

COHERENCE FOR ELEMENTARY AMENABLE GROUPS

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ABSTRACT. We prove that for an elementary amenable group, coherence of the group, homological coherence of the group, and coherence of the integral group ring are all equivalent. This generalises a result of Bieri and Strebel for finitely generated soluble groups.

1. INTRODUCTION

A group G is *type F_n* if G admits a model for a $K(G, 1)$ with finite n -skeleton, *type F_∞* if G is type F_n for all n , and *type F* if G admits a finite model for a $K(G, 1)$. If G has a finite index subgroup of type F , then we say that G is *type VF* . Replacing ‘a model for a $K(G, 1)$ ’ in the previous definitions with ‘a projective resolution of \mathbb{Z} over $\mathbb{Z}G$ ’, one obtains the properties FP_n , FP_∞ , FP , and VFP , respectively.

It is known from the work of Bestvina and Brady [BB97] that the properties of a group being finitely presentable and being of type FP_2 are not equivalent. In this note we investigate the equivalence of local versions of these properties.

A group G is *coherent* if every finitely generated subgroup is finitely presented and *homologically coherent* if every finitely generated subgroup is of type FP_2 . A ring R is *coherent* if every finitely generated left ideal is finitely presented. These various notions of coherence are being very actively investigated – let us mention here the remarkable recent article of Jaikin-Zapirain–Linton [JZL23] that establishes coherence of one-relator groups via the homological variants.

If $\mathbb{Z}G$ is coherent or G is coherent, then G is homologically coherent. Neither converse has been established and in particular there is no known implication between the properties of $\mathbb{Z}G$ being coherent and G being coherent.

In [BS79], Bieri and Strebel prove that the properties of coherence of the group, homological coherence, and coherence of the group ring are equivalent for a finitely generated soluble group. In doing so they relate the properties to ascending HNN extensions, these being groups which are generated by a *vertex group B* and *stable letter t* such that $t^{-1}Bt \supseteq B$, corresponding to actions on trees with one orbit of edges and a marked end. The HNN extension is *properly ascending* if $t^{-1}Bt \supset B$.

Note that Groves [Gro78] independently established the equivalence of (2) and (3) below.

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Theorem 1.1 (Bieri–Strebel). *For a finitely generated soluble group G the following are equivalent:*

- (1) G is coherent;
- (2) G is homologically coherent;
- (3) $\mathbb{Z}G$ is coherent;
- (4) G is polycyclic or G is a properly ascending HNN extension with polycyclic vertex group.

Our main theorem (Theorem 1.2) concerns elementary amenable groups. Recall that the class of *elementary amenable groups* is the smallest class of groups containing all finite and abelian groups that is closed under extensions and direct limits (and isomorphisms). In this paper, classes of groups are implicitly assumed to be closed under isomorphism. The class of elementary amenable groups is embryonically present in von Neumann’s seminal work [vN29], then reintroduced by the short name of *elementary groups* in [Day57] and studied further in [Cho80]. It can readily be shown that the class is subgroup closed and quotient closed, and that it contains all soluble groups. Thus the following generalises Bieri and Strebel’s result.

Theorem 1.2. *For a finitely generated elementary amenable group G the following are equivalent:*

- (1) G is coherent;
- (2) G is homologically coherent;
- (3) $\mathbb{Z}G$ is coherent;
- (4) either G is virtually polycyclic or G is a properly ascending HNN extension with virtually polycyclic vertex group.

We combine the above with the following three observations: firstly, that ascending HNN extensions of soluble groups are soluble (see Lemma 3.9); secondly, that an ascending HNN extension of a group of type F_∞ is also of type F_∞ ; and thirdly, that by [KMPN09, Theorem 1.1] an elementary amenable group of type F_∞ is type VF. In this way we obtain the following.

Corollary 1.3. *Every (homologically) coherent elementary amenable group is virtually soluble and of type VF.*

Although the conclusions of Theorems 1.1 and 1.2 are very similar, the methods of proof are quite different.

Bieri–Strebel prove that any homologically coherent soluble group G has finite Prüfer rank — namely, there exists an r such that every subgroup can be generated by r elements. From here they apply results of Mal’cev and the theory of nilpotent groups to deduce that G is polycyclic or that G is a properly ascending HNN extension with polycyclic vertex group.

In comparison, we introduce the notion of G being *indicably coherent* (Definition 3.1) and prove that this property is inherited by quotients of groups with no non-abelian free subgroups, and so in particular by elementary amenable groups. We prove a general virtual splitting principle for elementary amenable groups as ascending HNN extensions of lower complexity groups. Using these two tools we show that every finitely generated elementary amenable indicably coherent group G is virtually soluble.

Since the technique we use is very much based on understanding the behaviour of epimorphisms to \mathbb{Z} and their kernels, and since we utilise it explicitly in the proof of Corollary 2.3, let us briefly discuss the Bieri–Neumann–Strebel invariant, introduced in [BNS87]; note that we will discuss a slightly restricted version of the invariant, over \mathbb{Z} rather than \mathbb{R} . For a group G , we will identify the first cohomology group $H^1(G; \mathbb{Z})$ with the set of homomorphisms $G \rightarrow \mathbb{Z}$. Under this isomorphism, primitive elements of $H^1(G; \mathbb{Z})$ correspond to epimorphisms $G \rightarrow \mathbb{Z}$. The *BNS invariant* $\Sigma^1(G)$ is the subset of the set of primitive elements of $H^1(G; \mathbb{Z})$ consisting of precisely those epimorphisms that are induced by ascending HNN extensions with finitely generated vertex groups [Bro87, Proposition 3.1 and Theorem 5.2]. The BNS invariant satisfies the following property: given an epimorphism $\phi: G \rightarrow \mathbb{Z}$, the kernel $\ker \phi$ is finitely generated if and only if both ϕ and $-\phi$ lie in $\Sigma^1(G)$ [BNS87, Theorem B1].

Two conjectures. We close the introduction with two conjectures. First, recall two related reformulations, one of the Noetherian property and one of coherence of rings: namely a ring R is *Noetherian* if and only if every finitely generated left or right R -module is type FP_∞ , and a ring R is *coherent* if and only if every finitely presented left or right R -module is of type FP_∞ . Put yet another way, Noetherian rings are the rings where the category of finitely generated (left or right) modules is abelian and coherent rings are the rings where the category of finitely presented (left or right) modules is abelian.

The first conjecture is a long-standing open problem attributed to Baer. It is well known to hold for elementary amenable groups (we include a proof of this fact in Theorem 4.1).

Baer’s Noetherian conjecture. *Let G be a group. Then $\mathbb{Z}G$ is Noetherian if and only if G is virtually polycyclic.*

Note that one direction of the conjecture is known: that the group ring of a virtually polycyclic group is Noetherian was established by Philip Hall in the nineteen-fifties [Hal54] and can be considered as a non-commutative version of Hilbert’s Basissatz. Hall’s result would surely have been an inspiration to Baer and others to begin asking if the converse is true. As observed by Kropholler–Lorensen [KL19], the fact that $\mathbb{Z}G$ is Noetherian implies that G must be amenable, and hence in Baer’s conjecture we can restrict our attention solely to amenable groups.

We offer the following, analogous conjecture. Let us say that the group ring $\mathbb{Z}G$ is *properly coherent* if it is coherent but not Noetherian.

Conjecture 1.4. *Let G be a finitely generated amenable group. Then $\mathbb{Z}G$ is properly coherent if and only if G is a properly ascending HNN extension with virtually polycyclic vertex group.*

Note that assumption of amenability is necessary since $\mathbb{Z}F_n$, where F_n denotes the free group of rank n , is a semifir, and hence is coherent. Our Theorem 1.2 verifies the conjecture for elementary amenable groups.

2. RADICALS

It will be convenient in the proofs that follow to make use of a variation on the Hirsch–Plotkin radical of a group. Recall that the Fitting subgroup of a group is the join of the nilpotent normal subgroups. Fitting’s lemma says that if H

and K are nilpotent normal subgroups of a group then HK is also a nilpotent normal subgroup, and consequently the Fitting subgroup is the directed union of the nilpotent normal subgroups. Note that the Fitting subgroup need not necessarily be nilpotent itself, and hence it can be desirable to consider the Hirsch–Plotkin radical of a group. This is defined to be the join of the normal locally nilpotent subgroups and it is a characteristic locally nilpotent subgroup. There is a lemma that underpins the Hirsch–Plotkin radical in the same way that Fitting’s lemma underpins the Fitting subgroup: this lemma says that if H and K are normal locally nilpotent subgroups then their join HK is locally nilpotent. The Hirsch–Plotkin radical is the join of all normal locally nilpotent subgroups and the lemma ensures that it is the directed union of the normal locally nilpotent subgroups: in particular the Hirsch–Plotkin radical is the unique largest normal locally nilpotent subgroup.

For our purpose, other radicals are convenient. The following generalises the Hirsch–Plotkin radical. If \mathcal{P} is a subgroup-closed class of groups then \mathbf{LP} denotes the class of those groups whose finitely generated subgroups belong to \mathcal{P} .

Lemma 2.1. *Let \mathcal{P} be a subgroup-closed class of finitely generated groups with the property that if H and K are normal \mathcal{P} -subgroups of a group then HK is also a \mathcal{P} -subgroup. Then every group contains a unique largest normal \mathbf{LP} -subgroup.*

Proof. It suffices to prove that the join of any two normal \mathbf{LP} -subgroups again belongs to \mathbf{LP} . Below we adopt the notation that for subsets X and Y of a group, X^Y denotes the subgroup generated by the conjugates $y^{-1}xy$, $x \in X, y \in Y$. Let H and K be normal \mathbf{LP} -subgroups. To prove that HK is an \mathbf{LP} -group, we must show that for finite subsets $X \subseteq H$ and $Y \subseteq K$ the subgroup $\langle X, Y \rangle$ belongs to \mathcal{P} . Fix such a choice of X and Y , and let $C = \{[x, y] = x^{-1}y^{-1}xy; x \in X, y \in Y\}$. Note that C is finite and is a subset of $H \cap K$ because H and K are normal. Therefore $\langle C, X \rangle$ is a finitely generated subgroup of H and so belongs to \mathcal{P} . Since \mathcal{P} is subgroup closed, we deduce that $C^{\langle X \rangle}$ belongs to \mathcal{P} and so it is finitely generated. Moreover, $C^{\langle X \rangle}$ is contained in $H \cap K$. Therefore $\langle Y, C^{\langle X \rangle} \rangle$ is a finitely generated subgroup of K and hence it belongs to \mathcal{P} . Therefore the subgroup $C^{\langle X \rangle \langle Y \rangle}$ also belongs to \mathcal{P} . Now [Rob96, 5.1.7] shows that $C^{\langle X \rangle \langle Y \rangle} = [\langle X \rangle, \langle Y \rangle]$, the subgroup generated by all the commutators $[z, w]$ with $z \in \langle X \rangle$ and $w \in \langle Y \rangle$, and then we have

$$\langle Y, C^{\langle X \rangle} \rangle = \langle Y, C^{\langle X \rangle \langle Y \rangle} \rangle = \langle Y, [\langle X \rangle, \langle Y \rangle] \rangle = Y^{\langle X \rangle}.$$

This shows that $Y^{\langle X \rangle}$ belongs to \mathcal{P} . By symmetry $X^{\langle Y \rangle}$ belongs to \mathcal{P} . Therefore $\langle X, Y \rangle = Y^{\langle X \rangle} X^{\langle Y \rangle}$ is the join of two normal \mathcal{P} -subgroups of $\langle X, Y \rangle$, and so belongs to \mathcal{P} . □

Corollary 2.2. *In any group there is a unique largest normal locally polycyclic subgroup and there is a unique largest normal locally {virtually polycyclic} subgroup, and both these subgroups are characteristic.*

Proof. Both the class of polycyclic groups and the class of virtually polycyclic groups are closed under subgroups, quotients, and extensions. Since virtually polycyclic groups are finitely generated, both classes satisfy the hypotheses of Lemma 2.1. □

Corollary 2.3. *Suppose that the group G has a subgroup H of finite index which is an ascending HNN extension over a virtually polycyclic vertex group. Then G itself*

is virtually polycyclic or is an ascending HNN extension over a virtually polycyclic vertex group.

Proof. Since finite-index subgroups of an ascending HNN extension over a virtually polycyclic vertex group are also ascending HNN extensions over virtually polycyclic vertex groups, we may assume that H is normal.

We may suppose that $H = B *_B t$ is an ascending HNN extension where B is a virtually polycyclic subgroup of H , where t is an element of H such that $B \subseteq B^t$, and where $K = \bigcup_{n \geq 0} B^{t^n}$ is a normal subgroup of H such that $H = K \langle t \rangle$. Being an ascending union of virtually polycyclic subgroups we know that K is a locally virtually polycyclic subgroup and therefore contained in a characteristic such subgroup: namely the radical subgroup L given by Lemma 2.1 and its immediate corollary. Clearly G/L has no non-trivial finite normal subgroups and is virtually cyclic, so G/L is either trivial, infinite cyclic, or infinite dihedral. When $G = L$, then G is locally virtually polycyclic. Since H is finitely generated, so is G , and thus G is virtually polycyclic.

If G/L is infinite dihedral, then $H/(H \cap L) \cong \mathbb{Z}$, and the two epimorphisms $H \rightarrow \mathbb{Z}$ with kernel K thought of as elements of the first cohomology group of H over \mathbb{Z} lie in the same orbit under the action of G . Since H is an ascending HNN extension with a finitely generated vertex group, one of these cohomology classes lies in the Bieri–Neumann–Strebel invariant $\Sigma^1(H)$; since the two classes are related via an automorphism of H , so does the other class. Therefore, the kernel of these two epimorphisms is finitely generated. This forces $K = B$, and so H is virtually polycyclic. It follows that G is virtually polycyclic as well.

Finally, if $G/L \cong \mathbb{Z}$ then using the fact that G is finitely presented and [BS78, Theorem A] we conclude that G is an HNN extension with vertex group being a finitely generated subgroup of L . Since G does not contain a non-abelian free group, the HNN extension is ascending. Also, finitely generated subgroups of L are virtually polycyclic, and we are done. \square

3. INDICABILITY, COHERENCE, AND CASCADING GROUPS

Definition 3.1. A group is *indicable* if it is trivial or maps onto \mathbb{Z} .

A group G is *indicably coherent* if for every finitely generated subgroup $H \leq G$ and every epimorphism $\phi: H \rightarrow \mathbb{Z}$, the group H splits as an HNN extension with finitely generated vertex group A in such a way that ϕ coincides with the quotient map dividing H by the normal closure of A ; in such a situation, we will say that the HNN extension *realises* the epimorphism ϕ .

Note that when G is indicably coherent and does not contain non-abelian free groups, we automatically get that the HNN extension is ascending or descending. Also, homologically coherent groups are indicably coherent, by [BS78, Theorem A].

Let us now state the crucial property of indicably coherent groups.

Proposition 3.2. *If G is indicably coherent and does not contain non-abelian free groups, then every quotient of G is also indicably coherent.*

This result follows immediately from Lemma 3.3.

Lemma 3.3. *Let G be an ascending HNN extension with vertex group A , inducing an epimorphism $\phi: G \rightarrow \mathbb{Z}$. If $\rho: G \rightarrow Q$ is an epimorphism such that ϕ factors through ρ , then Q is an ascending HNN extension with vertex group $\rho(A)$.*

Proof. Let $t \in G$ be the stable letter of the given ascending HNN extension, and let $\iota: A \rightarrow A$ be the associated monomorphism. Observe that

$$\iota(\ker \rho|_A) = t^{-1} \ker \rho|_A t \leq \ker \rho;$$

it is immediate that $\iota(\ker \rho|_A) \leq \iota(A) \leq A$, and so

$$\iota(\ker \rho|_A) \leq \ker \rho|_A.$$

Therefore, ι descends to a homomorphism $\iota': \rho(A) \rightarrow \rho(A)$. Moreover, if $\iota'(a') = 1$ for some $a' \in \rho(A)$, then for every $a \in \rho^{-1}(a')$ we have $\iota(a) \in \ker \rho$ and so $a = t\iota(a)t^{-1} \in \ker \rho$ as well, which in turn implies $a' = \rho(a) = 1$. Hence, ι' is injective.

We may define an ascending HNN extension H with vertex group $\rho(A)$, stable letter t , and associated monomorphism ι' . It is clear that ρ induces a quotient map $G \rightarrow H$ which can be followed with a quotient map $\sigma: H \rightarrow Q$ to become ρ .

Take $h \in \ker \sigma$. Since $h \in H$, we may write

$$h = t^{n_0} x_1 t^{n_1} \dots x_m t^{n_m}$$

with $x_i \in \rho(A)$, $n_i \in \mathbb{Z}$. Since ϕ factors through ρ , we must have $\sum n_i = 0$, and so we may rewrite $h = \prod_{i=1}^m t^{-n'_i} x_i t^{n'_i}$. Let $N = \max_i |n'_i|$. We now have $t^{-N} h t^N \in \ker \sigma$ being a product of conjugates of elements of $\rho(A)$ by non-negative powers of t . But each such conjugate is obtained from an element of $\rho(A)$ by applying ι' a suitable number of times, and hence lies in $\rho(A)$ itself. Thus,

$$h \in \ker \sigma \cap \rho(A) = \{1\}.$$

We conclude that σ is an isomorphism, as desired. □

Definition 3.4 (Elementary amenable hierarchy). Define EA_0 to be the class of groups which are finite or abelian. For each ordinal $\alpha > 0$ define EA_α to be the class of groups G that are of one of the following forms:

- $G = \bigcup_i G_i$, where $G_i \in \text{EA}_{\beta_i}$ with $\beta_i < \alpha$;
- and $G = N.Q$, where \cdot is any group extension and $N, Q \in \text{EA}_\beta$ with $\beta < \alpha$.

A very easy proof by transfinite induction shows that every class EA_α is closed under taking subgroups.

Before proceeding, we will need one technical definition.

Definition 3.5 (Cascading groups). For an ordinal α , we define the class of α -cascading groups inductively:

- The class of 0-cascading groups consists of finite groups;
- For $\alpha > 0$, a group G is α -cascading if it admits a finite index subgroup H and an epimorphism $H \rightarrow \mathbb{Z}$ such that for every splitting of H as an HNN extension realising the epimorphism, if the vertex group is finitely generated, then it is β -cascading for some $\beta < \alpha$.

A group is *cascading* if it is α -cascading for some ordinal α .

Observe that many groups will satisfy Definition 3.5 vacuously, since they will have no finitely generated vertex groups at all. The definition really comes to life when combined with indicable coherence.

Lemma 3.6. *Finitely generated free-abelian groups are cascading, and the property of being cascading passes to finite index overgroups.*

The proof of the first part is an obvious induction on the rank. The second part is even easier.

Lemma 3.7. *Let α and β be two given ordinals with $\beta < \alpha$. Let G be a finitely generated group in EA_α , and suppose that G fits into the group extension*

$$K \rightarrow G \xrightarrow{\rho} Q,$$

where K lies in EA_β . If Q and all finitely generated groups in EA_β are cascading, then so is G .

Proof. Suppose that Q is γ -cascading. The proof is an induction on γ . If $\gamma = 0$, then Q is finite, and hence K is finitely generated. It lies in EA_β , and hence it is cascading, forcing G to be cascading as well.

Now, suppose that $\gamma > 0$, and that the result holds for all ordinals strictly smaller than γ . Let R be a finite index subgroup of Q , and let $\phi: R \rightarrow \mathbb{Z}$ be an epimorphism such that for every HNN extension decomposition of R realising ϕ , if the vertex group is finitely generated, then it is δ -cascading for some $\delta < \gamma$. Let $H = \rho^{-1}(R)$, and $\psi = \phi \circ \rho|_H: H \rightarrow \mathbb{Z}$. Observe that ψ is an epimorphism.

Suppose that H splits as an HNN extension realising ψ with finitely generated vertex group A . Then, using the fact that this HNN extension must be ascending or descending, Lemma 3.3 implies that R splits as an HNN extension realising ϕ with vertex group $\rho(A)$. Since A is finitely generated, so is $\rho(A)$, and hence $\rho(A)$ is δ -cascading for some $\delta < \gamma$. But now A fits into the exact sequence

$$K \cap A \rightarrow A \rightarrow \rho(A),$$

and induction tells us that A is ϵ_A -cascading, for some ordinal ϵ_A . It follows that G is $(\bigcup_A \epsilon_A + 1)$ -cascading, where the union runs over all finitely generated vertex groups A . □

Corollary 3.8. *If G is a finitely generated elementary amenable group, then G is cascading.*

Proof. Let $G \in EA_\alpha$ for some ordinal α , and suppose that G is finitely generated. The proof is by transfinite induction on α .

If $\alpha = 0$, then G is virtually abelian, and hence cascading.

Now suppose that $\alpha > 0$, and that the result holds for all ordinals strictly smaller than α . As G is finitely generated, if it is obtained from groups lower in the hierarchy as a directed union, then it itself lies lower in the hierarchy, and we are done by the inductive hypothesis. Hence we may assume the existence of an epimorphism $\rho: G \rightarrow Q$ with Q and $K = \ker \rho$ lower in the hierarchy. Since G is finitely generated, so is Q , and hence Q is cascading. We may now use Lemma 3.7. □

We will need the following result.

Lemma 3.9. *A non-empty directed union of soluble groups of derived length c is soluble of derived length c .*

Proof. Since being soluble of class at most c is defined by a group law, it is a local property, and hence is inherited by directed unions. Our union is soluble of derived length equal to c , since it contains a soluble subgroup of class c . □

Theorem 3.10. *Let G be a finitely generated elementary amenable group. If G is indicably coherent, then G is virtually soluble.*

Proof. We have already shown that G is α -cascading. We now proceed by induction on α .

If $\alpha = 0$, then G is finite and we are done. Otherwise, we have a finite-index subgroup $H \leq G$ and an epimorphism $\phi: H \rightarrow \mathbb{Z}$ from the definition of cascading. Since G is indicably coherent, we find an HNN extension of H realising ϕ with finitely generated vertex group A . Now, A is β -cascading for $\beta < \alpha$, and hence, by the inductive hypothesis, it is virtually soluble. Also, the HNN extension is ascending (without loss of generality). Let t denote its stable letter, and $\iota: A \rightarrow A$ the associated monomorphism.

Since A is virtually soluble, there exists a finite group Q such that the intersection B of the kernels of all homomorphisms $A \rightarrow Q$ is soluble. Moreover, since A is finitely generated, the subgroup B has finite index in A . Now, crucially, ι composed with any homomorphism $A \rightarrow Q$ is also such a homomorphism, and so $\iota(B) \leq B$. This allows us to construct a new ascending HNN extension, say K , with vertex group B and stable letter t . It is immediate that $\bigcup_{i \in \mathbb{N}} t^i B t^{-i}$ is a normal subgroup of $\bigcup_{i \in \mathbb{N}} t^i A t^{-i}$, since B is normal in A . Moreover, the image of every $t^i A t^{-i}$ in the quotient $\bigcup_{i \in \mathbb{N}} t^i A t^{-i} / \bigcup_{i \in \mathbb{N}} t^i B t^{-i}$ is finite and has cardinality bounded above by $|A/B|$. Since the quotient is a union of such images, it is finite. Hence K is a finite-index subgroup of H . It is also soluble by Lemma 3.9. \square

Corollary 3.11. *Let G be a finitely generated elementary amenable group. If G is homologically coherent, then G is virtually soluble.*

Proof. Since G is homologically coherent, for every finitely generated subgroup H of G and every epimorphism $H \rightarrow \mathbb{Z}$ we may write H as an HNN extension with finitely generated (in fact FP_2) vertex group by [BS78, Theorem A]. Thus, G is indicably coherent. The result follows from Theorem 3.10. \square

We are now ready to prove the main result.

Theorem 1.2. *For a finitely generated elementary amenable group G the following are equivalent:*

- (1) G is coherent;
- (2) G is homologically coherent;
- (3) $\mathbb{Z}G$ is coherent;
- (4) either G is virtually polycyclic or G is a properly ascending HNN extension with virtually polycyclic vertex group.

Proof. Clearly, each of (1) and (3) implies (2). If we assume (4), then the group G is virtually soluble by Lemma 3.9, and hence Theorem 1.1 shows that G admits a finite index subgroup H with $\mathbb{Z}H$ being coherent. Coherence of rings can be reformulated as follows: a ring R is coherent if and only if every R -homomorphism between finitely generated free R -modules has finitely generated kernel. From this point of view it is immediate that $\mathbb{Z}G$ is also coherent, and (2) follows.

Finally, assume (2). Corollary 3.11 implies that G has a finite-index soluble subgroup which is also homologically coherent. Items (1) and (3) follow from Theorem 1.1, and the observation that coherence of groups passes to finite-index overgroups, and similarly for group rings. For (4) we additionally need to use Corollary 2.3. \square

4. NOETHERIAN GROUPS

For context, let us also mention the following result, which is well known but seems to be hard-to-locate in the literature. Note that a group is called *Noetherian* or *slender* when all its subgroups are finitely generated, or equivalently when it satisfies the ascending chain condition \max - s on subgroups.

Theorem 4.1. *Every Noetherian elementary amenable group G is virtually polycyclic.*

Proof. The proof is an induction on the elementary amenable hierarchy. Every finite group is virtually polycyclic, and so is every finitely generated (and hence every Noetherian) abelian group. This shows the result for the class \mathbf{EA}_0 .

Now suppose that G lies in \mathbf{EA}_α , and the result holds for all groups in \mathbf{EA}_β for all $\beta < \alpha$. Since G is Noetherian, it is finitely generated. As shown above, this implies that G lies in \mathbf{EA}_β for some $\beta < \alpha$, or that we have an epimorphism $\rho: G \rightarrow Q$ with $\ker \rho = K$, and both K and Q lie lower in the hierarchy. Since being Noetherian passes to subgroups and quotients, both K and Q are polycyclic-by-finite. By passing to a finite-index subgroup of G , we may assume that Q is polycyclic.

Let N be a finite-index normal subgroup of K that is polycyclic. The group G/N is then finite-by-polycyclic, and hence polycyclic-by-finite. We conclude that G is virtually polycyclic. \square

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