A NOTE ON ASYNCHRONOUSLY AUTOMATIC GROUPS AND NOTIONS OF NON-POSITIVE CURVATURE

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Abstract. We prove groups acting cocompactly on locally finite trees with hyperbolic vertex stabilisers are asynchronously automatic. Combining this with previous work of the authors we obtain an example of a group satisfying several non-positive curvature properties (being a CAT(0) group, an injective group, a hierarchically hyperbolic group, and having quadratic Dehn function) which is asynchronously automatic but not biautomatic.

1. Introduction

Studying languages and automata related to presentations of groups has been one of the driving motivations of combinatorial and geometric group. This has given rise to many classes of groups — biautomatic groups, automatic groups, asynchronously automatic groups, semihyperbolic groups (introduced in [AB95]), and so on. The reader is referred to [ECH$^+$92] for background on automaticity and [Ree22] for a more recent survey.

Applying the theory of languages and automata to groups has seen a number of successes such as: giving effective solutions to word problem in many 3-manifold groups [ECH$^+$92] and many other groups (for example mapping class groups [Mos95], CAT(0) cubical groups [NR98], systolic groups [JS06], Helly groups [CCG$^+$20], and Coxeter groups [MOP22, OP22]); characterising virtually free groups [MS83] and hyperbolic groups [Pap95, HNS22] via languages; as well elucidating many structural properties of groups admitting stronger language or automata related properties [GS91, AB95].

There are still large gaps in our understanding of how various forms of non-positive curvature relate with various versions of automaticity. For example a recent breakthrough of Leary and Minasyan gave the first examples of CAT(0) groups which are not biautomatic [LM21] and an analogous result involving other forms of non-positive curvature was obtained by the authors in [HV22]. It is still an open question if CAT(0) groups are necessarily (asynchronously) automatic.

In this note we will examine the class *asynchronously automatic groups*, which we will define in Section 2, and its interaction with various classes of non-positively curved groups. Namely, groups with quadratic Dehn function,

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CAT(0) groups, hierarchically hyperbolic groups (HHGs), and groups acting geometrically on injective metric spaces (injective groups). See [BH99, BHS17, BHS19, Lan13] for definitions of the various classes and [HHP20, HV22] for their interactions. Our main technical result is a combination theorem relating groups acting on locally finite trees, hyperbolic groups, and the class of asynchronously automatic groups.

Proposition 1.1. *Let Γ be a group. If Γ acts cocompactly on a locally finite tree with hyperbolic vertex stabilisers, then Γ is asynchronously automatic.*

The proposition can be applied to prove generalised Baumslag–Solitar groups (GBS_1 groups) are asynchronously automatic and to give another proof that hyperbolic-by-free groups are asynchronously automatic (note that a more general result on split extensions of hyperbolic groups was obtained by Bridson [Bri93]). The result can be deduced by combining work of Shapiro [Sha92] and Gersten–Short [GS91]. However, we give a direct proof to make explicit the asynchronous structure.

We highlight the next corollary due to its relation to other results in the literature which we will explain below. Let H be the isometry group of a proper CAT(0) space X. Recall that a *uniform lattice* Γ in H is a discrete subgroup of H such that X/Γ is compact.

Corollary 1.2. *Let H be one of $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ or F_4^{-20} with $n \geqslant 2$ and let T be the automorphism group of a locally finite tree. Suppose T is non-discrete and cocompact. If Γ is a uniform lattice in $H \times T$, then Γ is asynchronously automatic.*

Proof. By [Hug21, Theorem A] the lattice Γ splits as a graph of groups with local groups covirtually isomorphic to uniform H-lattices. That is the stabilisers have a finite normal subgroup and the quotient is isomorphic to a uniform H-lattice These stabilisers act geometrically on the associated symmetric space of H, which is hyperbolic since H has rank one. The result follows from Proposition 1.1. □

Any H as in the previous corollary has an associated rank one symmetric space isometric to $\mathbf{H}^n_{\mathbb{R}}$, $\mathbf{H}^n_{\mathbb{C}}$, $\mathbf{H}^n_{\mathbb{H}}$, or $\mathbf{H}^2_{\mathbb{O}}$. In particular, Γ is quasi-isometric to the product of a hyperbolic symmetric space and a tree. Now, A. Margolis [Mar22, Theorem J] has proven that any group Λ quasi-isometric to a product of hyperbolic graphs $\prod_{i=1}^n X_i$ is biautomatic, provided that none of the X_i are quasi-isometric to a (possibly Euclidean) non-compact symmetric space. Note that by [LM21, Theorem 1.1] and [HV22, Theorem A] this result of Margolis is sharp, namely, there exists uniform lattices in both $\mathrm{Isom}(\mathbb{E}^2)\times T_{10}$ and $\mathrm{PSL}_2(\mathbb{R}) \times T_{24}$ which are not biautomatic.

Combining the previous corollary with the main result of [HV22] we obtain a group with a strange combination of properties. In particular, the group Γ has very strong non-positive curvature properties: being a hierarchically hyperbolic group, a CAT(0) group, acting geometrically on an injective metric space, and therefore having quadratic Dehn function. But Γ fails to be biautomatic. Here we show Γ satisfies the weaker property of being asynchronously automatic. This gives the first example of a group satisfying any of the previously mentioned geometric properties which is not biautomatic but is asynchronously automatic.

Theorem 1.3. *There exists a torsion-free non-residually finite uniform lattice* $\Gamma < \mathrm{PSL}_2(\mathbb{R}) \times T_{24}$ *which is a hierarchically hyperbolic group, an injective group, a* CAT(0) *group, and is not biautomatic. However,* Γ *is asynchronously automatic.*

As far as the authors are aware this is one of the first examples of a quadratic Dehn function group which is asynchronously automatic but not biautomatic. The only other examples are some free-by-cyclic groups claimed to be not automatic by Brady, Bridson, and Reeves [BBR06] announced in Bridson's ICM notes [Bri06].

We raise the following set of questions.

Question 1.4. *Is every (a) hierarchically hyperbolic group, (b) injective group, or (c)* CAT(0) *group asynchronously automatic?*

We do not know of any possible candidate counterexamples to the first two questions, however, the Leary–Minasyan groups introduced in [LM21] (and related groups [Hug21, Val23]) may be a good starting point to disprove (c). Note that by [But22] the Leary–Minasyan groups are not hierarchically hyperbolic groups. They were classified up to isomorphism in [Val22].

Question 1.5. *Which Leary–Minasyan groups are asynchronously automatic?*

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2. Preliminaries

2.1. Automata.

The definitions in this section are standard and have been taken from [ECH$^+$92, Chapters 1 and 7].

Let A be a finite set and let $A^\star$ be the free monoid generated by A. We denote the nullstring by ϵ. A *language* over the alphabet A is a subset $\mathcal{L} \subseteq A^\star$. Let W_L, W_R be words over A. A *shuffle* of (W_L, W_R) is a string $W \in A^\star$ and a map $\{1, \dots, |W|\} \to \{L, R\}$ such that if we substitute the nullstring ϵ in W for each element that maps to R we get W_L and if we substitute ϵ in W for each element that maps to L we get W_R.

Definition 2.1 (Finite state automaton)**.** A *finite state automaton* (FSA) $\mathcal{M}$ over the alphabet A consists a finite directed graph $\mathcal{G}(\mathcal{M})$, together with a (directed) edge label function $\ell\colon E^+(\mathcal{G}(\mathcal{M})) \to A$, a chosen vertex $o \in V(\mathcal{G}(\mathcal{M}))$ called the *initial state* and a subset $F \subseteq V(\mathcal{G}(\mathcal{M}))$ of *final states*. The vertices of $\mathcal{G}(\mathcal{M})$ are often referred to as *states*.

Let $\mathcal{M}$ be an FSA over an alphabet A. We say a string $W \in A^\star$ is *accepted* by $\mathcal{M}$ if and only if there is an oriented path γ in $\mathcal{G}(\mathcal{M})$ starting from o and ending in a vertex $q \in F$ such that γ is labelled by W. A language $\mathcal{L}$ over A is *regular* if and only if there exists an FSA $\mathcal{M}$ such that $\mathcal{L}$ coincides with the strings of $A^\star$ accepted by $\mathcal{M}$. We denote the regular language accepted by $\mathcal{M}$ by $\mathcal{L}(\mathcal{M})$

Definition 2.2 (Asynchronous automaton)**.** An *asynchronous (deterministic two-tape) automaton* $\mathcal{M}$ over A is a partial deterministic automaton over $A \cup \{\$\}$ where the states are partitioned into five subsets, denoted S_L, $S_L^\$$, S_R, $S_R^\$$ and $S^\$$. The set $S^\$$ consists of exactly one state $s^\$$ which will be the unique final state for $\mathcal{M}$. A directed edge e labelled by an element of A with initial vertex in $S_L \cup S_R$ has its terminal vertex in $S_L \cup S_R$; if e has initial vertex in $S_L^\$$ or $S_R^\$$, then it has its terminal vertex in the same set. A directed edge e labelled by \$ with initial vertex in S_L has terminal vertex in $S_R^\$$, and similarly with S_R and $S_L^\$$; if e has initial vertex in $S_L^\$ \cup S_R^\$$, then its terminal vertex is $s^\$$.

We say that $\mathcal{M}$ *accepts* a pair of strings $(W_L, W_R) \in A^\star \times A^\star$ if there is a shuffle W of $(W_L\$, W_R\$)$ which is accepted by the automaton $\mathcal{M}$.

2.2. **Automaticity.** We are interested in studying when a group Γ is asynchronously automatic; we briefly introduce the necessary definitions and basic results on the property below, and refer the interested reader to [ECH+92] for a more comprehensive account.

Let Γ be a group with a finite generating set A. We view A as a finite set together with a function $\pi_A^0 \colon A \to \Gamma$ that extends to a surjective monoid homomorphism $\pi_A \colon A^\star \to \Gamma$, where $A^\star$ is the free monoid on A. We say that a word $W \in A^\star$ *labels* or *represents* the element $\pi_A(W) \in \Gamma$. For simplicity, we will assume that A is symmetric, namely, $\pi_A(A) = \pi_A(A)^{-1}$, and contains the identity, that is $\pi_A(1) = 1_\Gamma$ for an element $1 \in A$. We denote by d_A the combinatorial metric on the Cayley graph $\mathrm{Cay}(\Gamma, A)$ of Γ.

We study combinatorial paths in $\mathrm{Cay}(\Gamma, A)$. Given a path P in $\mathrm{Cay}(\Gamma, A)$ and an integer $t \in \{0, \ldots, |P|\}$, where $|P|$ is the length of P, we denote by $P(t) \in \Gamma$ the t-th vertex of P, so that $P(0)$ and $P(|P|)$ are the starting and ending vertices of P, respectively. We further define $P(t) \in \Gamma$ for any $t \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ by setting $P(t) = P(|P|)$ whenever $t > |P|$.

Given a word $W \in A^\star$, we denote by $\widehat{W}$ the path in $\mathrm{Cay}(\Gamma, A)$ starting at 1_Γ and labelled by W. In particular, for any $t \in \mathbb{Z}_{\geqslant 0}$, we write $\widehat{W}(t)$ for the element of Γ represented by the prefix of W of length $\min\{t, |W|\}$.

Definition 2.3 (Asynchronously automatic group)**.** Let Γ be a group with finite symmetric generating set A containing the identity. An *asynchronous automatic structure* on Γ consists of the set A, a finite state automaton $\mathcal{M}$ over A, and asynchronous automata $\mathcal{M}_x$ for $x \in A$ satisfying:

(1) The map $\pi_A|_{\mathcal{L}(\mathcal{M})} \colon \mathcal{L}(\mathcal{M}) \to \Gamma$ is surjective;
(2) for $x \in A$, we have $(W_L, W_R) \in \mathcal{L}(\mathcal{M}_x)$ if and only if $\pi_A(W_L x) = \pi_A(W_R)$ and both W_L and W_R are elements of $\mathcal{L}(\mathcal{M})$.

Definition 2.4 (Departure function)**.** Let Γ be a group with finite symmetric generating set A containing the identity and let $\mathcal{L}$ be a regular language over A that maps onto Γ. A *departure function* for $(\Gamma, A, \mathcal{L})$ is any function $\mathcal{D} \colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ such that if $W \in \mathcal{L}$, $r, s \geqslant 0$, $t \geqslant \mathcal{D}(r)$, and $s + t \leqslant |W|$, then $d_A(\widehat{W}(s), \widehat{W}(s+t)) \geqslant r$.

The next definition is really a characterisation of boundedly asynchronous groups proven in [ECH+92, Theorem 7.2.8] (see also [Mos95, Proposition on p. 309]).

Definition 2.5 (Boundedly asynchronous group)**.** Let Γ be a group with finite symmetric generating set A containing the identity and let $\mathcal{L}$ be a regular language over A that maps onto Γ under π_A. We say $(A, \mathcal{L})$ is a *boundedly asynchronous structure* if

(a) there exists a departure function $\mathcal{D}$ for $(\Gamma, A, \mathcal{L})$;
(b) there exists a constant $\kappa > 0$, such that, for every pair of strings $V, W \in \mathcal{L}$ with $d_A(\pi_A(V), \pi_A(W)) \leqslant 1$, we have that the Hausdorff distance of the paths $\widehat{V}$ and $\widehat{W}$ is at most κ.

We say Γ is *boundedly asynchronous* if Γ admits a boundedly asynchronous structure.

The definitions of asynchronously automatic group and boundedly asynchronous group are equivalent in the following sense: every boundedly asynchronous group is immediately an asynchronously automatic group and by [ECH+92, Theorem 7.2.4] if a group admits an asynchronously automatic structure, then it also admits a boundedly asynchronous structure.

2.3. **Graphs of groups.** We are interested in groups acting on simplicial trees. We outline the main results we are using below, and refer the interested reader to [Ser03] for a more comprehensive account.

We first introduce the notion of graphs of groups and their fundamental groups. Given a finite undirected graph $\mathcal{G}$ (with loops and multiple edges allowed), we write $V(\mathcal{G})$ and $E^+(\mathcal{G})$ for its sets of vertices and directed edges, respectively. We also write $E^+(\mathcal{G}) \to E^+(\mathcal{G}), e \mapsto \overline{e}$ for the function changing the direction of edges, and $\iota\colon E^+(\mathcal{G}) \to V(\mathcal{G})$ for the starting vertex function, so that an edge e has endpoints $\iota(e)$ and $\iota(\overline{e})$.

Definition 2.6 (Fundamental group of graph of groups)**.** A *graph of groups* is a connected finite undirected graph $\mathcal{G}$ together with a collection of groups $\{\Gamma_v \mid v \in V(\mathcal{G})\}$, a collection of groups $\{\Gamma_e \mid e \in E^+(\mathcal{G})\}$ satisfying $\Gamma_e = \Gamma_{\overline{e}}$, and a collection $\{i_e\colon \Gamma_e \to \Gamma_{\iota(e)} \mid e \in E^+(\mathcal{G})\}$ of injective homomorphisms.

Given a graph of groups $(\mathcal{G}, \{\Gamma_v\}, \{\Gamma_e\}, \{i_e\})$ as above, its *fundamental group* is the group generated by $\left(\bigsqcup_{v \in V(\mathcal{G})} \Gamma_v\right) \sqcup E^+(\mathcal{G})$ with defining relations $\overline{e} = e^{-1}$ for $e \in E^+(\mathcal{G})$, $i_{\overline{e}}(g) = e^{-1} i_e(g) e$ for $e \in E^+(\mathcal{G})$ and $g \in \Gamma_e$, and $e = 1$ for $e \in E^+(T_{\mathcal{G}})$, where $T_{\mathcal{G}}$ is a maximal subtree of $\mathcal{G}$.

Let $(\mathcal{G}, \{\Gamma_v\}, \{\Gamma_e\}, \{i_e\})$ be a graph of groups, and let Γ be its fundamental group. Given $e \in E^+(\mathcal{G})$, let $\mathcal{S}(e)$ be a left transversal of Γ_e in $\Gamma_{\iota(e)}$ containing $1_\Gamma \in \Gamma$; pick also a base vertex $c \in V(\mathcal{G})$. Let $\mathcal{W}_{\mathcal{G}}$ be the set of all expressions of the form $s_1 e_1 s_2 e_2 \cdots s_n e_n$, where $e_1 e_2 \cdots e_n$ is a directed loop in $\mathcal{G}$ based at c, and $s_i \in \mathcal{S}(e_i)$ with $s_i \neq 1_\Gamma$ whenever $e_{i-1} = \overline{e_i}$. It is then well-known [Ser03, Theorem 11 on p. 45] that every element $g \in \Gamma$ has a unique expression of the form $g = Uh$ with $U \in \mathcal{W}_{\mathcal{G}}$ and $h \in \Gamma_c$. We call such an expression the *normal form* for g.

Now let Γ be a group acting cocompactly on a simplicial tree $\mathcal{T}$. Without loss of generality (subdividing edges of $\mathcal{T}$ if necessary), we may assume that the action is without edge inversions, i.e. any element of Γ fixing an edge of $\mathcal{T}$ also fixes its endpoints. In that case, using Bass–Serre theory (see [Ser03, Theorem 13 on p. 55]), Γ can be described as the fundamental group of a graph of groups $(\mathcal{G}, \{\Gamma_v\}, \{\Gamma_e\}, \{i_e\})$, where $\mathcal{G} = \mathcal{T}/\Gamma$, the groups Γ_v and Γ_e are the stabilisers (under the action of Γ) of lifts $\widetilde{v} \in V(\mathcal{T})$ and $\widetilde{e} \in E^+(\mathcal{T})$ of $v \in V(\mathcal{G})$ and $e \in E^+(\mathcal{G})$, respectively, and $i_e \colon \Gamma_e \to \Gamma_{\iota(e)}$ is the inclusion of Γ_e into the stabiliser of $\iota(\widetilde{e})$ composed with an inner automorphism of Γ. Moreover, if $g = Uh$ is a normal form of $g \in \Gamma$, where $U = s_1e_1 \cdots s_ne_n \in \mathcal{W}_\mathcal{T}$ and $h \in \Gamma_c$, then $e_1 \cdots e_n$ is the projection of the geodesic path in $\mathcal{T}$ from $\widetilde{c}$ to $g \cdot \widetilde{c}$; here and later, we write $\mathcal{W}_\mathcal{T}$ for $\mathcal{W}_\mathcal{G}$.

3. Proof of Proposition 1.1

Suppose a group Γ acts cocompactly on a locally finite tree $\mathcal{T}$ without edge inversions (so that we can use the notation of Section 2.3), and suppose that the vertex stabilisers under this action are (Gromov) hyperbolic. Let B be a finite symmetric generating set of Γ_c, and write $\mathcal{W}_B \subseteq B^\star$ for the set of all geodesic words over B. Consider the generating set $A' := E^+(\mathcal{G}) \sqcup B \sqcup \bigsqcup_{e \in E^+(\mathcal{G})} \mathcal{S}(e)$ for Γ and its "symmetric closure" $A = A' \sqcup (A')^{-1}$; note that since $\mathcal{T}$ is locally finite, we have $|\mathcal{S}(e)| < \infty$ for all $e \in E^+(\mathcal{G})$, and hence A' (and therefore A) is finite. Consider the language

$$\mathcal{L} := \{U_\mathcal{T} U_B \mid U_\mathcal{T} \in \mathcal{W}_\mathcal{T}, U_B \in \mathcal{W}_B\}$$

of words over A.

We note that $\mathcal{L}$ is a regular language over A: see Figure 1. We also note that there are finitely many words in $\mathcal{L}$ representing any given element of $g \in \Gamma$, since the normal form for g is unique and there are finitely many geodesic words over B representing any given element of Γ_c. We claim that $(A, \mathcal{L})$ is an boundedly asynchronous structure for Γ.

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Lemma 3.1. *There exists a departure function $\mathcal{D}$ for $(\Gamma, A, \mathcal{L})$.*

Proof. Let $\mathcal{M}$ be a finite state automaton over A accepting $\mathcal{L}$. Suppose, without loss of generality (removing states from $\mathcal{M}$ if necessary), that each state $s \in \mathcal{M}$ is accessible (there exists a word $U_s \in A^\star$ labelling a path from the initial state of $\mathcal{M}$ to s) and live (there exists a word $W_s \in A^\star$ labelling a path from s to a final state of $\mathcal{M}$). Given any two states $s, t \in V(\mathcal{G}(\mathcal{M}))$ and an element $g \in \Gamma$, write $\mathcal{V}_{s,t}(g)$ for the set of words $V \in A^\star$ labelling a path in $\mathcal{M}$ from s to t such that $\pi_A(V) = g$.

Now, as there are finitely many words in $\mathcal{L}$ representing any given element of Γ, it follows that $\mathcal{V}_{s,t}(g)$ is finite for all s, t and g, since $\{U_sVW_t \mid V \in \mathcal{V}_{s,t}(\gamma)\}$ is a collection of words in $\mathcal{L}$ representing $\pi_A(U_s)g\pi_A(W_t) \in \Gamma$. In particular, since A and the number of states in $\mathcal{M}$ are finite, the set $\mathcal{U}_r := \bigcup\{\mathcal{V}_{s,t}(g) \mid s,t \in V(\mathcal{G}(\mathcal{M})), g \in \Gamma, d_A(1,g) < r\}$ is also finite for any $r \geqslant 0$. We can therefore define a function $\mathcal{D} \colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by setting $\mathcal{D}(r)$ to be larger than the length of any word in $\mathcal{U}_r$. It is then clear from the construction that $\mathcal{D}$ is a departure function for $(\Gamma, A, \mathcal{L})$, as required. □

Proposition 3.2. *There exists a constant $\kappa > 0$ with the following property: for every $V, W \in \mathcal{L}$ with $d_A(\pi_A(V), \pi_A(W)) \leqslant 1$, the Hausdorff distance between $\widehat{V}$ and $\widehat{W}$ is at most κ.*

The idea of the proof is as follows: first, it can be shown that any vertex of $\widehat{W}$ that is not a vertex of $\widehat{V}$ is bounded distance away from the left coset

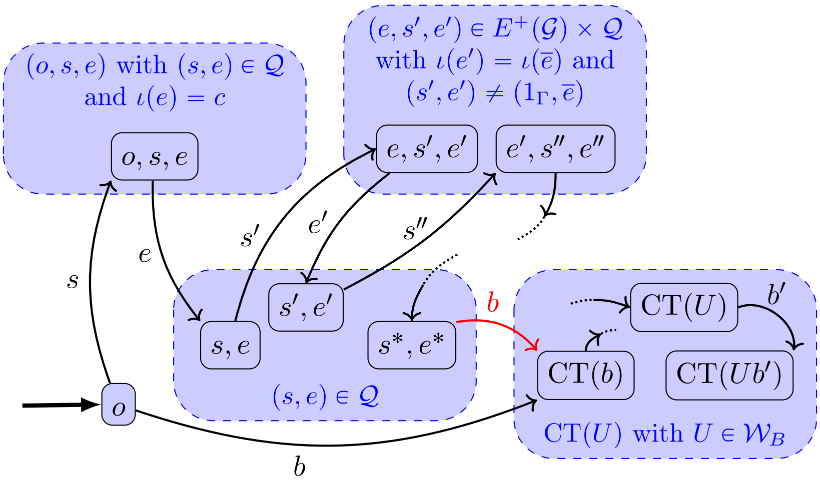

FIGURE 1. A finite state automaton over A accepting $\mathcal{L}$. Here we set $\mathcal{Q} := \{(s, e) \mid e \in E^+(\mathcal{G}), s \in \mathcal{S}(e)\}$. We use b, b' for letters in B, and the red arrow exists if and only if $\iota(\overline{e^*}) = c$. We set $\mathrm{CT}(U) := \{V \in B^* \mid UV \in \mathcal{W}_B\}$ to be the *cone type* of U; it is well-known that since Γ_c is hyperbolic it has finitely many cone types [ECH+92, Theorem 3.2.1]. The initial state of the automaton is (o), and the final states are (o), $(s, e) \in \mathcal{Q}$ with $\iota(\overline{e}) = c$, and $\mathrm{CT}(U)$ for $U \in \mathcal{W}_B$.

$\pi_A(V)\Gamma_c$. The resulting vertices of $\pi_A(V)\Gamma_c$ therefore form a quasi-geodesic, and the result then follows from stability of quasi-geodesics in the hyperbolic group Γ_c. The proof of Proposition 3.2 is illustrated in Figure 2.

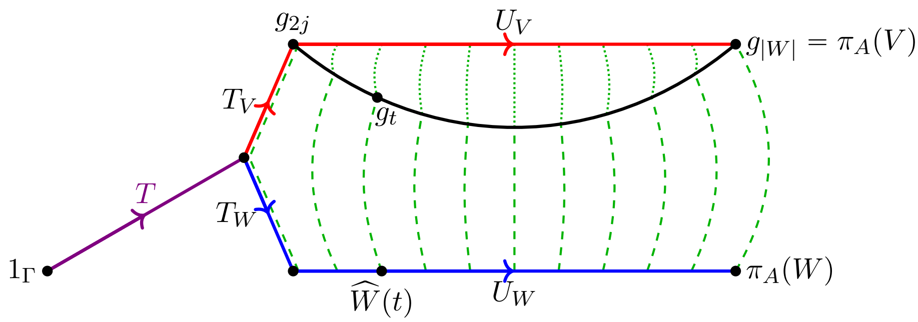

FIGURE 2. The proof of Proposition 3.2, with $\widehat{V}$ (red and purple) and $\widehat{W}$ (blue and purple) shown. The black path is a (ζ, ζ)-quasi-geodesic in $\pi_A(V)\Gamma_c$ (with respect to the metric d_B), the green dashed lines represent paths of length $\leqslant 2\eta$ in $\mathrm{Cay}(\Gamma, A)$, and the green dotted lines are mapped by $\pi_A(V)^{-1}$ to paths of length $\leqslant \kappa - 2\eta$ in $\mathrm{Cay}(\Gamma_c, B)$.

Proof. Let $\eta = \max\{d_{\mathcal{T}}(\tilde{c}, a \cdot \tilde{c}) \mid a \in A\}$, and let $\zeta \geqslant 4\eta + 1$ be such that for any $g \in \Gamma_c$ with $d_A(1, g) \leqslant 4\eta + 1$ we have $d_B(1, g) \leqslant \zeta$ (such η and ζ exist since A is finite). Since Γ_c is hyperbolic, there exists $\kappa \geqslant 2\eta$ such that any two (ζ, ζ)-quasi-geodesics in the Cayley graph $\mathrm{Cay}(\Gamma_c, B)$ are Hausdorff distance $\leqslant \kappa - 2\eta$ away from each other [CDP06, Chapitre 3, Théorème 1.3].

Now, we can write $V = s_1 e_1 \cdots s_n e_n U_V$ and $W = s'_1 e'_1 \cdots s'_m e'_m U_W$, where $s_1 e_1 \cdots s_n e_n, s'_1 e'_1 \cdots s'_m e'_m \in \mathcal{W}_{\mathcal{T}}$ and $U_V, U_W \in \mathcal{W}_B$. Let $j \leqslant \min\{n, m\}$ be the largest integer such that $(s_i, e_i) = (s'_i, e'_i)$ for all $i \leqslant j$; we write $T = s_1 e_1 \cdots s_j e_j$, $T_V = s_{j+1} e_{j+1} \cdots s_n e_n$ and $T_W = s'_{j+1} e'_{j+1} \cdots s'_m e'_m$, so that $V = T T_V U_V$, and $W = T T_W U_W$. Since $d_A(\pi_A(V), \pi_A(W)) \leqslant 1$, it then follows that $d_{\mathcal{T}}(\widetilde{c}, \pi_A(T_V^{-1} T_W) \cdot \widetilde{c}) = d_{\mathcal{T}}(\pi_A(V) \cdot \widetilde{c}, \pi_A(W) \cdot \widetilde{c}) \leqslant \eta$, and therefore $(n - j) + (m - j) \leqslant \eta$.

We aim to show that for any $t \in \mathbb{Z}_{\geqslant 0}$, the element $\widehat{W}(t)$ is distance $\leqslant \kappa$ away from $\widehat{V}$ in $\mathrm{Cay}(\Gamma, A)$; the proof is analogous if the roles of $\widehat{V}$ and $\widehat{W}$ are swapped. The result is clear for $t < 2j$, as in that case $\widehat{W}(t) = \widehat{V}(t)$. Therefore, we may assume, without loss of generality, that $t \geqslant 2j$.

We first claim that there exists an element $g_t \in \pi_A(V)\Gamma_c$ such that we have $d_A(\widehat{W}(t), g_t) \leqslant 2\eta$. Indeed, if $t \leqslant 2m$ then we can take $g_t := \pi_A(T T_V)$: then $g_t^{-1} \widehat{W}(t)$ is labelled by a subword of $T_V^{-1} T_W$, and we have $|T_V| = 2(n - j)$ and $|T_W| = 2(m - j)$, implying that $d_A(\widehat{W}(t), g_t) \leqslant 2(n-j) + 2(m-j) \leqslant 2\eta$. Otherwise, we have $\widehat{W}(t) \in \pi_A(W)\Gamma_c$ and therefore $d_A(\widehat{W}(t)^{-1} \pi_A(V) \cdot \widetilde{c}, \widetilde{c}) = d_A(\pi_A(V) \cdot \widetilde{c}, \pi_A(W) \cdot \widetilde{c}) \leqslant \eta$, implying that $\widehat{W}(t)^{-1} \pi_A(V)$ has normal form $T'U'$, where $T' = s''_1 e''_1 \cdots s''_\ell e''_\ell \in \mathcal{W}_{\mathcal{T}}$ for some $\ell \leqslant \eta$ and $U' \in \mathcal{W}_B$; we then set $g_t := \widehat{W}(t)^{-1} \pi_A(T')$, so that $d_A(\widehat{W}(t), g_t) = d_A(1, \pi_A(T')) \leqslant 2\ell \leqslant 2\eta$, as claimed. Without loss of generality, we may also assume that $g_{|W|} = \pi_A(V)$, since we have $d_A(\pi_A(V), \widehat{W}(|W|)) \leqslant 1$.

Now let $g'_t := \pi_A(T T_V)^{-1} g_t$ for $2j \leqslant t \leqslant |W|$. This yields a collection $g'_{2j}, g'_{2j+1}, \ldots, g'_{|W|} \in \Gamma_c$ such that $d_A(\widehat{W}(t), \pi_A(T T_V) g'_t) \leqslant 2\eta$ for all t. In particular, for $2j \leqslant t < |W|$ we have

$$\begin{aligned} d_A(g'_t, g'_{t+1}) &= d_A(g_t, g_{t+1}) \\ &\leqslant d_A(g_t, \widehat{W}(t)) + d_A(\widehat{W}(t), \widehat{W}(t+1)) + d_A(\widehat{W}(t+1), g_{t+1}) \\ &\leqslant 2\eta + 1 + 2\eta = 4\eta + 1, \end{aligned}$$

implying that $d_B(g'_t, g'_{t+1}) \leqslant \zeta$ by the choice of ζ. In particular, the points $g'_{2j}, \ldots, g'_{|W|}$ are vertices of a (ζ, ζ)-quasi-geodesic in $\mathrm{Cay}(\Gamma_c, B)$ starting at $g'_{2j} = 1_{\Gamma_c}$ and ending at $g'_{|W|} = \pi_A(U_V)$. Therefore, by the choice of κ, the element g'_t is distance at most $\kappa - 2\eta$ away from $\widehat{U_V}$ in $\mathrm{Cay}(\Gamma_c, B)$, and hence in $\mathrm{Cay}(\Gamma, A)$. This implies that $g_t = \pi_A(T T_V) g'_t$ is distance at most $\kappa - 2\eta$ away from $\pi_A(T T_V)\widehat{U_V} \subseteq \widehat{V}$, and thus $\widehat{W}(t)$ is distance at nost $(\kappa - 2\eta) + d_A(\widehat{W}(t), g_t) \leqslant \kappa$ away from $\widehat{V}$, as required. □

We have verified all of the properties for $(A, \mathcal{L})$ to be a boundedly asynchronous structure for Γ. □

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