HOMOLOGY GROWTH OF POLYNOMIALLY GROWING MAPPING TORI

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Abstract. We prove that residually finite mapping tori of polynomially growing automorphisms of hyperbolic groups, groups hyperbolic relative to finitely many virtually polycyclic groups, right-angled Artin groups (when the automorphism is untwisted), and right-angled Coxeter groups have the cheap rebuilding property of Abert, Bergeron, Fraczyk, and Gaboriau. In particular, their torsion homology growth vanishes for every Farber sequence in every degree.

1. Introduction

Let Γ be a residually finite group of type F. By Lück's celebrated approximation theorem, the i^{th} ℓ^2-Betti number $b_i^{(2)}(\Gamma)$ of Γ is a measure of the growth of the i^{th} homology of Γ with rational coefficients [Lüc94]. More precisely, if $(\Gamma_k)_{k\in\mathbb{N}}$ is a descending sequence of finite index normal subgroups of Γ such that $\bigcap_{k\in\mathbb{N}}\Gamma_k = 1$, then

$$b_i^{(2)}(\Gamma) = \lim_{k\to\infty} \frac{\dim_{\mathbb{Q}} H_i(\Gamma_k, \mathbb{Q})}{[\Gamma \colon \Gamma_k]}.$$

The ℓ^2-Betti numbers are important group invariants which have found many applications in topology and group theory (see [Lüc02] and the references therein for a comprehensive account). It is thus natural to study the growth of other homology groups associated to Γ, as well as the growth of the torsion part $|H_i(\Gamma_k, \mathbb{Z})|_{\mathrm{tors}}$ of the homology.

This paper is concerned with the growth of the mod-p Betti numbers of certain groups Γ, as well as the *homology torsion growth*, which is defined to be

$$t_j(\Gamma; \Gamma_k) = \lim_{k\to\infty} \frac{\log |H_j(\Gamma_k, \mathbb{Z})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]},$$

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where $(\Gamma_k)_{k\in\mathbb{N}}$ is a Farber sequence of Γ. The *integral torsion* $\rho^{\mathbb{Z}}(\Gamma)$ of Γ is given by

$$\rho^{\mathbb{Z}}(\Gamma) := \sum_{j\geqslant 0} t_j(\Gamma;\Gamma_k).$$

The invariant has largely evaded computation, aside from right-angled Artin groups and certain graph products [AOS21, OS21] (see also [FHL22] where non-vanishing is proven for certain Bestvina–Brady groups), the only other known results the authors are aware of prove its vanishing in various instances [BV13, Sau16, KKN17, ABFG21]. Although we mention there is an upper bound for negatively curved Riemannian manifolds of dimension at least four in terms of their volume [BGS20]. An important conjecture of Lück relates the ℓ^2-torsion $\rho^{(2)}(\Gamma)$ of Γ with the integral torsion $\rho^{\mathbb{Z}}(\Gamma)$.

Conjecture 1.1 ([Lüc13, Conjecture 1.11(3)]). *Let Γ be an infinite residually finite ℓ^2-acyclic group of type* VF. *Then, $\rho^{(2)}(\Gamma) = \rho^{\mathbb{Z}}(\Gamma)$.*

The conjecture is known to hold for some classes of groups, including amenable groups by the work of Kar–Kropholler–Nikolov [KKN17] and Li–Thom [LT14], and fundamental groups of closed aspherical manifolds which admit non-trivial S^1-action, or contain a nontrivial elementary amenable normal subgroup by the work of Lück [Lüc13].

Recently, the first, third and fourth authors proved that the conjecture holds for all mapping tori of polynomially growing outer automorphisms of finite rank free groups [AHK22]. Specifically, they showed that such groups have vanishing homology torsion growth; the vanishing of the ℓ^2-torsion was proven by Clay [Cla17]. The aim of this paper is to extend the homological torsion growth vanishing to a much larger family of polynomially growing mapping tori.

To that end, fix a finite generating set S of a group G. For any $g \in G$, let $\|[g]\|_S$ denote the length of the shortest word in the conjugacy class $[g]$ of g, and $|g|_S$ the length of the shortest word equivalent to g in G. An outer automorphism $\Phi \in \mathrm{Out}(G)$ is said to *grow polynomially*, if for every conjugacy class c in G, there exists some integer $d \geqslant 0$ and real number $C > 0$ such that for all $n \in \mathbb{N}$,

$$\|\Phi^n(c)\|_S \leqslant Cn^d + C.$$

An automorphism $\phi \in \mathrm{Aut}(G)$ is said to *grow polynomially*, if for every element $g \in G$, there exists some integer $d \geqslant 0$ and $C > 0$ such that for all $n \in \mathbb{N}$,

$$|\phi^n(g)|_S \leqslant Cn^d + C.$$

Note that both of these notions are independent of the choice of a finite generating set for G.

The growth of (outer) automorphisms has been widely studied in the literature, most notably for automorphisms of free abelian groups thanks to the Jordan decomposition, surface groups with the Nielsen–Thurston decomposition [FM11, Theorem 13.2], or free groups thanks to the train track theory of Bestvina–Handel [BH92]. We note that, for a large family of groups $\mathcal{G}$ including free abelian groups or hyperbolic groups, an automorphism of $G \in \mathcal{G}$ has either polynomial or at least exponential growth (see [Cou22, Theorem 1.1]). This dichotomy is even finer in the case of free groups where Levitt [Lev09] gave a complete classification of all possible types of growth for automorphisms of free groups. However, as shown by Coulon [Cou22], the automorphisms of some groups may have more exotic types of growth.

Our main result is the following:

Theorem A. *Let Γ be a residually finite group isomorphic to one of*

- *$G \rtimes_\Phi \mathbb{Z}$ with G hyperbolic;*
- *$G \rtimes_\Phi \mathbb{Z}$ with G is hyperbolic relative to a finite collection of virtually polycyclic groups;*
- *$A_L \rtimes_\Phi \mathbb{Z}$ where A_L is a right-angled Artin group and $\Phi \in \mathrm{Out}(A_L)$ is untwisted (see Section 6); or*
- *$W_L \rtimes_\Phi \mathbb{Z}$ where W_L is a right-angled Coxeter group.*

If Φ is polynomially growing, then for every Farber sequence $(\Gamma_k)_{k\in\mathbb{N}}$ of Γ, every $j \geqslant 0$ and every field $\mathbb{K}$, we have

$$\lim_{k\to\infty} \frac{\dim_{\mathbb{K}} H_j(\Gamma_k, \mathbb{K})}{[\Gamma : \Gamma_k]} = 0 \quad \textit{and} \quad \lim_{k\to\infty} \frac{\log |H_j(\Gamma_k, \mathbb{Z})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]} = 0.$$

As explained above, the automorphisms of most of the groups involved in Theorem A have either exponential or polynomial growth, so that Theorem A deals with a substantial class of cyclic extensions of such groups.

As in [AHK22], the key tool in proving Theorem A is the *cheap rebuilding property* developed in a recent breakthrough of Abert–Bergeron–Fraczyk–Gaboriau [ABFG21]. The property ensures the vanishing of homology torsion growth of residually finite groups Γ which satisfy it. Crucially for us, the cheap rebuilding property of Γ can be deduced whenever Γ admits a sufficiently nice action on a CW-complex with stabilisers which satisfy the cheap rebuilding property (see Section 2.2 for more details, and specifically Theorem 2.11). The strategy introduced by Abert–Bergeron–Fraczyk–Gaboriau, which allows for inductive arguments, turns out to be sufficiently flexible to be applied in various distinct situations. Indeed, such a technique enables the control of the (torsion) homology growth of for instance $\mathrm{SL}_n(\mathbb{Z})$, mapping class groups [ABFG21], $\mathrm{Out}(W_n)$ [GGH22] or of inner-amenable groups [Usc22].

When the fibre G is one-ended and relatively hyperbolic we deploy the theory of JSJ decompositions [GL17] following Guirardel–Levitt. In order to tackle the case where the fibre G in $G \rtimes \mathbb{Z}$ is infinitely-ended, we prove a combination-type theorem, which uses the key observation that if a polynomially growing outer automorphism preserves a free product decomposition, then after possibly passing to a power, the outer automorphism preserves a *sporadic* free factor system (see Proposition 3.1). The terminology and machinery we build upon comes from the work of Guirardel–Horbez [GH22].

Theorem 3.2. *Let $G = G_1 * \ldots * G_k * F_N$ be a residually finite free product of groups. Fix $\alpha \in \mathbb{N}$. Suppose that for every $i \in \{1, \ldots, k\}$, and every polynomially growing automorphism $\phi_i \in \mathrm{Aut}(G_i)$, the group $G_i \rtimes_{\phi_i} \mathbb{Z}$ has the cheap α-rebuilding property.*

Then for every polynomially growing automorphism $\phi \in \mathrm{Aut}(G)$ preserving the conjugacy classes of the groups G_i with $i \in \{1, \ldots, k\}$, the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property.

The work to extend this theorem (and the first two parts of Theorem A) to groups which might have torsion is done in Section 5.

To prove Theorem A in the case of right-angled Artin and Coxeter groups, we use the work of Fioravanti on coarse median preserving automorphisms of groups [Fio21]. Whilst we do not explicitly use any coarse median geometry, this does explain the extra hypothesis of *untwisted* in the case of right-angled Artin groups. The key here is that Fioravanti provides us with an action on a product of trees invariant under the automorphism. From here, we use an inductive argument involving Theorem 3.2 to deduce the cheap rebuilding property.

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2. Background

2.1. Free products, free factor systems and the graph of free factors.

Let $G_1, \ldots, G_k$ be a finite collection of finitely generated groups, let F_N be a free group of rank N and let

$$G = G_1 * \ldots * G_k * F_N.$$

We denote by $\mathcal{F}$ the set of conjugacy classes of the groups G_i with $i \in \{1, \ldots, k\}$. We refer to the pair $(G, \mathcal{F})$ as a *free product*.

Definition 2.1. The pair $(G, \mathcal{F})$ is a *sporadic free product* if one of the following holds:

(1) we have $k = 0$ and $G = \mathbb{Z}$;
(2) we have $k = 1$ and $G = G_1$ or $G = G_1 * \mathbb{Z}$;
(3) we have $k = 2$ and $G = G_1 * G_2$.

Otherwise, the pair $(G, \mathcal{F})$ is a *nonsporadic free product*.

Given a free product $(G, \mathcal{F})$, an element $g \in G$ is *peripheral* if there exists $[A] \in \mathcal{F}$ with $g \in A$. Otherwise, we say that g is *nonperipheral*.

A *free factor system of* $(G, \mathcal{F})$ is a finite collection $\mathcal{F}' = \{[A_1], \ldots, [A_\ell]\}$ of conjugacy classes of finitely generated subgroups of G such that:

(1) for every $i \in \{1, \ldots, k\}$, there exists a subgroup A of G with $G_i \subseteq A$ and $[A] \in \mathcal{F}'$;
(2) there exists a subgroup B of G such that $G = A_1 * \ldots * A_\ell * B$.

A free factor system of $(G, \mathcal{F})$ is *proper* if it is distinct from $\mathcal{F}$ and $\{[G]\}$.

There is a natural partial order on the set of free factor systems of $(G, \mathcal{F})$, where $\mathcal{F}_1 \leqslant \mathcal{F}_2$ if, for every $[A] \in \mathcal{F}_1$, there exists $[B] \in \mathcal{F}_2$ with $A \subseteq B$.

Note that $\mathcal{F}$ is minimal for this partial order. A *free factor* of $(G, \mathcal{F})$ is a free factor system consisting of a unique element.

Definition 2.2. Let $(G, \mathcal{F})$ be a free product. The *free factor graph of* $(G, \mathcal{F})$, denoted by $\mathrm{FF}(G, \mathcal{F})$, is the graph whose vertices are the proper free factors of $(G, \mathcal{F})$, two free factors $\mathcal{F}_1$ and $\mathcal{F}_2$ being adjacent if $\mathcal{F}_1 < \mathcal{F}_2$ or $\mathcal{F}_2 < \mathcal{F}_1$.

By results of Guirardel and Horbez [GH22, Proposition 2.11] (see also the work of Bestvina and Feighn [BF14] and Handel and Mosher [HM14] for the case $G = F_N$), the graph $\mathrm{FF}(G, \mathcal{F})$ is Gromov hyperbolic.

The group $\mathrm{Out}(G, \mathcal{F})$ of outer automorphisms of G which preserve $\mathcal{F}$ has a natural action by isometries on the graph $\mathrm{FF}(G, \mathcal{F})$ induced by its action on the set of free factors of $(G, \mathcal{F})$. The next result describes the loxodromic elements of $\mathrm{FF}(G, \mathcal{F})$. Recall that an element $\phi \in \mathrm{Out}(G, \mathcal{F})$ is fully irreducible if no power of ϕ fixes a free factor system of $(G, \mathcal{F})$.

Theorem 2.3. [GH22, Theorem 4.1] *Let $(G, \mathcal{F})$ be a nonsporadic free product. An element $\phi \in \mathrm{Out}(G, \mathcal{F})$ is a loxodromic element of $\mathrm{FF}(G, \mathcal{F})$ if and only if ϕ is fully irreducible.*

The next theorem gives an existence condition of fully irreducible elements in subgroups of $\mathrm{Out}(G, \mathcal{F})$.

Theorem 2.4. [GH22, Theorem 7.1] *Let $(G, \mathcal{F})$ be a nonsporadic free product and let H be a finitely generated subgroup of $\mathrm{Out}(G, \mathcal{F})$. If H does not virtually preserve a proper $(G, \mathcal{F})$-free factor system then H contains a fully irreducible outer automorphism.*

We now describe the Gromov boundary of $\mathrm{FF}(G, \mathcal{F})$. A $(G, \mathcal{F})$*-tree* is an $\mathbb{R}$-tree T equipped with an action of G by isometries such that, for every $i \in \{1, \ldots, k\}$, the group G_i fixes a point in T. Given a $(G, \mathcal{F})$-tree T and a point $x \in T$, we denote by G_x the stabiliser of x.

A $(G, \mathcal{F})$-tree is *very small* if tripod stabilisers are trivial and arc stabilisers are cyclic (maybe trivial), nonperipheral and root-closed.

A *Grushko* $(G, \mathcal{F})$*-tree* is a very small $(G, \mathcal{F})$-tree T such that T is simplicial, the action of G is minimal, edge stabilisers are trivial and, for every vertex v, either G_v is trivial or $[G_v] \in \mathcal{F}$. Recall that minimal means that G does not preserve a proper subtree of T.

Note that, if $\{[A]\}$ is a free factor of $(G, \mathcal{F})$, the free factor system $\mathcal{F}$ induces a free factor system $\mathcal{F}|_A$ of A. A $(G, \mathcal{F})$*-arational tree* is a very small $(G, \mathcal{F})$-tree T which is not a Grushko tree and such that, for every free factor $\{[A]\}$ of $(G, \mathcal{F})$ the action of $(A, \mathcal{F}|_A)$ on its minimal tree in T induces a Grushko $(A, \mathcal{F}|_A)$-tree.

The following theorem relates the Gromov boundary of $\mathrm{FF}(G, \mathcal{F})$ with the $(G, \mathcal{F})$-arational trees.

Proposition 2.5. [GH22, Theorem 3.4] *Let $(G, \mathcal{F})$ be a nonsporadic free product and let H be a finitely generated subgroup of $\mathrm{Out}(G, \mathcal{F})$. If H has a finite orbit in $\partial_\infty \mathrm{FF}(G, \mathcal{F})$, then H has a finite index subgroup which fixes the homothety class of a $(G, \mathcal{F})$-arational tree.*

If T is a $(G, \mathcal{F})$-arational tree, we denote by $[T]$ its homothety class and by $\mathrm{SF}\colon \mathrm{Stab}([T]) \to \mathbb{R}_+^*$ the stretching factor homomorphism.

Lemma 2.6. [GH22, Lemma 6.1, Corollary 6.7] *Let T be a $(G, \mathcal{F})$-arational tree. For every $\phi \in \mathrm{Stab}([T])$, we have $\mathrm{SF}(\Phi) \neq 1$ if and only if Φ is fully irreducible.*

Lemma 2.7. *Let $(G, \mathcal{F})$ be a nonsporadic free product and let $\Phi \in \mathrm{Out}(G, \mathcal{F})$ be polynomially growing. Then Φ is not fully irreducible.*

Proof. Suppose towards a contradiction that Φ is fully irreducible. By Theorem 2.3, the element Φ is a loxodromic element of $\mathrm{FF}(G, \mathcal{F})$. In particular, Φ acts on $\mathrm{FF}(G, \mathcal{F})$ with North-South dynamics and has exactly two finite orbits in $\partial_\infty \mathrm{FF}(G, \mathcal{F})$ consisting of its attracting and repelling fixed points. By Proposition 2.5, up to taking a power of Φ, we may suppose that Φ fixes the homothety class of a $(G, \mathcal{F})$-arational tree. By Lemma 2.6, the stretching factor λ of Φ is distinct from 1. We may suppose, up to considering Φ^{-1} that $\lambda > 1$.

Let $g \in G$ and let $\ell([g])$ be the translation length of the conjugacy class of g in T. Since Φ preserves the homothety class of T, for every $m \in \mathbb{N}$, we have $\ell(\Phi^m([g])) = \lambda^m \ell([g])$. But $\ell(\Phi^m([g]))$ is bounded from above by a multiple of $\|\Phi^m([g])\|$ (see for instance [CM87, Propositions 1.5, 1.8]). As Φ is polynomially growing, this implies that, for every $g \in G$, we have $\ell([g]) = 0$. Since G is finitely generated, the group G fixes a point in T (see for instance [CM87, Section 3]), a contradiction. Thus, Φ is not fully irreducible. $\square$

2.2. **The cheap rebuilding property.** Let $\alpha \in \mathbb{N}$. In this section, we give the relevant background regarding the *cheap α-rebuilding property*, which was introduced by Abert, Bergeron, Fraczyk and Gaboriau [ABFG21] to prove that certain groups have vanishing (torsion) homology growth. Although we will not state the complete definition of this property, we list in the following propositions the properties which we will use in the rest of the paper. This property is relevant for our considerations by the following theorem.

Theorem 2.8. [ABFG21, Theorem 10.20] *Let Γ be a residually finite countable group and let $\alpha \in \mathbb{N}$. Suppose that Γ has the cheap α-rebuilding property. For every Farber sequence $(\Gamma_k)_{k \in \mathbb{N}}$ of Γ, each $j \leqslant \alpha$ and every coefficient field $\mathbb{K}$, we have*

$$\lim_{k \to \infty} \frac{\dim_{\mathbb{K}} H_j(\Gamma_k, \mathbb{K})}{[\Gamma : \Gamma_k]} = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{\log |H_j(\Gamma_k, \mathbb{Z})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]} = 0.$$

We refer to [ABFG21] for the definition of a Farber sequence. Examples include decreasing sequences of finite index normal subgroups with trivial intersection.

Proposition 2.9. [ABFG21, Corollary 10.13] *Let Γ be a residually finite countable group and let $\alpha \in \mathbb{N}$. The following statements hold.*

1. *Let $\Gamma' \subseteq \Gamma$ be a finite index subgroup. Then Γ has the cheap α-rebuilding property if and only if Γ' does.*
2. *If Γ has an infinite normal subgroup N such that Γ/N is of type F_α and N has the cheap α-rebuilding property, then Γ has the cheap α-rebuilding property.*

(3) *For every $m \in \mathbb{N}^*$ and every $\alpha \in \mathbb{N}$, the group $\mathbb{Z}^m$ has the cheap α-rebuilding property.*

Lemma 2.10. [ABFG21, Corollary 10.13(4)] *Infinite virtually polycyclic groups have the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$.*

Theorem 2.11. [ABFG21, Corollary 10.13] *Let Γ be a residually finite group acting on a CW-complex Ω in such a way that any element stabilising a cell fixes it pointwise. Let $\alpha \in \mathbb{N}$. Suppose that the following conditions hold:*

(1) *$\Gamma\backslash\Omega$ has finite α-skeleton;*
(2) *Ω is $(\alpha-1)$-connected;*
(3) *for each cell $\omega \in \Omega$ of dimension $j \leqslant \alpha$ the stabiliser $\mathrm{Stab}_\Gamma(\omega)$ has the cheap $(\alpha - j)$-rebuilding property.*

Then Γ itself has the cheap α-rebuilding property.

3. A combination theorem for the cheap rebuilding property of mapping tori

Let $(G, \mathcal{F})$ be a free product. The main result of this section, Theorem 3.2, is a combination theorem which allows us to deduce the cheap rebuilding property for some mapping tori of G, assuming that it holds for the mapping tori of the factors $\mathcal{F}$. We will use the combination theorem in subsequent sections to prove the cheap α-rebuilding property for a large family of mapping tori with polynomially growing monodromy.

The main step in the proof of Theorem 3.2 is to construct, for every polynomially growing outer automorphism $\Phi \in \mathrm{Out}(G)$, a Φ-invariant splitting of G. This is done in Proposition 3.1, which is reminiscent of the Kolchin theorem for elements of $\mathrm{Out}(F_n)$, due to Bestvina, Feighn and Handel [BFH05].

Proposition 3.1. *Let $(G, \mathcal{F})$ be a free product and let $\Phi \in \mathrm{Out}(G, \mathcal{F})$ be polynomially growing. There exists $k \in \mathbb{N}$ such that Φ^k preserves a sporadic free factor system $\mathcal{F}'$. In particular, Φ^k preserves the Bass-Serre tree associated to $\mathcal{F}'$.*

Proof. If $\mathcal{F}$ is sporadic, we may set $\mathcal{F}' = \mathcal{F}$. Otherwise, let $\mathcal{F}'$ be a maximal Φ-periodic proper free factor system. It suffices to prove that $\mathcal{F}'$ is sporadic. Suppose towards a contradiction that $\mathcal{F}'$ is nonsporadic. Up to taking a power of Φ, we may suppose that $\mathcal{F}'$ is Φ-invariant. Thus, we may view Φ as an element of $\mathrm{Out}(G, \mathcal{F}')$. By maximality of $\mathcal{F}'$ and Theorem 2.4, the element Φ is fully irreducible. This contradicts Lemma 2.7. Thus, the free factor system $\mathcal{F}'$ is sporadic. □

Theorem 3.2. *Let $G = G_1 * \ldots * G_k * F_N$ be a residually finite free product of groups. Fix $\alpha \in \mathbb{N}$. Suppose that for every $i \in \{1, \ldots, k\}$, and every polynomially growing automorphism $\phi_i \in \mathrm{Aut}(G_i)$, the group $G_i \rtimes_{\phi_i} \mathbb{Z}$ has the cheap α-rebuilding property.*

Then for every polynomially growing automorphism $\phi \in \mathrm{Aut}(G)$ preserving the conjugacy classes of the groups G_i with $i \in \{1, \ldots, k\}$, the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property.

Proof. The proof is by induction on the Grushko rank $k + N$ of G. If $k = 1$ and $N = 0$, then the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property by hypothesis. If $k = 0$ and $N = 1$, then $G \rtimes_\phi \mathbb{Z}$ is virtually isomorphic to $\mathbb{Z}^2$. By Proposition 2.9 (1) (3), for every $\alpha \in \mathbb{N}$, the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property.

Suppose now that $k + N \geqslant 2$. Let Φ be the outer class of ϕ and let $\mathcal{F}$ be a sporadic free factor sytem given by Proposition 3.1. Let T be the canonical Bass-Serre tree of G associated to $\mathcal{F}$. The tree T has a unique orbit of edges and its edge stabilisers in G are trivial. In particular, vertex stabilisers in G of T are proper free factors of G, hence have a smaller Grushko rank than the one of G. Since T is canonical, it is preserved by Φ. Up to taking a power of Φ we may suppose that Φ acts trivially on the underlying graph of $G\backslash T$.

The actions of G and Φ on T induce an action of $G \rtimes_\phi \mathbb{Z}$ on T. Edge stabilisers in $G \rtimes_\phi \mathbb{Z}$ are infinite cyclic and the stabiliser of a vertex v of T is isomorphic to $G_v \rtimes_{\phi|_{G_v}} \mathbb{Z}$, where G_v is the vertex stabiliser of v in G and $\phi|_{G_v}$ is the automorphism of G_v induced by a representative of Φ preserving G_v.

We now prove that $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property by applying Theorem 2.11 to the action of $G \rtimes_\phi \mathbb{Z}$ on T. Since the action is cocompact and since T is a tree, it suffices to prove that the stabiliser of any vertex of T has the cheap α-rebuilding property and that the stabiliser of any edge of T has the cheap $(\alpha - 1)$-rebuilding property.

Since edge stabilisers in $G \rtimes_\phi \mathbb{Z}$ are infinite cyclic, they have the cheap $(\alpha - 1)$-rebuilding property by Proposition 2.9 (3). Let v be a vertex of T. Note that, since ϕ is a polynomially growing automorphism, so is $\phi|_{G_v} \in \mathrm{Aut}(G_v)$. Since the Grushko rank of G_v is smaller than the one of G, by induction hypothesis, the group $G_v \rtimes_{\Phi|_{G_v}} \mathbb{Z}$ has the cheap α-rebuilding property. Thus by Theorem 2.11, the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property, which concludes the proof. $\square$

Remark 3.3. Note that the following fact follows from the proof. Let $G = G_1 * \ldots * G_k * F_N$ be a residually finite free product of groups and let ϕ be a polynomially growing automorphism of G which preserves the conjugacy classes of the factors G_i. If for every $i \in \{1, \ldots, k\}$ and every $\alpha \in \mathbb{N}$ the group $G_i \rtimes_{\phi|_{G_i}} \mathbb{Z}$ has the cheap α-rebuilding property, then $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$.

4. The cheap rebuilding property for residually finite torsion free (relatively) hyperbolic groups

In this section,we prove the cheap rebuilding property for mapping tori of residually finite (relatively) hyperbolic groups. Theorem 3.2 is the main step in order to prove the cheap rebuilding property for infinitely ended hyperbolic groups. The one-ended case requires the use of the JSJ decomposition of the group, whose properties are presented after Lemma 4.2, following Guirardel–Levitt [GL17].

Proposition 4.1. *Let G be a finitely generated residually finite group of type F_∞. Let $\mathcal{P}$ be a collection of conjugacy classes of subgroups of G. Suppose that the group $\mathrm{Out}(G, \mathcal{P})$ of outer automorphism of G preserving $\mathcal{P}$ is finite.*

For every $\phi \in \mathrm{Aut}(G)$ with $[\phi] \in \mathrm{Out}(G, \mathcal{P})$ and every $\alpha \in \mathbb{N}$, the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property.

Proof. Since $\mathrm{Out}(G, \mathcal{P})$ is finite, the group $G \rtimes_\phi \mathbb{Z}$ has a finite index subgroup isomorphic to $G \times \mathbb{Z}$. By Proposition 2.9 (2) (3), for every $\alpha \in \mathbb{N}$ the group $G \times \mathbb{Z}$ has the cheap α-rebuilding property. By Proposition 2.9 (1), for every $\alpha \in \mathbb{N}$, the $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property. □

Lemma 4.2. *Let S be a compact, connected hyperbolic surface and let $G = \pi_1(S)$. Let $D \in \mathrm{MCG}(S)$ be a Dehn multi-twist. For every $\alpha \in \mathbb{N}$, the group $G \rtimes_D \mathbb{Z}$ has the cheap α-rebuilding property.*

Proof. Let $\beta = (\beta_1, \ldots, \beta_k)$ be the multi-curve associated to D, where, for any distinct $i, j \in \{1, \ldots, k\}$, the curves β_i and β_j are disjoint and not parallel. Then β induces a decomposition of the surface S which is preserved by D. This decomposition also induces a tree T equipped with an action of G with cyclic stabilisers which is preserved by D. Thus, the group $G \rtimes_D \mathbb{Z}$ also acts on T.

In order to prove Lemma 4.2, we use Theorem 2.11 applied to the action of $G \rtimes_D \mathbb{Z}$ on T. As in the proof of Theorem 3.2, it suffices to prove that, for every $\alpha \in \mathbb{N}$ and every cell $w \in T$, the stabiliser of w in $G \rtimes_D \mathbb{Z}$ has the cheap α-rebuilding property.

Edge stabilisers in $G \rtimes_D \mathbb{Z}$ are isomorphic to $\mathbb{Z}^2$, hence have the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$ by Proposition 2.9 (3). Vertex stabilisers in $G \rtimes_T \mathbb{Z}$ are isomorphic to $G_v \times \mathbb{Z}$, where G_v is the fundamental group of a connected component of $S \backslash \beta$, hence have the cheap α-rebuilding property for every α by Proposition 2.9 (2). Thus, by Theorem 2.11, for every $\alpha \in \mathbb{N}$, the group $G \rtimes_D \mathbb{Z}$ has the cheap α-rebuilding property. □

Let $\mathcal{G}$ be the family of torsion free groups which are hyperbolic relative to a finite family of virtually polycyclic groups. This includes for instance the class of toral relatively hyperbolic groups, which are torsion free groups hyperbolic relative to a finite collection $\mathcal{P}$ of conjugacy classes of finitely generated abelian subgroups. Let $G \in \mathcal{G}$ and suppose that G is one-ended relative to $\mathcal{P}$. Using the work of Guirardel and Levitt [GL17, Corollary 9.20] (see also [GL15, Section 3.3]), one can construct a canonical JSJ tree T_G for G, that is, a simplicial tree equipped with an action of G which is preserved by $\mathrm{Aut}(G)$. The group $\mathrm{Aut}(G)$ has a finite index subgroup $\mathcal{K}(T_G)$ which acts as the identity on the underlying graph of $G \backslash T_G$. Edge stabilisers of T_G in G are virtually polycyclic. If v is a vertex of T_G, its stabiliser G_v in G satisfies one of the following:

(1) the group G_v is isomorphic to the fundamental group of a compact hyperbolic surface S and the image of the natural homomorphism $\mathcal{K}(T_G) \to \mathrm{Out}(G_v)$ is contained in the mapping class group $\mathrm{MCG}(S)$ of S;
(2) there exists $[P] \in \mathcal{P}$ with $G_v = P$. In particular, the group G_v is virtually polycyclic;
(3) the image of the natural homomorphism $\mathcal{K}(T_G) \to \mathrm{Out}(G_v)$ is finite.

Theorem 4.3. *Let $G \in \mathcal{G}$ and let $\Phi = [\phi] \in \mathrm{Out}(G)$ be a polynomially growing outer automorphism. Then $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property for all α.*

Proof. By Theorem 3.2, it suffices to prove the result when G is one-ended. Let T_G be the JSJ tree associated to G described above. Since T_G is preserved by $\mathrm{Aut}(G)$, the action of G and ϕ on T_G induces an action of $G \rtimes_\phi \mathbb{Z}$ on T_G. Up to taking a power of ϕ, we may suppose that $\phi \in \mathcal{K}(T_G)$.

We prove Theorem 4.3 by applying Theorem 2.11 to the action of $G \rtimes_\phi \mathbb{Z}$ on T_G. As in the proof of Theorem 3.2, it suffices to prove that, for every cell $\omega \in T_G$, the group $\mathrm{Stab}(\omega)$ has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$.

Edge stabilisers in $G \rtimes_\phi \mathbb{Z}$ are virtually polycyclic. Thus, by Lemma 2.10, they have the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$.

Let $v \in T_G$. There are three cases for the vertex stabilisers.

Case 1:. *G_v is the fundamental group of a compact hyperbolic surface S and $\mathcal{K}(T_G) \to \mathrm{Out}(G_v)$ has image contained in* $\mathrm{MCG}(S)$.

Since ϕ is polynomially growing, its image in $\mathrm{MCG}(S)$ is in fact a Dehn multi-twist D. Thus, the stabiliser in $G \rtimes_\phi \mathbb{Z}$ of v is isomorphic to $\pi_1(S) \rtimes_D \mathbb{Z}$. By Lemma 4.2, it has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$. ◆

Case 2:. *G_v is virtually polycyclic.*

Here the stabiliser of v in $G \rtimes_\phi \mathbb{Z}$ is virtually polycyclic-by-$\mathbb{Z}$ (hence, polycyclic). Thus, it has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$ by Lemma 2.10. ◆

Case 3:. *The image of the natural homomorphism $\mathcal{K}(T_G) \to \mathrm{Out}(G_v)$ is finite.*

Up to taking a power of ϕ, we may suppose that the stabiliser of v in $G \rtimes_\phi \mathbb{Z}$ is isomorphic to $G_v \times \mathbb{Z}$. By for instance [GL15, Lemma 3.8], the group G_v belongs to $\mathcal{G}$. By results of Dahmani [Dah03, Theorem 0.1], the group G_v is of type F_∞. By Proposition 2.9 (2), for every $\alpha \in \mathbb{N}$, the group $G_v \times \mathbb{Z}$ has the cheap α-rebuilding property. ◆

Thus, Theorem 2.11 implies that $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property for every $\alpha \in \mathbb{N}$. ☐

Remark 4.4. The restriction to virtually polycyclic subgroups is also used in Case (3); to deduce that G_v is of type F_∞ requires that arbitrary subgroups in $\mathcal{P}$ themselves are of finite type.

5. Extending to groups with torsion

In this section we prove two results, allowing the methods in Sections 3 and 4 to apply to groups which have torsion, and therefore can be infinitely ended without splitting as a free product.

First, we extend Theorem 4.3 to all residually finite groups hyperbolic relative to finitely many virtually polycyclic subgroups, which requires only one more passage to a finite index subgroup.

Lemma 5.1. *Let G be a residually finite group which is hyperbolic relative to a finite collection of virtually polycyclic groups. Then G is virtually torsion free.*

Proof. It is well known, using for instance the action of G on a relative Rips complex (see [Dah03]), that for a relatively hyperbolic group G, there exists a finite number of finite subgroups $F_1, \ldots, F_k$ of G, such that any finite-order element $g \in G$ can be conjugated into some F_i, or into a peripheral subgroup of G. Since polycyclic groups contain finitely many conjugacy classes of finite order elements by [BCRS91, Theorem 7.1], it follows that G also has finitely many conjugacy classes of finite order elements. Now since G is residually finite, there exists a finite index normal subgroup G' of G which does not contain any of the finite conjugacy classes, and thus is torsion free. □

Theorem 5.2. *Suppose G is a finitely generated residually finite group, hyperbolic relative to a finite set $\mathcal{P}$ of conjugacy classes of virtually polycyclic groups, and suppose $\Phi \in \mathrm{Out}(G)$ is polynomially growing. Then $G \rtimes_\Phi \mathbb{Z}$ has the cheap α-rebuilding property for all α.*

Proof. Let G' be a finite index torsion free characteristic subgroup of G as in Lemma 5.1. By Proposition 2.9(1), it suffices to show that the finite index subgroup $G' \rtimes_{\Phi|_{G'}} \mathbb{Z} \leqslant_f G \rtimes_\Phi \mathbb{Z}$ has the cheap α-rebuilding property for all α. Note that $G' \leqslant_f G$ is one ended, torsion free and hyperbolic relative to virtually polycyclic subgroups, and thus by Theorem 4.3 it has the cheap α-rebuilding property for all α. □

Since it may be of independent interest, we also record a version of Theorem 3.2 for infinitely ended and accessible (for instance, finitely presented) groups. Note that such a group is not necessarily virtually torsion free.

Recall that an infinitely ended group splits over a finite subgroup [Sta71]; that the group is accessible when such a splitting can be iteratively refined until the vertex groups are one-ended or finite, and that finitely presented groups are accessible [Dun85]. Two such splittings of an accessible infinitely ended group will have the same set of elliptic subgroups, although it is possible that there are finite subgroups stabilising a vertex in one splitting and not in another. However, the same one-ended vertex stabilisers occur in all such splittings, and passing to a *reduced* splitting removes even this ambiguity [GL17].

Theorem 5.3. *Let G be a residually finite, infinitely ended, and accessible group. Fix $\alpha \in \mathbb{N}$. Let $\{G_i\}$ be (representatives of each conjugacy class of) the one-ended groups occurring in a Stallings–Dunwoody decomposition of G, and suppose that for every G_i and every polynomially growing automorphism $\phi_i \in \mathrm{Aut}(G_i)$, the group $G_i \rtimes_{\phi_i} \mathbb{Z}$ has the cheap α-rebuilding property.*

Then for every polynomially growing automorphism $\phi \in \mathrm{Aut}(G)$ the group $G \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property.

First, we prove a lemma reducing the situation above to a free product:

Lemma 5.4. *Suppose a finitely generated group G is infinitely-ended, residually finite and accessible. Then G has a finite index subgroup H which is a free product of one-ended groups and a finitely generated free group, and the suspension $G \rtimes_\phi \mathbb{Z}$ has a finite index subgroup $H \rtimes_{\phi'} \mathbb{Z}$.*

Proof. Since G is accessible, it has a Stallings–Dunwoody decomposition: a splitting over finite groups where the vertex groups are finite or one-ended. Consider a graph of groups with Bass–Serre tree T realising such a splitting.

Consider the set of of non-trivial elements which either stabilise an edge in a fixed fundamental domain or are contained in a finite group stabilising a vertex in that fundamental domain. (The cases overlap; equivalently, one can take the set of non-trivial elements in the edge groups and finite vertex groups of the graph of groups.) This set is finite, and so G has a finite index normal subgroup H that intersects this set trivially.

We consider the action of H on T. Since H was chosen to be normal, it has trivial intersection with all edge stabilisers (they are conjugates of those that were excluded), and so this action is with trivial edge stabilisers. For the vertex stabilisers, there are two possibilities: either G_v was finite, in which case $H_v = H \cap G_v$ is trivial, or G_v was one-ended, in which case H_v is finite index in the original stabiliser G_v, so is again one ended. The quotient graph of groups — giving a splitting of H — is finite, has trivial edge groups, and has one-ended or trivial vertex groups, so H satisfies the first conclusion.

For the statement about the suspension, note that since G is finitely generated it has only finitely many subgroups of a given finite index, and so some power of the automorphism ϕ preserves H. Let ϕ' be the restriction to H, and observe that $H \rtimes_{\phi'} \mathbb{Z}$ is finite index in $G \rtimes_{\phi} \mathbb{Z}$ as required. □

Proof of Theorem 5.3. Using Lemma 5.4, we see that $G \rtimes_{\phi} \mathbb{Z}$ has a finite index subgroup $H \rtimes_{\phi'} \mathbb{Z}$ where H is a free product of one-ended groups $\{H_j\}$ and a finitely generated free group. This is a Grushko decomposition of H, and so up to passing to a finite index subgroup corresponding to taking a higher power of ϕ we can assume that the conjugacy class of every H_j is preserved by ϕ'. We now apply Theorem 2.11 to $H \rtimes_{\phi'} \mathbb{Z}$ together with Proposition 2.9(1) to obtain the conclusion. □

6. Right angled Artin and Coxeter groups

Let L be a flag complex on $[m] = \{1, \ldots, m\}$ and recall that A_L and W_L, respectively, denote the *right-angled Artin* and *Coxeter groups* on L. We say an automorphism of A_L is *untwisted* if it is contained in the subgroup $U(L) \leqslant \mathrm{Aut}(A_L)$ where $U(L)$ is the subgroup generated by the following automorphisms:

(1) *graph automorphisms*, namely automorphisms of L;
(2) *inversions*, namely $\iota_v \colon A_L \to A_L$ by $v \mapsto v^{-1}$ and $u \mapsto u$ for $u \neq v$ and $u, v \in L^{(0)}$;
(3) *partial conjugations*, namely $k_{w,C}$ for $w \in L^{(0)}$ and a connected component C of $L \backslash \mathrm{st}(w)$. We have $k_{w,C}(u) = w^{-1}uw$ if $u \in C^{(0)}$ and $k_{w,C}(u) = u$ if $u \in L^{(0)} \backslash C$; and
(4) *folds*, namely $\tau_{v,w}$ for $v, w \in L^{(0)}$ with $\mathrm{lk}\, v \subseteq \mathrm{st}\, w$ and $\mathrm{lk}\, v \subseteq \mathrm{lk}\, w$. They are defined by $\tau_{v,w}(v) = vw$ and $\tau_{v,w}(u) = u$ for all $u \in L^{(0)} \backslash \{v\}$.

Note that by [Fio21, Proposition A(3)] untwisted automorphisms of A_L are exactly the automorphisms which preserve the standard coarse median structure on A_L. We will not use this fact or any results about coarse medians explicitly but we note that it underpins much of our work in this section.

We denote the finite index subgroup of $U(A_L)$ generated by inversions, folds and partial conjugations by $U_0(A_L)$. Similarly we denote the finite index subgroup of $\mathrm{Aut}(W_L)$ generated by all automorphisms except graph automorphisms by $U_0(W_L) = \mathrm{Aut}_0(W_L)$. This notation is justified since all automorphisms of W_L are coarse median preserving by [Fio21, Proposition A(2)]. That these subgroups are indeed finite index is given by [Fio21, Remark 3.27] in the case of A_L and [SS19, Proposition 1.2] in the case of W_L.

Theorem 6.1. *Let L be a flag complex on $[m]$ and let $\Gamma = A_L \rtimes_\phi \mathbb{Z}$. If ϕ is an untwisted and polynomially growing automorphism of A_L, then Γ has the cheap α-rebuilding property for all α.*

Proof. We proceed by induction on m, the number of vertices of L. When $m = 1$ we have that A_L is isomorphic to $\mathbb{Z}$. In this case Γ is virtually $\mathbb{Z}^2$ and so the result follows from Proposition 2.9. We now suppose $m > 1$. Note that if $K \subset L$ is a full subcomplex then any untwisted automorphism of A_L preserving A_K restricts to an untwisted automorphism of A_K. To prove the inductive step there are three cases to consider.

Case 1:. *A_L is freely reducible.* In this case A_L admits a Grushko splitting $A_{K_1} * \cdots * A_{K_k} * F_n$ where each K_i and $[n]$ is a full subcomplex of L. A sufficiently high power of an automorphism preserves the conjugacy class of every factor A_{K_i}, and so after passing to such a power and applying Proposition 3.1 we may assume this Grushko decomposition is ϕ-invariant. Moreover, each such subcomplex contains at least one vertex but strictly less than m vertices. Thus, by the inductive hypothesis each $A_{K_i} \rtimes_{\phi|_{A_{K_i}}} \mathbb{Z}$ satisfies the cheap α-rebuilding hypothesis for all α. The conclusion follows from Remark 3.3. ◆

Case 2:. *A_L is freely and directly irreducible.*

As in the previous case, up to taking a power of ϕ, we may assume that ϕ is contained in the subgroup $U_0(A_L)$. By [Fio21, Proposition D] A_L splits as an amalgamated free product $A_J *_{A_{J\cap K}} A_K$ with each $J, K, J \cap K \subset L$ non-empty such that the Bass–Serre tree T is ϕ-invariant. Moreover, for $X \in \{J, K, J \cap K\}$ we see $\phi(A_X) = A_X$ (see [Fio21, Lemma 5.3]). It follows Γ acts on T with stabilisers conjugate to $A_X \rtimes_{\phi|_{A_X}} \mathbb{Z}$. By the inductive hypothesis, the subgroups $A_X \rtimes_{\phi|_{A_X}} \mathbb{Z}$ satisfy the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$. The conclusion follows from Theorem 2.11. ◆

Case 3:. *A_L is directly reducible.*

Now, A_L splits as $\prod_i A_{K_i}$ where each A_{K_i} is non-trival and either freely reducible or freely and directly irreducible. Thus, A_L acts on a product of trees $X = \prod_i T_i$. Here each T_i either arises from Proposition 3.1 if A_{K_i} is freely reducible, or from the amalgamated product splitting given by [Fio21, Proposition D] as in Case 2 when A_{K_i} is both directly and freely irreducible.

As in the previous two cases, up to passing to a power we may assume $\phi|_{A_{K_i}}$ preserves the stabilisers of A_{K_i} acting on T_i. Since the stabilisers of A_L acting on X are products of the stabilisers of the A_{K_i} acting on T_i we see that $\phi(\mathrm{Stab}_{A_L}(\sigma)) = \mathrm{Stab}_{A_L}(\sigma)$ for each cell $\sigma \in X$. Thus, $A_L \rtimes_\phi \mathbb{Z}$ acts on X with the stabiliser of a cell σ isomorphic to $A_{J_\sigma} \rtimes_{\phi|_{A_{J_\sigma}}} \mathbb{Z}$ for some RAAG A_{J_σ} where J_σ is non-empty and has strictly less vertices than L. The conclusion follows from Theorem 2.11. ◆

This completes the proof of the inductive step and the theorem. ☐

Theorem 6.2. *Let L be a flag complex on $[m]$ and let $\Gamma = W_L \rtimes_\phi \mathbb{Z}$. If ϕ is polynomially growing, then Γ has the cheap α-rebuilding property for all α.*

Proof. The proof is entirely analogous to Theorem 6.1 with the following modifications. In the case where $m = 1$ we have that $\Gamma = \mathbb{Z}/2 \times \mathbb{Z}$ which has the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$ by Proposition 2.9. By [Fio21, Proposition A(2)] the group $U_0(W_L) = \mathrm{Aut}_0(W_L)$ has finite index in $\mathrm{Aut}(W_L)$. The three cases are now identical, taking into account the remarks after Theorem E and at the start of Section 5 of [Fio21], since the results we use for A_L also hold for W_L. □

We can actually say more regarding the cheap α-rebuilding property for mapping tori of automorphisms of RAAGs. Indeed, if L is $(\alpha-1)$-connected, then A_L has the cheap α-rebuilding property [ABFG21, Theorem I]. In particular, if L is contractible then for any automorphism ϕ of A_L, the group $A_L \rtimes_\phi \mathbb{Z}$ has the cheap α-rebuilding property for all α. On the other hand, by [AOS21, OS21], the mod-p torsion homology growth of A_L equals the reduced mod-p Betti numbers of L shifted by a degree. In turn, by [FHL22, Theorem B], this is equal to the dimension of the homology of A_L with coefficients in a certain universal division ring $\mathcal{D}_{\mathbb{F}_p A_L}$ (sometimes called agrarian homology). Suppose $\widetilde{H}^*(L;\mathbb{Q}) = 0$ and $\widetilde{H}^*(L;\mathbb{F}_p) \neq 0$. Let $\Gamma = A_L \rtimes_\phi \mathbb{Z}$ and suppose $\mathcal{D}_{\mathbb{F}_p\Gamma}$ exists (this will be true if Γ is residually finite rationally solvable for instance). In this case the ℓ^2-torsion of Γ will vanish by [DL03] and [Lüc02, Theorem 7.27(7)]. But, in [HK21, Section 4] the authors introduce agrarian torsion $\tau^{\mathcal{D}_{\mathbb{F}_p\Gamma}}(\Gamma)$ taking values in $(\mathcal{D}_{\mathbb{F}_p\Gamma})^\times_{\mathrm{ab}}/\{\pm 1\}$. In light of this we raise the following question.

Question 6.3. *Let L be a flag complex on $[m]$ such that $\widetilde{H}^*(L;\mathbb{Q}) = 0$ and $\widetilde{H}^*(L;\mathbb{F}_p) \neq 0$ for some prime p. Let $\Gamma = A_L \rtimes_\phi \mathbb{Z}$ where ϕ is exponentially growing. Does $\mathcal{D}_{\mathbb{F}_p\Gamma}$ exist and if so is $\tau^{\mathcal{D}_{\mathbb{F}_p\Gamma}}(\Gamma) \neq 0$?*

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