one-ended hyperbolic group relative to $\mathcal{P}$ and let $\phi = [\Phi] \in \mathrm{Out}(G, \mathcal{P})$. Let ϕ^N be an almost rotationless power of ϕ. For every rigid vertex $v \in VT^{\mathrm{Per}}$, there exists $[H] \in \mathrm{Per}_{\mathrm{NP}}(\phi^N)$ such that $\mathrm{Stab}(v) = H$. Conversely, for every $[H] \in \mathrm{Per}_{\mathrm{NP}}(\phi^N)$, there exists a rigid vertex $v \in VT^{\mathrm{Per}}$ with $\mathrm{Stab}(v) = H$.*

Proof. Up to taking a power of ϕ, we may assume that ϕ is almost rotationless (see Remark 5.13). Let $v \in VT^{\mathrm{Per}}$ be rigid. We claim that there exists $[H] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$ with $H \leqslant \mathrm{Stab}(v)$. Consider the pre-image of v in T^{ref}. If it is a vertex, then $\mathrm{Stab}(v)$ also stabilises a non-elementary vertex w of T^{ϕ}. If w is rigid, then $\langle\phi\rangle^0 \to \mathrm{Out}(G_v)$ has finite image, so $\mathrm{Stab}(v)$ is periodic for some ϕ^k, and so for ϕ (since we assume it is almost rotationless).

Otherwise w is a QH with fibre vertex. Note that all elementary subgroups of $\mathrm{Stab}(w)$ are virtually cyclic. In particular, the edges adjacent to $v \in VT^{\mathrm{Per}}$ are all virtually cyclic, hence finitely generated. Thus, $\mathrm{Stab}(v)$ is a finitely generated subgroup of $\mathrm{Stab}(w)$. Since w is a QH with fibre vertex, it is locally quasi-convex. Thus, $\mathrm{Stab}(v)$ is hyperbolic.

Note that, since $\mathrm{Stab}(v)$ is hyperbolic and a subgroup of $\mathrm{Stab}(w)$, the restriction $\mathcal{P} \cup \mathrm{Per}_{\mathrm{NP}}(\phi)|_{\mathrm{Stab}(v)}$ of $\mathcal{P} \cup \mathrm{Per}_{\mathrm{NP}}(\phi)$ to $\mathrm{Stab}(v)$ is a finite family of virtually cyclic subgroups of $\mathrm{Stab}(v)$ by Lemma 5.2. Thus, we can apply [GL15, Theorem 3.9] to show that $\mathrm{Out}(\mathrm{Stab}(v), \mathcal{P} \cup \mathrm{Per}_{\mathrm{NP}}(\phi)|_{\mathrm{Stab}(v)} \cup \mathrm{Inc}_v)$ (which has $\mathrm{Out}(\mathrm{Stab}(v), \mathcal{P} \cup (\mathrm{Per}_{\mathrm{NP}}(\phi)|_{\mathrm{Stab}(v)} \cup \mathrm{Inc}_v)^{(t)})$ as a finite index subgroup) is infinite if and only if $\mathrm{Stab}(v)$ has an $(\mathcal{A}, \mathcal{P} \cup \mathrm{Per}_{\mathrm{NP}}(\phi)|_{\mathrm{Stab}(v)} \cup \mathrm{Inc}_v)$-splitting. Since v is rigid, no such splitting of $\mathrm{Stab}(v)$ exists. Therefore, $\mathrm{Out}(\mathrm{Stab}(v), \mathcal{P}|_{\mathrm{Stab}(v)} \cup \mathrm{Per}_{\mathrm{NP}}(\phi)|_{\mathrm{Stab}(v)} \cup \mathrm{Inc}_v)$ is finite. In particular, the group $\mathrm{Stab}(v)$ is a nonelementary periodic subgroup of some power of ϕ.

Now suppose that the pre-image of v contains an edge. Since v is nonelementary, in fact this pre-image must be the minimal invariant tree $T^{\mathrm{ref}}_{\mathrm{Stab}(v)}$.

If this is contained in T_H^{ref} for some $H \in \text{Per}_{\text{NP}}(\phi)$, then v is the unique vertex of T^{Per} stabilised by H, and by Lemma 5.18 $\text{Stab}(v)$ (as the stabiliser of this vertex) is equal to H.

So suppose $T_{\text{Stab}(v)}^{\text{ref}}$ is not contained in any T_H^{ref}. Since the T_H^{ref} are disjoint by Proposition 5.15, there is some edge of $T_{\text{Stab}(v)}^{\text{ref}}$ contained in no T_H^{ref}, and hence collapsing all the T_H^{ref} will give an $(\mathcal{A}, \mathcal{P} \cup \text{Per}_{\text{NP}}(\phi))$ tree where $\text{Stab}(v)$ is not elliptic, contradicting Theorem 4.2(7).

Conversely, let $[H] \in \text{Per}_{\text{NP}}(\phi)$. By construction of T^{Per}, the group H fixes a vertex v of T^{Per}. Since H is nonelementary, such a vertex is unique and is rigid by Theorem 4.2 (10). By Lemma 5.18, we have $\text{Stab}(v) = H$. □

Corollary 5.20. *Let G be a one-ended hyperbolic group relative to $\mathcal{P}$ and let $\Phi \in \text{Aut}(G, \mathcal{P})$. Let $\mathcal{H} = \bigcup_{n \in \mathbb{N}^*} \text{Per}_{\text{NP}}(\phi^n)$.*

1. *For every geodesic edge path γ of T^{Per} of length 3 and every automorphism $\Psi \in \phi$ preserving γ, there exist a vertex v of γ and $g \in G_v$ of infinite order fixed by a power of Ψ.*
2. *The group G_Φ acts acylindrically on T^{Per}.*

Proof. Let γ be a geodesic edge path of length 3 in T^{Per} preserved by an automorphism $\Psi \in \phi$. Suppose that there exists an edge e of γ whose stabiliser is virtually cyclic (this applies in particular when one of the vertices of γ is QH with fibre). Hence $\text{Out}(G_e)$ is finite. Thus, Ψ has a power acting as the identity on the infinite cyclic subgroup of G_e.

Thus, we may suppose that γ only contains elementary and rigid vertices. Since γ has length 3 and since T^{Per} is bipartite, it contains an interior vertex v which is rigid. Let e_1, e_2 be the two edges of γ adjacent to v. Then $G_{e_1} \cap G_{e_2}$ is finite by Theorem 4.2 (8) and for every $i \in \{1, 2\}$ the group G_{e_i} is its own normaliser in G_v.

Let ϕ^N be an almost rotationless power of ϕ. By Theorem 5.19, there exists $[H] \in \text{Per}_{\text{NP}}(\phi^N)$ such that $\text{Stab}(v) = H$. Let $\Phi_H \in \phi^N$ be such that $\text{Per}(\Phi_H) = H$. Since Ψ preserves H, and since H is its own normaliser by Lemma 5.18, there exists $h \in H$ such that $\Psi^N = \text{ad}_h \Phi_H$. As Ψ^N preserves both G_{e_1} and G_{e_2}, we see that $h \in N(G_{e_1}) \cap N(G_{e_2})$, which is finite as explained above. Up to taking a power of Φ_H and Ψ^N, we may assume that both Φ_H and Ψ^N act trivially on $N(G_{e_1}) \cap N(G_{e_2})$. Taking further powers of Ψ^N and Φ_H shows that there exists $M \in \mathbb{N}$ such that $\Psi^M = \Phi_H^M$.

Let $g \in H$ be infinite order, which exists since H is nonelementary. Then Ψ has a power which fixes g. This proves Assertion (1).

We now prove that the action of G_Φ on T^{Per} is acylindrical. Note that, if a finite index subgroup of G_Φ acts acylindrically on T^{Per}, so does G_Φ. Thus, we may assume that ϕ is almost rotationless.

By Theorem 4.2 (2), the action of G on T^{Per} is acylindrical. Thus, by Lemma 4.3 (3) (which we can apply by Assertion (1)), it suffices to prove that, for every $n \in \mathbb{N}^*$, every $\Psi \in \phi^n$ and every $g \in \text{Fix}(\Psi^n)$, the element g is elliptic in T.

Since ϕ is almost rotationless, in order to prove Assertion (2), it suffices to prove that, for every $n \in \mathbb{N}^*$, every $\Psi \in \phi$ and every $g \in \text{Fix}(\Psi^n)$, the element g is elliptic in T.

Let g be as above. If g is peripheral, then g fixes a point by construction of T^{Per}.

Suppose now that g is nonperipheral. Suppose towards a contradiction that g is loxodromic in T^{Per}. Since $\Psi^n(g) = g$, the characteristic set of the isometry F_{Ψ^n} contains the axis of g. By Lemma 4.3 (2), up to taking a power of ϕ and changing the representative Ψ, the isometry F_{Ψ^n} fixes pointwise the axis of g. By Lemma 4.3 (1), up to taking a power of Ψ, the automorphism Ψ fixes elementwise a nonabelian free group of loxodromic elements. In particular, since every peripheral element fixes a point in T^{Per}, we see that Ψ fixes a nonabelian free group of nonperipheral elements. Since ϕ is almost rotationless, for every $n \geqslant 1$, we have $\mathrm{Per}_{\mathrm{NP}}(\phi) = \mathrm{Per}_{\mathrm{NP}}(\phi^n)$, we see that $[\mathrm{Per}(\Psi)] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$. Thus, there exists $[H] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$ such that $g \in \mathrm{Fix}(\Psi) \subseteq H$. In that case, the element g is elliptic in T^{Per} by construction of T^{Per}, a contradiction.

Therefore, the element g is elliptic in T^{Per} and we can apply Lemma 4.3 (3) to prove that the action of G_Φ on T^{Per} is acylindrical. □

Let G be a finitely generated group and let $\Phi \in \mathrm{Aut}(G)$. Suppose that Φ has a power which preserves the conjugacy class of a malnormal subgroup F of G. We then denote by F_Φ the group $F \rtimes_{\mathrm{ad}_g \circ \Phi^{n_F}} \mathbb{Z}$, where n_F is the minimal positive integer such that Φ^{n_F} preserves the conjugacy class of F and $g \in G$ is such that $\mathrm{ad}_g \circ \Phi^{n_F}(F) = F$. Since F is malnormal, the group F_Φ does not depend on g. Note that the group F_Φ only depends on the outer class of Φ.

Corollary 5.21. *Let G be a one-ended hyperbolic group relative to $\mathcal{P}$ and let $\Phi \in \mathrm{Aut}(G, \mathcal{P})$. If for every $[P] \in \mathcal{P}$ the group P_Φ is in $\mathbf{FJC_X}$, then G_Φ is in $\mathbf{FJC_X}$.*

Proof. Consider the action of G_Φ on T^{Per}. By Corollary 5.20, the action of G_Φ on T^{Per} is acylindrical. Up to taking a power of Φ (which does not change the conclusion by Theorem 2.1), we may assume that $\phi = [\Phi]$ is almost rotationless.

By Theorem 2.5, it suffices to show that, for every $v \in VT^{\mathrm{Per}}$, the stabiliser $(G_\Phi)_v$ of v in G_Φ belongs to $\mathbf{FJC_X}$. Note that, for every $v \in VT^{\mathrm{Per}}$, the group $(G_\Phi)_v$ can be written as a semi direct product $G_v \rtimes \mathbb{Z}$, where G_v is the stabiliser of v in G.

Suppose first that G_v is elementary. If $G_v \subseteq P$ for some $[P] \in \mathcal{P}$, then $(G_\Phi)_v$ is a subgroup of P_Φ. In particular, it belongs to $\mathbf{FJC_X}$ by Theorem 2.1 (1). If G_v is infinite virtually cyclic, then $(G_\Phi)_v$ belongs to $\mathbf{FJC_X}$.

Suppose now that G_v is QH with fibre. Then G_v fits in a short exact sequence

$$1 \to F \to G_v \to \pi_1(\Sigma_v) \to 1,$$

where Σ_v is a hyperbolic 2-orbifold and F is finite. Moreover, up to taking a power of Φ (which is possible by Theorem 2.1 (2)), since $\mathrm{Out}(G_v)$ has a finite index subgroup acting as the identity on F, there exists $\Psi \in \phi$ preserving G_v and fixing F elementwise. Thus, we have a short exact sequence

$$1 \to F \to (G_\Phi)_v \to \pi_1(\Sigma_v) \rtimes \mathbb{Z} \to 1.$$

The groups F and $\pi_1(\Sigma_v) \rtimes \mathbb{Z}$ belong to $\mathbf{FJC_X}$. For the latter one may use that surface bundles over the circle are locally CAT(0) and apply [Weg12]. Moreover, for every virtually cyclic group $Q \subseteq \pi_1(\Sigma_v) \rtimes \mathbb{Z}$, the preimage of Q in $(G_\Phi)_v$ is virtually cyclic, and hence belongs to $\mathbf{FJC_X}$. Thus, the group $(G_\Phi)_v$ belongs to $\mathbf{FJC_X}$.

Suppose that G_v is rigid. By Theorem 5.19, since ϕ is almost rotationless, there exists $[H] \in \mathrm{Per}_{\mathrm{NP}}(\phi)$ such that $G_v = H$.

Suppose that $\Phi \in \phi$ is such that $\mathrm{Per}(\Phi) = H$. Then $(G_\Phi)_v$ is isomorphic to $\mathrm{Per}(\Phi) \rtimes_\Phi \mathbb{Z}$. By Lemma 2.6, we have $(G_\Phi)_v \in \mathbf{FJC_X}$.

As we have ruled out every case, for every $v \in VT^{\mathrm{Per}}$, the group $(G_\Phi)_v$ belongs to $\mathbf{FJC_X}$. By Theorem 2.5, the group G_Φ belongs to $\mathbf{FJC_X}$. □

5.C. **An aside on slender peripherals.** We also isolate here an interesting consequence of Theorem 5.19 for automorphisms of groups hyperbolic relative to slender groups. Recall that a group is *slender* if all its subgroups are finitely generated.

Theorem 5.22. *Let G be a hyperbolic group relative to a collection $\mathcal{P}$ of slender groups and let $\Phi \in \mathrm{Aut}(G)$. There exists $N \in \mathbb{N}^*$ such that $\mathrm{Per}(\Phi) = \mathrm{Fix}(\Phi^N)$ and $\mathrm{Per}(\Phi)$ is finitely generated.*

Proof. See also the proof of [GL15, Theorem 8.2]. We claim that it suffices to prove that $\mathrm{Per}(\Phi)$ is finitely generated. Indeed, suppose that $\mathrm{Per}(\Phi)$ is generated by $a_1, \ldots, a_n$. For every $i \in \{1, \ldots, n\}$, let k_i be such that $\Phi^{k_i}(a_i) = a_i$. Let $N = k_1 \ldots k_n$. Then $\mathrm{Per}(\Phi) = \mathrm{Fix}(\Phi^N)$.

So we prove that $\mathrm{Per}(\Phi)$ is finitely generated. Note that slender groups are NRH groups, so that $\mathrm{Aut}(G) = \mathrm{Aut}(G, \mathcal{P})$. Note also that, since $\mathcal{P}$ is a set of conjugacy classes of slender groups, every elementary subgroup of G is finitely generated. Thus, we may suppose that $\mathrm{Per}(\Phi)$ is not elementary.

Let $\phi = [\Phi] \in \mathrm{Out}(G)$. Suppose first that G is one-ended relative to $\mathcal{H} = \mathrm{Per}_{\mathrm{NP}}(\phi)$. Let T^{Per} be the associated JSJ tree.

Since edge stabilisers of T^{Per} are elementary, they are all finitely generated. Thus, every vertex stabiliser of T^{Per} is also finitely generated.

By Theorem 5.19, the group $\mathrm{Per}(\Phi)$ is equal to the stabiliser of a vertex of T^{Per}, hence is finitely generated.

Suppose now that G is not one-ended relative to $\mathcal{H}$ and consider a minimal reduced Stallings-Dunwoody decomposition S of G such that, for every $[H] \in \mathcal{P} \cup \mathcal{H}$, the group H is contained in the stabiliser of a vertex of S. Recall that edge stabilisers in S are all finite. Since $\mathrm{Per}(\Phi)$ is infinite, it fixes a unique vertex v. Since the deformation space of S (see [GL07]) is invariant by Φ, the group $\Phi(G_v)$ also fixes a unique vertex w in S. As $\mathrm{Per}(\Phi) \subseteq \Phi(G_v)$ and as $\mathrm{Per}(\Phi)$ only fixes v, we see that $v = w$. This shows that $\Phi(G_v) = G_v$.

Note that, by minimality of S, the group G_v is one-ended hyperbolic relative to the restriction $\mathcal{P}_v$ of $\mathcal{P}$ in G_v. Since $\mathcal{P}$ is a set of conjugacy classes of slender groups, so is $\mathcal{P}_v$. Thus, the conclusion follows from the one-ended case applied to the restriction $\Phi|_{G_v}$. □

We remark that Minasyan–Osin [MO12, Corollary 1.3] also proved that the fixed subgroup of the automorphism of any hyperbolic group relative to slender groups is finitely generated.

6. A combination theorem for the Farrell–Jones conjecture

Let G be a finitely generated group and let $\Phi \in \mathrm{Aut}(G)$. If F is a malnormal subgroup of G whose conjugacy class is Φ-periodic, recall the definition of F_Φ from just above Corollary 5.21. In this section, we prove the following combination theorem.

Theorem D. *Let $G = G_1 * \ldots * G_k * F_N$ be a free product of finitely generated groups, let $\mathcal{F}' = \{[G_1], \ldots, [G_k]\}$ and let $\Phi \in \mathrm{Aut}(G, \mathcal{F}')$. If for each $i \in \{1, \ldots, k\}$, the group $(G_i)_\Phi$ is in $\mathbf{FJC_X}$, then $G \rtimes_\Phi \mathbb{Z}$ is in $\mathbf{FJC_X}$.*

The proof of Theorem D is by induction on $k + N$. Let $\mathcal{F}' \leqslant \mathcal{F}$ be a maximal proper Φ-periodic free factor system. Up to taking a power of Φ (which does not change the conclusion of Theorem D by Theorem 2.1 (2)), we may suppose that $\Phi \in \mathrm{Aut}(G, \mathcal{F})$. We will distinguish between two cases, according to whether $\mathcal{F}$ is sporadic or not.

6.A. The nonsporadic case.

Lemma 6.1. *Let $G = G_1 * \ldots * G_k * F_N$ be a free product of groups, let $\mathcal{F} = \{[G_1], \ldots, [G_k]\}$ and let $\Phi \in \mathrm{Aut}(G, \mathcal{F})$ be fully irreducible. If for each $i \in \{1, \ldots, k\}$, the group $(G_i)_\Phi$ is in $\mathbf{FJC_X}$, then $G \rtimes_\Phi \mathbb{Z}$ is in $\mathbf{FJC_X}$.*

Proof. Let S be the Grushko $(G, \mathcal{F})$-tree given by Lemma 3.3. Let $\mathcal{P}_S(\Phi)$ be the $\|.\|_S$-maximal polynomial subgroups of Φ. By Theorem 3.2, up to taking a power of Φ (which does not change the conclusion of Lemma 6.1 by Theorem 2.1 (2)) the group $G \rtimes_\Phi \mathbb{Z}$ is hyperbolic relative to the suspension of $\mathcal{P}_S(\Phi)$. By Proposition 3.4, for every $[P] \in \mathcal{P}_S(\Phi)$, either $[P] \in \mathcal{F}$ or P is infinite cyclic. In either case P_Φ is contained in $\mathbf{FJC_X}$. By Theorem 2.4, the group $G \rtimes_\Phi \mathbb{Z}$ is contained in $\mathbf{FJC_X}$. □

6.B. The sporadic case.

This section follows [BFW23, Proof of Proposition 4.1]. Let $(G, \mathcal{F})$ be a sporadic free product and let $\Phi \in \mathrm{Aut}(G, \mathcal{F})$. Since $(G, \mathcal{F})$ is sporadic, the automorphism Φ induces a G-equivariant homeomorphism of the Bass-Serre tree $T_\mathcal{F}$ associated with $\mathcal{F}$. This induces an action of $G \rtimes_\Phi \mathbb{Z}$ on $T_\mathcal{F}$. However, this action is not necessarily acylindrical. In order to apply Theorem 2.5, we will consider the action of $G \rtimes_\Phi \mathbb{Z}$ on the tree of cylinders of $T_\mathcal{F}$ associated with an admissible relation that we now describe.

Let t be a generator of the $\mathbb{Z}$-factor. Up to taking a finite index subgroup of $G \rtimes_\Phi \mathbb{Z}$ (which is possible by Theorem 2.1 (2)), we may suppose that t fixes an edge e. In that case, edge stabilisers of the action of $G \rtimes_\Phi \mathbb{Z}$ are all infinite cyclic, generated by conjugates of t. Therefore, the commensurability relation is an admissible equivalence relation, and we define the tree of cylinders T_c of $T_\mathcal{F}$ relative to this admissible relation.

Lemma 6.2. *Let Y be the cylinder of $T_\mathcal{F}$ containing e. The stabiliser of Y in $G \rtimes_\Phi \mathbb{Z}$ is isomorphic to $\langle \mathrm{Fix}(\Phi^n) \rangle_{n \in \mathbb{N}} \rtimes_\Phi \mathbb{Z}$.*

Proof. Note that any element $h \in G \rtimes_\Phi \mathbb{Z}$ can be written uniquely as $w^{-1}t^j$, where $w \in G$ and $j \in \mathbb{Z}$. Let $w^{-1}t^j \in \mathrm{Stab}(Y)$ with $w \in G$ and $j \in \mathbb{Z}$ and let $e' = w^{-1}t^j e$. Then we have $G_{e'} = \langle w^{-1}tw \rangle$. Moreover, since $e, e' \in EY$, by definition of the commensurability relation, there exist $n, m \in \mathbb{N}$ such that

$$t^n = w^{-1}t^m w = w^{-1}\Phi^m(w)t^m.$$

In particular, we see that $n = m$ and $\Phi^m(w) = w$. □

Lemma 6.3. *The action of $G \rtimes_\Phi \mathbb{Z}$ on T_c is acylindrical.*

Proof. The proof is identical to [BFW23, Lemma 4.6]. Let $v, v' \in VT_c$ with $d_{T_c}(v, v') \geqslant 6$. We may suppose that v and v' correspond to vertices w and w' in $T_{\mathcal{F}}$ up to considering adjacent vertices in the path between them. Even after this operation, we have $d_{T_c}(v, v') \geqslant 4$.

Let $g \in G_v \cap G_{v'}$. Then g fixes the path in $T_{\mathcal{F}}$ between w and w'. Since $d_{T_c}(v, v') \geqslant 4$, the path in $T_{\mathcal{F}}$ between w and w' must contain two edges in distinct cylinders. Hence g fixes two edges in distinct cylinders. Since edge stabilisers in $T_{\mathcal{F}}$ are infinite cyclic and since we are considering the commensurability relation, two edges in $T_{\mathcal{F}}$ are in the same cylinder if and only if the intersection of their stabilisers is nontrivial. In particular, this shows that g is trivial and that the action of $G \rtimes_\Phi \mathbb{Z}$ on T_c is acylindrical. □

Lemma 6.4. *Let $(G, \mathcal{F})$ be a sporadic free product of groups and let $\Phi \in \mathrm{Aut}(G, \mathcal{F})$. If for each $[A] \in \mathcal{F}$, the group A_Φ is in $\mathbf{FJC_X}$, then $G \rtimes_\Phi \mathbb{Z}$ is in $\mathbf{FJC_X}$.*

Proof. Let $T_{\mathcal{F}}$ be the Bass-Serre tree associated with $\mathcal{F}$ and let T_c be its tree of cylinders relative to the commensurability relation. We want to apply Theorem 2.5 to the action of $G \rtimes_\Phi \mathbb{Z}$ on T_c. This action is acylindrical by Lemma 6.3. Thus, it suffices to prove that every vertex stabiliser belongs to $\mathbf{FJC_X}$.

Recall that we have a bipartition of $VT_c = V_0T_c \coprod V_1V_c$, where vertices in V_0T_c correspond to vertices of $T_{\mathcal{F}}$ and vertices in V_1T_c correspond to cylinders of T_c.

If $v \in V_0T_c$, then its stabiliser is isomorphic to A_Φ. In that case, the stabiliser of v belongs to $\mathbf{FJC_X}$ by hypothesis.

Suppose now that $v \in V_1T_c$. By Lemma 6.2, the stabiliser of v is isomorphic to $\mathrm{Per}(\Phi) \rtimes_\Phi \mathbb{Z}$. By Lemma 2.6, the group $\mathrm{Per}(\Phi) \rtimes_\Phi \mathbb{Z}$ belongs to $\mathbf{FJC_X}$.

Thus, every vertex stabiliser of the action of $G \rtimes_\Phi \mathbb{Z}$ on T_c belongs to $\mathbf{FJC_X}$. By Theorem 2.5, the group $G \rtimes_\Phi \mathbb{Z}$ belongs to $\mathbf{FJC_X}$. □

6.C. **End of the proof of Theorem D.** Let $G = G_1 * \ldots * G_k * F_N$ be a free product of groups, let $\mathcal{F}' = \{[G_1], \ldots, [G_k]\}$ and let $\Phi \in \mathrm{Aut}(G, \mathcal{F}')$. We prove by induction on $k + N$ that $G \in \mathbf{FJC_X}$.

Suppose first that $k+N = 1$. If $N = 0$, then $G = G_1$ and $G \rtimes_\Phi \mathbb{Z} \in \mathbf{FJC_X}$ by hypothesis. If $k = 0$, then $G = \mathbb{Z}$, $G \rtimes_\Phi \mathbb{Z}$ is solvable and the result follows from [Weg15]. This proves the base case.

Suppose now that $k+N \geqslant 2$ and let $\mathcal{F}$ be a maximal Φ-periodic free factor system. We may assume, up to taking a power of Φ, that $\mathcal{F}$ is Φ-invariant, so that we can view Φ as an element of $\mathrm{Aut}(G, \mathcal{F})$. This is possible by Theorem 2.1 (2) as, for every $n \in \mathbb{N}$, the group $G \rtimes_{\Phi^n} \mathbb{Z}$ is a finite index subgroup of $G \rtimes_\Phi \mathbb{Z}$.

By induction hypothesis, for every $[A] \in \mathcal{F}$, the group A_Φ belongs to $\mathbf{FJC_X}$. Combining the nonsporadic case (Lemma 6.1) and the sporadic case (Lemma 6.4), we conclude that $G \rtimes_\Phi \mathbb{Z}$ belongs to $\mathbf{FJC_X}$. This concludes the proof. □

6.D. **Proving Theorem A.** We first record a corollary of Theorem D.

Corollary 6.5. *Let $(G, \mathcal{P})$ be a virtually torsion-free relatively hyperbolic group with $\mathcal{P}$ finite and let $\Phi \in \mathrm{Aut}(G, \mathcal{P})$. If for every $[P] \in \mathcal{P}$ we have $P_\Phi \in \mathbf{FJC_X}$, then $G_\Phi \in \mathbf{FJC_X}$.*

Proof. By Theorem 2.1 we may assume G is torsion-free. Let $\mathcal{F}$ be the minimal free factor system of G such that, for every $[P] \in \mathcal{P}$, there exists $[A] \in \mathcal{F}$ with $P \subseteq A$. Since $\Phi \in \mathrm{Aut}(G, \mathcal{P})$, by minimality of $\mathcal{F}$, we have $\Phi \in \mathrm{Aut}(G, \mathcal{F})$. Let $[A] \in \mathcal{F}$. We denote by $\mathcal{P}_A$ the peripheral structure of A induced by $\mathcal{P}$. Since G is torsion-free, the group A is one-ended hyperbolic relative to $\mathcal{P}_A$. By Corollary 5.21 the group A_Φ belongs to $\mathbf{FJC_X}$. By Theorem D, the group G_Φ belongs to $\mathbf{FJC_X}$. □

Finally, combining Corollary 5.21 and Corollary 6.5 proves our first theorem from the introduction.

Theorem A. *Let $(G, \mathcal{P})$ be a virtually torsion-free or one-ended relatively hyperbolic group with $\mathcal{P}$ finite and let $\Phi \in \mathrm{Aut}(G, \mathcal{P})$. If for every $[P] \in \mathcal{P}$ we have $P_\Phi \in \mathbf{FJC_X}$, then $G_\Phi \in \mathbf{FJC_X}$.*

We now discuss the (minor) changes to the proof used to prove the following theorem.

Theorem E. *Suppose $(G, \mathcal{P})$ is one-ended or virtually torsion free, and hyperbolic relative to finitely many conjugacy classes of slender subgroups. Then for every automorphism Φ of G, $\Gamma := G \rtimes_\Phi \mathbb{Z}$ is in $\mathcal{AC}(\mathbf{VNil})$.*

Proof. Knopf's work on acylindrical actions of trees applies equally well in the setting of $\mathcal{AC}(\mathbf{VNil})$ (see [Kno19, Corollary 4.2] and [BFW23, Theorem 2.4], note that Knopf does not state this but it is implicit in her work). We use the same trees as every step of the proof of Theorem A. Whenever a vertex group is identified as $\mathrm{Per}(\Phi) \rtimes_\Phi \mathbb{Z}$, use Theorem 5.22 to further identify it as some $\mathrm{Fix}(\Phi^k) \rtimes_\Phi \mathbb{Z}$. As this has a finite index subgroup isomorphic to $\mathrm{Fix}(\Phi^k) \times \mathbb{Z}$, and $\mathcal{AC}(\mathbf{VNil})$ passes to both subgroups and direct products, this vertex group lies in $\mathcal{AC}(\mathbf{VNil})$. □

7. Proofs of the applications

Our first application is to extensions of groups with relatively hyperbolic kernel.

Corollary B. *Let $(N, \mathcal{P})$ be a virtually torsion-free or one-ended relatively hyperbolic group such that $\mathcal{P}$ consists of finitely many conjugacy classes of groups which are NRH and whose suspensions $P \rtimes_\Psi \mathbb{Z}$ are in $\mathbf{FJC_X}$ for all automorphisms Ψ of P. Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence. If Q is in $\mathbf{FJC_X}$, then Γ is in $\mathbf{FJC_X}$.*

Proof. Since for all $[P] \in \mathcal{P}$ the group P is NRH we have that $\mathrm{Aut}(G, \mathcal{P})$ is a finite index subgroup of $\mathrm{Aut}(G)$. Let $\Phi \in \mathrm{Aut}(G)$. The suspension G_Φ has a finite index subgroup G_{Φ^n} such that $\Phi^n \in \mathrm{Aut}(G, \mathcal{P})$. Now, Theorem A implies that G_{Φ^n} is in $\mathbf{FJC_X}$. It follows from Theorem 2.1 that G_Φ is in $\mathbf{FJC_X}$. The result now follows from Theorem 2.2. □

Our other application is that $\mathrm{Aut}(G)$ is in $\mathbf{FJC_X}$ for G a one-ended group hyperbolic relative to finitely many conjugacy classes of polycyclic subgroups. Before we prove this, we collect some results.

Theorem 7.1 ([GL15, Theorem 4.3]). *Let $(G, \mathcal{P})$ be a relatively hyperbolic group. Suppose for every $[P] \in \mathcal{P}$, the group P is finitely generated. If G is one-ended relative to $\mathcal{P}$, then there is a short exact sequence*

$$1 \to \mathfrak{T} \to \mathrm{Out}_0(G, \mathcal{P}) \to \prod_{i=1}^{p} \mathrm{MCG}_0(S_i) \times \prod_{j} \mathrm{Out}(P_j, \mathrm{Inc}^{(t)}_{P_j}) \to 1$$

where

(1) $\mathfrak{T}$ is a quotient of a finite direct product where each factor is virtually cyclic or contained in some P for $P \in \mathcal{P}$;
(2) $\mathrm{MCG}_0(S_i)$ maps onto a finite index subgroup of the extended mapping class group $\mathrm{MCG}^*(S_i)$ with finite kernel (they are virtually isomorphic).

Proposition 7.2. *If G is a virtually polycyclic group, then G, $\mathrm{Out}(G)$ and $\mathrm{Aut}(G)$ are in $\mathbf{FJC_X}$.*

Proof. By [BG06, Theorem 1.1] we see that $\mathrm{Out}(G)$ is an arithmetic group. Hence, $\mathrm{Out}(G)$ is in $\mathbf{FJC_X}$ by [BFL14]. Technically they only prove the conjecture for K- and L-theory but it follows for A-theory by [Rüp16], [Kno19, Proof of Theorem 1.8(a)], and [ELP+18, Theorem 6.19]. Alternatively, one may use [ELP+18] and [KUWW18].

Since G is virtually soluble it is in $\mathbf{FJC_X}$ by [Weg15] (for K- and L-theory), [KUWW18] (for A-theory), and Theorem 2.1. Now, a virtually polycyclic group is poly-{virtually cyclic}, so any extension $G \rtimes \mathbb{Z}$ is also virtually polycyclic. Thus, $G \rtimes \mathbb{Z}$ is in $\mathbf{FJC_X}$. Further, note that $G/Z(G)$ is virtually polycyclic and so in $\mathbf{FJC_X}$. Combining these observations with Theorem 2.2 shows that $\mathrm{Aut}(G)$ is in $\mathbf{FJC_X}$. □

Proposition 7.3. *The mapping class group of a hyperbolic 2-orbifold is in $\mathbf{FJC_X}$.*

Proof. This follows from the result for (orientable) surfaces [BB19] and assembling results in the literature. Note that Bartels–Bestvina only prove FJC for K- and L-theory but the result for A-theory follows (as usual) from [Kno19, Proof of Theorem 1.8(a)], and [ELP+18, Theorem 6.19].

Let S be a hyperbolic orbifold, and let Σ be an orientable surface covering S with finite degree so that $\pi_1(\Sigma)$ is characteristic in $\pi_1(S)$. (This can be achieved by taking any covering surface, passing to its orientation cover if necessary, and then taking the characteristic core of the corresponding subgroup and realising the covering surface.) By [KE21] there is an injective map from $\mathrm{Aut}_{\mathrm{geom}}(\pi_1(S))$ to $\mathrm{Aut}_{\mathrm{geom}}(\pi_1(\Sigma))$, where these *geometric* automorphism groups are exactly the lifts of the mapping class groups.

Restricting to the image, $\mathrm{Inn}(\pi_1(S))$ will be normal, and by the third isomorphism theorem the quotient is isomorphic to $C/(\mathrm{Inn}(\pi_1(S))/\mathrm{Inn}(\pi_1(\Sigma)))$, where C is a subgroup of $\mathrm{Aut}_{\mathrm{geom}}(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma))$, the mapping class group of Σ. The quotient of inner automorphism groups is finite (in fact isomorphic to the deck transformations $\pi_1(S)/\pi_1(\Sigma)$, so we have realised $\mathrm{MCG}(S)$ as an extension

$$1 \to F \to C \to MCG(S) \to 1.$$

Since mapping class groups of surfaces are residually finite by [Gro74] (and residual finiteness passes to subgroups), we may apply Lemma 2.3 to obtain the conclusion. □

Proposition 7.4. *If G is a one ended group hyperbolic relative to finitely many conjugacy classes of virtually polycyclic groups, then $\mathrm{Out}(G)$ is in $\mathbf{FJC_X}$.*

Proof. By Theorem 7.1 there is a finite index subgroup $\mathrm{Out}_0(G)$ fitting into a short exact sequence

$$1 \to \mathfrak{T} \to \mathrm{Out}_0(G) \to \prod_{i=1}^{p} \mathrm{MCG}_0(S_i) \times \prod_j \mathrm{Out}(P_j, \mathrm{Inc}^{(t)}_{P_j}) \to 1.$$

We want to apply Theorem 2.2 to this short exact sequence. First we check the kernel $\mathfrak{T}$: this is a quotient of a direct product of virtually polycyclic groups, and hence is itself virtually polycyclic, and so in $\mathbf{FJC_X}$ by Proposition 7.2.

Now consider the image. The subgroups $\mathrm{Out}(P_j, \mathrm{Inc}^{(t)}_{P_j})$ are subgroups of $\mathrm{Out}(P_j)$ for a virtually polycyclic P_j, and hence are in $\mathbf{FJC_X}$ by Proposition 7.2. Each $\mathrm{MCG}(S_i)$ maps with finite kernel onto the mapping class group of a hyperbolic 2-orbifold. By Lemma 2.3 it is enough to consider these mapping class groups. These are in $\mathbf{FJC_X}$ by Proposition 7.3. Then the product is in $\mathbf{FJC_X}$ by Theorem 2.1.

To apply Theorem 2.2 it remains to check the preimages of elements. These are of the form $\mathfrak{T} \rtimes \mathbb{Z}$, which are in $\mathbf{FJC_X}$ by Proposition 7.2 since they are again virtually polycyclic. □

Theorem C. *If G is a one-ended group hyperbolic relative to finitely many conjugacy classes of virtually polycyclic groups, then $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ are in $\mathbf{FJC_X}$.*

Proof. Since $\mathcal{P}$ contains only finitely many conjugacy classes of virtually polycyclic groups, by [DS05, Corollary 1.14], we may modify $\mathcal{P}$ such that that for every $[P] \in \mathcal{P}$, the group P is NRH. Observe that since G is hyperbolic relative to finitely many conjugacy classes of virtually polycyclic subgroups its centre is finite. Hence, $\mathrm{Inn}(G)$ is quasi-isometric to G and again hyperbolic relative to finitely many conjugacy classes of NRH virtually polycyclic subgroups by [BDM09]. The result now follows from applying Corollary B to the short exact sequence

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1.$$

□

Theorem C allows to prove that the outer automorphism groups of some small complexity relatively hyperbolic groups also belong to $\mathbf{FJC_X}$.

Corollary 7.5. *Let $G = A *_C B$, where A and B are one-ended hyperbolic groups relative to finitely many conjugacy classes of virtually polycyclic groups and C is a finite group. The groups $\mathrm{Out}(G)$ and $\mathrm{Aut}(G)$ are in $\mathbf{FJC_X}$.*

Proof. We prove the result for $\mathrm{Out}(G)$, the proof for $\mathrm{Aut}(G)$ being identical to the proof of Theorem C (this uses Proposition 8.3 when C is nontrivial and Theorem A otherwise). Let $\mathrm{Out}^0(G)$ be the index (at most) 2 subgroup of $\mathrm{Out}(G)$ preserving the conjugacy classes of A and B. By [For02], every element $\phi \in \mathrm{Out}^0(G)$ has a representative $\Phi \in \phi$ such that $\Phi(A) = A$ and $\Phi(B) = B$. Moreover, the map sending ϕ to Φ defines an isomorphism between $\mathrm{Out}^0(G)$ and $\mathrm{Aut}(A, C) \times \mathrm{Aut}(B, C)$. By Theorem C, the groups $\mathrm{Aut}(A, C)$ and $\mathrm{Aut}(B, C)$ belong to $\mathbf{FJC_X}$. By Theorem 2.1, the groups $\mathrm{Out}^0(G)$ and $\mathrm{Out}(G)$ belong to $\mathbf{FJC_X}$. □

Corollary 7.6. *Let $G = A*_C$, where A is a one-ended hyperbolic group relative to finitely many conjugacy classes of virtually polycyclic groups and C is a finite group. The groups $\mathrm{Out}(G)$ and $\mathrm{Aut}(G)$ are in $\mathbf{FJC_X}$.*

Proof. As above we only prove the result for $\mathrm{Out}(G)$. Let t be a stable letter for the HNN extension $A*_C$. By [Lev05], the group $\mathrm{Out}(G)$ has an index 2 subgroup $\mathrm{Out}^0(G)$ such that any element $\phi \in \mathrm{Out}^0(G)$ has a representative $\Phi \in \phi$ such that $\Phi(A) = A$ and $\Phi(t) = ta$ for some $a \in A$. Moreover, the map sending ϕ to Φ induces an isomorphism between $\mathrm{Out}^0(G)$ and $A \rtimes \mathrm{Aut}(A, C)$. Thus, $\mathrm{Out}^0(G)$ fits in a short exact sequence

$$1 \to A \to \mathrm{Out}^0(G) \to \mathrm{Aut}(A, C) \to 1.$$

The group $\mathrm{Aut}(A, C)$ belongs to $\mathbf{FJC_X}$ by Theorem C. Moreover, for every infinite cyclic subgroup $Q \in \mathrm{Aut}(A, C)$, the preimage of Q in $\mathrm{Out}^0(G)$ belongs to $\mathbf{FJC_X}$ by Theorem A. Thus, by Theorem 2.2, the groups $\mathrm{Out}^0(G)$ and $\mathrm{Out}(G)$ belong to $\mathbf{FJC_X}$. □

One more infinitely ended case is known, since $\mathrm{Out}(F_2) \cong \mathrm{GL}(2, \mathbb{Z})$ is an arithmetic group, and the same extension arguments as above will give the result for $\mathrm{Aut}(F_2)$. Our techniques do not seem to extend to $\mathrm{Out}(F_n)$, which will be necessary to make any further progress.

8. Torsion in the infinitely ended case

The aim in this section is to prove as much of Section 6 as possible without the assumption that G is virtually torsion free. To this end we give two propositions. The first allows us – in many cases – to pass from an infinitely-ended group to a finite index subgroup which is a free product of one-ended groups and a finitely generated free group, while the second is a generalisation of the sporadic case (of Section 6.B) to graphs of groups with one, finitely stabilised, edge.

Proposition 8.1. *Suppose G is finitely generated, infinitely-ended and accessible. Further suppose that for each one-ended vertex group G_v in a Stallings–Dunwoody decomposition of G, there is a normal finite index subgroup K_v that trivially intersects all the incident edge groups. Then G has a finite index subgroup K that is a free product of one-ended groups and a finitely generated free group.*

Furthermore, for any automorphism Φ, the suspension G_Φ has a finite index subgroup K_Ψ, where Ψ is the restriction of some power of Φ to H.

Note that [AGHK23, Lemma 5.4] is a similar result, but the statement and proof are considerably simplified by the underlying assumption of that paper that all groups considered are residually finite. Here we prove necessary and sufficient conditions on the vertex groups and incident edge groups for the existence of such a subgroup.

Remarks 8.2. We record a number of observations about the hypotheses.

- Accessibility is only used to ensure that the vertex groups are one-ended. More generally, given a splitting over finite edge groups and where the vertex groups satisfy the separability hypothesis, one can find a finite index free product where the non-trivial vertex groups are finite index subgroups of (conjugates of) the original infinite vertex groups.
- The sufficient conditions given in the statement are also necessary: any finite index K satisfying the conclusion will have intersections $K \cap G_v$, finite index in G_v avoiding the edge groups. Passing to the normal core recovers the finite index normal subgroup having the desired property.
- Finite generation is only used in the "furthermore"; a sufficient condition on Φ would be that it has such a power.

The proof involves Bass–Serre covering theory, see [Bas93] for details. Since it is independent of the rest of the paper we do not provide a self-contained description of the theory here, though references to the appropriate results and definitions will be made as necessary.

Proof. Let $\mathcal{G}$ be the graph of groups corresponding to a Stallings–Dunwoody decomposition of G. We will produce a graph of groups $\mathcal{K}$ which covers $\mathcal{G}$ (in the sense of [Bas93, Definition 2.6]). To do this we need to produce a graph with a map to the underlying graph of $\mathcal{G}$, and its vertex (and edge, though these will be trivial) groups, together with monomorphisms to the appropriate vertex group of $\mathcal{G}$. Abusing notation, we refer to all these graph and group maps as f, trusting that it will be clear from context which is meant. For each edge (and inverse edge) e of $\mathcal{K}$ we also need to define an element $\delta_e \in G_{f(\iota(e))}$. These allow us to assemble the "star maps" at a vertex v of $\mathcal{K}$ (see [Bas93, Proposition 2.4]):

$$\begin{array}{rcl} \bigsqcup_{e\in f^{-1}(e'),\iota(e)=v} K_\iota(e) & \to & G_{\iota(e')}/G_e \\ (e,k) & \mapsto & \delta_e k G_{e'} \end{array}$$

which we are required to check are bijective to ensure we defined a covering. (Note that, in general, there are compatibility conditions to check on the δ_e. However, these are vacuous when K_e are trivial, as they will be for us – see [Bas93, Definition 2.1].)

By [Bas93, Proposition 2.7], once we have defined $\mathcal{K}$ and f so that we have a covering, there is an inclusion (defined using f) from $H = \pi_1(\mathcal{K})$ to $G = \pi_1(\mathcal{G})$, and the Bass–Serre trees of are the same, with K acting as a subgroup of G. In this case, we can use [And23, Lemma 3.4] to calculate the index $[G : K]$: it is the sum (as x ranges over the preimages of any vertex or edge) of the indices $[G_x : K_x]$. In particular, to get a finite index subgroup, we need to ensure that all these indices are finite, and that the sums agree.

To facilitate this, let $K_v \trianglelefteq G_v$ be as given in the statement when G_v is infinite, and if G_v is finite, let K_v be trivial. Set $d_v = [G_v : K_v]$, and set $d = \mathrm{lcm}(\{d_v\})$. (Observe that for $v = \iota(e)$, by the tower law and second isomorphism theorem, $|G_e| = [G_e : K_v \cap G_e] = [K_vG_e : K_v]$ divides $[G_v : K_vG_e] \cdot [K_vG_e : K_v] = [G_v : K_v] = d_v$, so this accounts for all edge groups as well.)

Now let the vertex set of $\mathcal{K}$ consist of d/d_v preimages of each v, each with vertex group K_v. Let each $f(K_v)$ be the inclusion into G_v. For the edge groups, let $v = \iota(e)$ and note that in order for the local action of K_v on its star to respect the index sum formula, we must have $d_v/|G_e|$ preimages of e adjacent to a preimage of v. Note that by the tower law argument above, $d_v/|G_e| = [G_v : K_vG_e]$. Summing across the d/d_v preimages of v, we will see the expected $d/|G_e|$ preimages of e adjacent to preimages of v. Exactly the same argument applies to $\tau(e)$, and so for every edge e there is a bijection between the "heads" at preimages of $\iota(e)$ and "tails" at preimages of $\tau(e)$ where we would like to attach a preimage of e. Picking any explicit bijection, give $\mathcal{K}$ edges joining the indicated heads and tails. Set all edge groups to be trivial.

Setting d to be the least common multiple ensures that the resulting graph is connected whatever choices are made. (If it has some smaller connected component, then summing indices over the orbits within it will provide a smaller common multiple.)

Given a vertex v of $\mathcal{K}$ and the set of incident preimages of some e', let the δ_e range over a set of coset representatives G_v/K_vG_e. We now investigate the star maps.

First, we see they are injective. Suppose that $\delta_{e_1}k_1G_e = \delta_{e_2}k_2G_e$. This (together with normality of K_v in G_v) implies that δ_{e_1} and δ_{e_2} represent the same K_vG_e coset, and so that $e_1 = e_2$. Cancelling the δ_{e_i}, we now have that $k_1G_e = k_2G_e$. This implies that $k_1^{-1}k_2$ is contained in G_e, but since the intersection $G_e \cap K_v$ is trivial, they must be equal.

To see surjectivity, consider some coset $gG_{e'}$. Let δ_e be the previously determined representative of the coset $gK_vG_{e'}$. Notice that $\delta_e^{-1}g$ is contained in K_vG_e, and so there is some element $g_{e'}$ of $G_{e'}$ so that $k = \delta_e^{-1}gg_{e'}$ is contained in K_v. By construction, the copy of k lying inside the K_v associated to the coset $\delta_eK_vG_{e'}$ is mapped to $\delta_e\delta_e^{-1}gg_{e'}G_e$ which is gG_e.

Since $\mathcal{K}$ is a finite graph of groups with trivial edge groups, K is a free product of its non-trivial vertex groups and a finitely generated free group. The non-trivial vertex groups are (conjugates of) finite index subgroups of the original vertex groups, so are themselves one-ended, as required. Finally, $\mathcal{K}$ covers $\mathcal{G}$ and the sums of indices over the preimages of any edge or vertex is d, so K has finite index in G.

For the final statement, note that a finitely generated group has only finitely many subgroups of a given finite index, and so some power $\Psi = \Phi^k$ preserves K. The suspension K_Ψ is finite index in G_Φ by one final application of the tower law. ☐

Even if this fails, we may still be able to proceed in some cases.

Proposition 8.3. *Suppose $G \cong A *_C B$ or $G \cong A*_C$ with C a finite group. Further suppose that $\Phi \in \mathrm{Aut}(G)$ preserves the conjugacy classes of A and*

B, and that the restrictions of Φ to A is such that $A \rtimes_{\Phi|_A} \mathbb{Z}$ belongs to $\mathbf{FJC_X}$ (and similarly for B, if applicable). Then $G \rtimes_\Phi \mathbb{Z}$ belongs to $\mathbf{FJC_X}$.

The proof being largely an elaboration of the arguments in Section 6.B, here we indicate the necessary changes and references.

Proof. First, we have to argue that Φ preserves the action on the Bass–Serre tree T for this splitting. This follows from [Lev05], or for the two vertex case already from [For02], which give that these one edge splittings are *rigid*: the unique reduced tree in their deformation spaces, together with the hypothesis that Φ preserves the vertex groups.

So we may consider the action of $G \rtimes_\Phi \mathbb{Z}$ on T. Use t to denote the generator of the $\mathbb{Z}$ factor. After possibly passing to a finite index subgroup (by taking the square of Φ, if necessary) we may suppose the quotient graphs are the same for both actions, and that t stabilises an edge e. Edge stabilisers are virtually cyclic, and admit a map to $\mathbb{Z}$ with a conjugate of t is mapped to the generator. For G_e, we may take this preimage to be t.

Note that by work of Wall [Wal67, Lemma 4.1] virtually cyclic groups act on the line and have a unique maximal finite normal subgroup which is the kernel of this action. Since G_e surjects onto $\mathbb{Z}$, in fact this is the unique maximal finite subgroup.

Following the proof in the free splitting case, we need to take a tree of cylinders to ensure that we have an acylindrical action. Just as in that case, the commensurablilty relation is admissible, and we take the tree of cylinders T_c relative to this relation.

We need to adapt the proofs of Lemma 6.2 and Lemma 6.3 to the new situation. In the first case, we assume that e and $w^{-1}t^j e = w^{-1}e$ are edges in the same cylinder. Then $\langle t \rangle$ and $\langle w^{-1}tw \rangle$ are finite index subgroups of the respective stabilisers, and again the commensurability relation implies that there are powers $n, m \in \mathbb{N}$ so that

$$t^n = w^{-1}t^m w = w^{-1}\Phi^m(w)t^m.$$

As before, we see that $n = m$ and w is periodic.

To adapt the proof of Lemma 6.3, make the adjustments of the first paragraph of that proof and then consider the whole intersection $G_v \cap G_{v'}$. This group fixes edges in two distinct cylinders, so is contained inside some $G_e \cap G_{e'}$, where this intersection is between two virtually cyclic subgroups that are not commensurable. In particular, this means the intersection is finite; in fact its cardinality is bounded by the size of the unique maximal finite subgroup (in either – they are conjugate). This means the action of $G \rtimes_\Phi \mathbb{Z}$ on T_c is acylindrical.

To finish the proof, we recall the bipartite nature of T_c, and observe that vertex stabilisers are either isomorphic to $A \rtimes_{\Phi|_A} \mathbb{Z}$ (or the same for B – the original vertex stabilisers), or a cylinder stabiliser $\mathrm{Per}(\Phi) \rtimes_\Phi \mathbb{Z}$. The first kind is in $\mathbf{FJC_X}$ by hypothesis; the second by Lemma 2.6. □

With this in hand, one can begin to try and run the induction argument of Section 6 on a Stallings–Dunwoody decomposition of a more general infinitely ended group. However, there seems as yet to be no analogy for the relative hyperbolicity argument used in the non-sporadic case, and so the induction will not be able to proceed if at some stage we encounter a maximal periodic "Stallings–Dunwoody type splitting" that has more than one edge, and at least one edge with non-trivial stabiliser.

A proof of the following conjecture, the analogy of Theorem 3.2 for general infinite ended groups, should complete the proof of Theorem A with no assumption on torsion.

Conjecture 8.4. *Suppose G is the fundamental group of a non-sporadic graph of groups with finite edge stabilisers, and $\Phi \in \mathrm{Aut}(G)$ is fully irreducible relative to this splitting. Then $G \rtimes_{\Phi^N} \mathbb{Z}$ is hyperbolic relative to the suspensions of polynomially growing subgroups of Φ.*

It may be necessary to assume accessibility in the previous conjecture but for now we do not. As in the free product case, the correct notion of growth should be with respect to the translation length function for the action of G on the Bass–Serre tree. The correct definition of *fully irreducible* appears to be that in any splitting (strictly) dominated by ours every power of Φ does not preserve the set of elliptic subgroups.

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