# HIGMAN–THOMPSON GROUPS AND PROFINITE PROPERTIES OF RIGHT-ANGLED COXETER GROUPS

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Abstract. We prove that every right-angled Coxeter group (RACG) is profinitely rigid amongst all Coxeter groups. On the other hand we exhibit RACGs which have infinite profinite genus amongst all finitely generated residually finite groups. We also establish profinite rigidity results for graph products of finite groups. Along the way we prove that the Higman–Thompson groups $V_n$ are generated by 4 involutions, generalising a classical result of Higman for Thompson's group $V$.

**Key words.** *Coxeter groups, profinite rigidity, Grothendieck rigidity, graph products, generating sets of Higman–Thompson groups*

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## 1. Introduction

For a group $G$ we denote by $\mathcal{F}(G)$ the set of isomorphism classes of finite quotients of $G$. Two groups $G$ and $H$ are said to have the same finite quotients if $\mathcal{F}(G) = \mathcal{F}(H)$. A group $G$ is called *profinitely rigid relative to a class of groups* $\mathcal{C}$ if $G \in \mathcal{C}$ and for any group $H$ in the class $\mathcal{C}$ whenever $\mathcal{F}(G) = \mathcal{F}(H)$, then $G \cong H$. By definition, a finitely generated residually finite group $G$ is called *profinitely rigid (in the absolute sense)* if $G$ is profinitely rigid relative to the class consisting of all finitely generated residually finite groups. If $G$ is not profinitely rigid then we say $G$ is *profinitely flexible*.

The *genus* of $G$, denoted by $\mathcal{G}(G)$, is defined as the set of isomorphism classes of finitely generated residually finite groups with the same finite quotients as $G$. A group $G$ is called *almost profinitely rigid* if $\mathcal{G}(G)$ is finite.

The study of profinite rigidity has motivated and been the subject of a plethora of research. For example, finitely generated nilpotent groups are almost profinitely rigid [Pic71] and so are polycyclic groups [GPS80]. There are metabelian groups with infinite genus [Pic74].

There has been tremendous progress with regard to 3-manifold groups; deep work of Bridson–McReynolds–Reid–Spitler shows that there are hyperbolic 3-manifolds groups which are profinitely rigid in the absolute sense [BMRS20] with more examples constructed in [CW22]. Note there are profinitely flexible, albeit with finite genus, torus bundle groups [Ste72, Fun13, Hem14].

For rigidity within the class of 3-manifold groups even more is known, we summarise some of the highlights: the Thurston geometry is detected by the

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profinite completion [WZ17], so are various decompositions [Wil18a, Wil18b, WZ19], fibring is a profinite invariant [JZ20] (see [HK22] for a generalisation), and in a recent breakthrough Yi Liu showed that finite volume hyperbolic 3-manifolds are almost profinitely rigid amongst 3-manifold groups [Liu23]. An analogous result for generic free-by-cyclic groups was obtained in [HK23].

On the other hand many full-size groups are not profinitely rigid. Platonov–Tavgen' showed that $F_2 \times F_2$ is profinitely flexible [PT86]. More examples were given by Bass–Lubotzky [BL00]. In a different vein Pyber [Pyb04] showed the genus could be uncountable. Next, Bridson–Grunewald gave examples of profinite flexibility amongst the class of finitely presented groups [BG04] and showed that $F_n \times F_n$ for $n \geqslant 3$ has infinite genus. We generalise this result by showing that $F_2 \times F_2$ also has infinite genus (Corollary 5.8). More recently, Bridson [Bri16] showed that the profinite genus amongst finitely presented groups can be infinite.

Despite Coxeter groups being ubiquitous with geometric group theory, the study of their profinite genus has remained elusive. To the authors knowledge the only work on this topic are the following results: Bessa–Grunewald–Zalesskii studied the genus of groups within the class of groups that are virtually the direct products of free and surface groups [BGZ14], Kropholler–Wilkes [KW16] proved that a right-angled Coxeter group (RACG) is profinitely rigid amongst RACGs, Santos Rego–Schwer proved all triangle Coxeter groups are distinguished from each other by their finite quotients [RS22], and the third and fourth author of this article proved irreducible affine Coxeter groups are profinitely rigid amongst Coxeter groups [MV23]. Finally, a small number of hyperbolic reflection groups are known to be profinitely rigid in the absolute sense [BMRS21]. To this end we raise and partially answer the following questions.

**Question 1.1.** *Are Coxeter groups profinitely rigid in the absolute sense? Are they rigid relative to the class of Coxeter groups?*

The second question has particular relevance to the widely studied but unsolved isomorphism problem for Coxeter groups. See [Mü06], [RS22] for surveys on the isomorphism problem. Indeed, knowing profinite rigidity amongst the class of Coxeter groups would give an algorithm to determine whether two Coxeter groups are not isomorphic (simply examine their finite quotients).

Our first result shows that graph products of finite groups can be distinguished from each other by their finite quotients. This vastly generalises a result of Kropholler–Wilkes [KW16] that RACGs are distinguished from each other by their finite quotients and also provides a completely new proof of their result. Indeed, their proof relies on the cup product structure of a RACG whereas we proceed using subgroup separability properties and profinite Bass–Serre theory.

**Theorem 3.6.** *Let $G_\Gamma$ be a graph product of finite groups. Then, $G_\Gamma$ is profinitely rigid relative to the class of graph products of finite groups.*

Returning to Coxeter groups, we show that RACGs are profinitely rigid amongst the class of Coxeter groups. This shows that the relative version of Question 1.1 has a positive answer for RACGs.

**Theorem 3.8.** *Let $W$ be a right-angled Coxeter group. Then, $W$ is profinitely rigid amongst the class of Coxeter groups.*

Our next major result shows a strong profinite flexibility statement for direct products of free products.

**Theorem 5.6.** *Let $\ell \geqslant 4$. Let $G_1, \ldots, G_\ell$ be finitely generated, residually finite groups such that for all $1 \leqslant j \leqslant \ell$ the centralizer of a nonabelian subgroup of $G_j$ is trivial. Assume also that at least four of the $G_j$ have a subgroup of index 2. Let $d \geqslant 2$. Then, the genus of*

$$\prod_{i=1}^{d} \left( \mathop{\ast}_{j=1}^{\ell} G_j \right)$$

*is infinite.*

As an immediate corollary we obtain that Question 1.1 has a negative answer in the general case.

**Corollary 1.2.** *The genus of the Coxeter group $(\ast_{i=1}^4 C_2) \times (\ast_{i=1}^4 C_2)$ is infinite.*

We can also show certain graph products, and in particular *irreducible* RACGs, that is RACGs which do not split as a non-trivial direct product, are not profinitely rigid (see Proposition 5.9 and Corollary 5.13).

Our proof utilises the strategy due to Platonov–Tavgen' to construct so called *Grothendieck pairs* out of fibre products of quotients to perfect groups with vanishing second integral homology and no finite quotients. A Grothendieck pair is a subgroup $P \leqslant G$ such that every finite quotient of $P$ is induced by one of $G$ and such that $P$ surjects onto every finite quotient of $G$.

Due to the fact Coxeter groups are generated by involutions we have to find infinite perfect groups which have vanishing second integral homology, no finite quotients, and are generated by involutions. To this end we utilise the Higman–Thompson groups $V_n$ and work of Kapoudjian [Kap02] showing that $H_2(V_n; \mathbb{Z}) = 0$. Note that the computation of all of the homology groups of $V_n$ was completed in a recent breakthrough of Szymik–Wahl [SW19]. Our main new contribution to the theory of $V_n$ is the following generalisation of a classical result of Higman about Thompson's group $V$ [Hig74].

**Theorem 4.1.** *The group $V_n$, with $n \geqslant 2$, is generated by four involutions.*

We also use the *3/2-generation* of the Higman–Thompson groups $V_n$ due to Donovan–Harper [DH20] to construct more examples of Grothendieck pairs (see Theorem 5.7 for the result and its preceding paragraph for the definition of 3/2-generation).

We end with a refinement of the first part of Question 1.1 which our methods leave open.

**Question 1.3.** *Let $W$ be a Coxeter group. Is $W$ profinitely rigid relative to finitely presented groups?*

**Structure of the paper.** In Section 2 we give the necessary background on Coxeter groups and graph products we will need to prove our results. We also prove that even Coxeter groups are finite subgroup separable (Lemma 2.2); see Section 2 for the relevant definitions.

In Section 3.A we give some properties of the profinite completion and recount some needed profinite Bass-Serre theory. In Section 3.B we prove Theorem 3.6. In Section 3.C we prove Theorem 3.8. In Section 3.D we show that a Coxeter group splitting as a non-trivial free product is detected by the profinite completion. We use this to prove (Theorem 3.10) that a free product of Coxeter groups each profinitely rigid amongst Coxeter groups is again profinitely rigid amongst Coxeter groups.

In Section 4 we describe the necessary background on Higman–Thompson groups $V_n$ and then prove Theorem 4.1.

In Section 5 we recount the Platonov–Tavgen' construction and combine it with Theorem 4.1 to prove Theorem 5.6. We provide several other examples of Grothendieck pairs using different properties of the groups $V_n$ (see Theorem 5.5 and Theorem 5.7). We then go on to show that many graph products of groups are not profinitely rigid in the absolute sense. See Proposition 5.9 and its corollaries.

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## 2. Coxeter groups and graph products of groups

In this section we briefly recall the basics on Coxeter groups and graph products of groups that we need for the later sections. More information about Coxeter groups can be found in [Dav08] and about graph products of groups in [Gre90].

Let $\Gamma = (V, E)$ denote a finite simplicial graph and $m\colon E \to \mathbb{N}_{\geqslant 2}$ an edge-labeling. We define the *Coxeter group* $W_\Gamma$ as

$$W_\Gamma := \left\langle V \mid v^2 \text{ for } v \in V, (vw)^{m(\{v,w\})} \text{ if } \{v,w\} \in E \right\rangle.$$

We say $W_\Gamma$ is *right-angled* if $m(e) = 2$ for every edge $e \in E$. If all edge labels are even, we call $W_\Gamma$ *even*. Given a subset $X \subseteq V$, we define the *special parabolic subgroup* $W_X$ as the subgroup $\langle X \rangle \subseteq W_\Gamma$. By [Bou68, p. 20], [Dav08, Theorem 4.1.6], this is well defined since $W_X$ is canonically isomorphic to the Coxeter group defined via the full subgraph spanned by $X$. By definition, any conjugate of a special parabolic subgroup is called *parabolic*.

Given a family of parabolic subgroups, their intersection is again a parabolic subgroup by [Sol76] and [Qi07]. Let $W_\Gamma$ be a Coxeter group and $A \subseteq W_\Gamma$ be a subgroup. By [Qi07, Thm. 1.2] there exists a unique minimal parabolic subgroup $gW_\Delta g^{-1}$ such that $A \subseteq gW_\Delta g^{-1}$. This parabolic subgroup is called the *parabolic closure* of $A$ and is denoted by $Pc(A)$. We note that if two subgroups $A$, $B$ are conjugate, then $Pc(A)$ is conjugate to $Pc(B)$.

The definition of a graph product of groups is similar. Let $\Gamma = (V, E)$ be a finite simplicial graph and let $f\colon V \to \{\text{non-trivial fin. gen. groups}\}$ be a vertex-labeling. The *graph product of groups* $G_\Gamma$ is defined as a quotient

$$G_\Gamma := \left( \mathop{\ast}_{v \in V} f(v) \right) \Big/ \langle\langle [a,b] \mid a \in f(v), b \in f(w) \text{ if } \{v,w\} \in E \rangle\rangle$$

The definitions of (special) parabolic subgroups and the parabolic closure of a subgroup are the same as for Coxeter groups. For more details see [Gre90] and [AM15].

Note that if every vertex group is isomorphic to $C_2$, the graph product of groups $G_\Gamma$ is a right-angled Coxeter group. If every vertex group is isomorphic to $\mathbb{Z}$, we call $G_\Gamma$ a *right-angled Artin group.*

It is known that Coxeter groups are linear, see [Bou68, p. 91], [Dav08, Cor. 6.12.11]. Hence, Coxeter groups are residually finite by [Mal40]. It was shown by Green in [Gre90, Cor. 5.4] that graph products of residually finite groups are residually finite. In particular, graph products of finite groups are residually finite.

Complete subgraphs are important for the study of graph products and Coxeter groups. In this article we adopt the well known graph-theoretic terminology and call them *cliques.* Note that we also call a parabolic subgroup $gG_\Delta g^{-1}$ a clique if $\Delta$ is a clique.

Finally, we call a graph product of groups $G_\Gamma$ (Coxeter group $W_\Gamma$) *reducible*, if there exists a partition of the vertex set $V = V_1 \cup V_2$ such that $\Gamma$ is the join of the induced subgraphs, that is there is an edge (with label 2) between every pair of vertices $\{v_1, v_2\}$ with $v_1 \in V_1$ and $v_2 \in V_2$. Otherwise we call the group *irreducible.*

Let $G$ be a group and $H, L \leqslant G$ be two non-conjugate subgroups. By definition $H$ is *conjugacy separable* from $L$ if there exists a homomorphism $\varphi\colon G \to F$, with $F$ finite, such that $\varphi(H)$ is not conjugate to $\varphi(L)$. A group $G$ is said to be *subgroup conjugacy separable* if any two finitely generated non-conjugate subgroups of $G$ are conjugacy separable. For example all limit groups are known to be subgroup conjugacy separable by [CZ16].

We call $G$ *finite subgroup conjugacy separable* if any two non-conjugate finite subgroups of $G$ are conjugacy separable. We call $G$ *conjugacy separable* if any two non-conjugate cyclic subgroups of $G$ are conjugacy separable.

Being (subgroup) conjugacy separable should not be confused with being *locally extended residually finite* (LERF) also known as *subgroup separable.* This latter property states that every finitely generated subgroup $H \leqslant G$ is separable from every element $g \in (G - H)$ in a finite quotient. Namely, for each such $g$, there exists $\alpha\colon G \to F$, with $F$ finite, such that $\alpha(g) \notin \alpha(H)$.

It was proven by Caprace and Minasyan in [CM13, Theorem 1.2] that an even Coxeter group $W_\Gamma$ is conjugacy separable if $\Gamma$ has no triangles isomorphic to the graph in Figure 1.

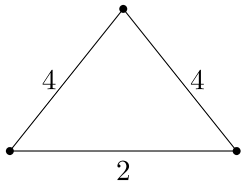


FIGURE 1. Coxeter graph of type $\widetilde{B}_2$.

We ask the following question

**Question 2.1.** *Is every Coxeter group finite subgroup conjugacy separable?*

**Lemma 2.2.** *Let $W_\Gamma$ be a Coxeter group. If $\Gamma$ is even, then $W_\Gamma$ is finite subgroup conjugacy separable.*

*Proof.* It is well known that for an even Coxeter group $W_\Gamma$ and a special parabolic subgroup $W_X$ the homomorphism $p_X\colon W_\Gamma \twoheadrightarrow W_X$ induced by $p_X(v) = v$ for all $v \in X$ and $p_X(w) = 1$ for $w \in (V - X)$ is a well-defined retraction, see [Gal05, Proposition 2.1] for more details on this.

Let $G$ and $H$ be two non-conjugate finite subgroups in $W_\Gamma$. By [Bou68, Chap. 5 § 4, Exercise 2] we know that there exist finite special parabolic subgroups $W_I$, $W_J$ and $g, h \in W_\Gamma$ such that $Pc(G) = gW_Ig^{-1}$ and $Pc(H) = hW_Jh^{-1}$.

Without loss of generality we replace $G$ and $H$ by $g^{-1}Gg$ and $h^{-1}Hh$ respectively. Furthermore, we can assume that $|J| \leqslant |I|$.

Now, we consider the canonical retraction $p_I : W_\Gamma \twoheadrightarrow W_I$. We get $p_I(G) = G$ and $p_I(H) \subseteq W_I \cap W_J = W_{I\cap J}$. If $J \subseteq I$, then $p_I(H) = H$ and $p_I(G)$ is not conjugate to $p_I(H)$. If $J \nsubseteq I$, then the cardinality of $I \cap J$ is smaller than the cardinality of $J$. Assume for a contradiction that $p_I(G)$ is conjugate to $p_I(H)$. Then $Pc(p_I(G)) = W_I$ is conjugate to $Pc(p_I(H)) \subseteq W_{I\cap J}$. The order of $W_{I\cap J}$ is smaller than the order of $W_I$, thus these groups can not be conjugate. □

The same proof strategy shows the following lemma.

**Lemma 2.3.** *Let $G_\Gamma$ be a graph product of finite groups. Then $G_\Gamma$ is finite subgroup conjugacy separable.*

The following lemma regarding the splitting of graph products of groups as amalgamated products will be useful later on.

**Lemma 2.4.** [Gre90, Lemma 3.20] *Let $G_\Gamma$ be graph product of finite groups. If there exist two vertices $v, w \in V$ such that $\{v, w\} \notin E$, then $G_\Gamma \cong G_{st(v)} *_{G_{lk(v)}} G_{V-\{v\}}$ where $lk(v) = \{w \in V \mid \{v, w\} \in E\}$ and $st(v) = lk(v) \cup \{v\}$.*

Denote by $\mathcal{CF}(G)$ the set of conjugacy classes of all finite subgroups in $G$. We define a partial order on $\mathcal{CF}(G)$ as follows: $[A] \leqslant [B]$ if there exists a

$g \in G$ such that $A \subseteq gBg^{-1}$ (see Theorem 4.1 in [Rad03] for the proof that this is indeed a partial order).

Recall the following definitions. Given a poset $(P, \leqslant)$ and a subset $A \subseteq P$, a *greatest lower bound of* $A$, denoted by $\bigwedge A$, is an element $x \in P$ such that $x \leqslant a$ for all $a \in A$ and if $y \leqslant a$ for all $a \in A$, then $y \leqslant x$. Further, a *least upper bound* of $A$, denoted by $\bigvee A$, is an element $x \in P$ such that $x \geqslant a$ for all $a \in A$ and if $y \geqslant a$ for all $a \in A$, then $y \geqslant x$. Note that if a greatest lower bound resp. a least upper bound exists, it is unique.

**Theorem 2.5.** *Let $G_\Gamma$ and $G_\Omega$ denote two graph products of groups with finite directly indecomposable vertex groups. The following are equivalent:*

*(1)* $G_\Gamma \cong G_\Omega$
*(2)* $\Gamma \cong \Omega$ *(thought of as graphs with vertices labelled by finite directly indecomposable groups)*
*(3) There exists an order isomorphism $\psi\colon \mathcal{CF}(G_\Gamma) \to \mathcal{CF}(G_\Omega)$ such that for any $[A]$ in $\mathcal{CF}(G_\Gamma)$ and $B \in \psi([A])$ we have $A \cong B$.*

*Proof.* It was proven in [Rad03, Thm. 5.4] that given two graphs $\Gamma$, $\Omega$ with finite directly indecomposable vertex groups such that $G_\Gamma = G_\Omega$, then $\Gamma \cong \Omega$. We note that the equivalence of (1) and (2) follows by replacing $=$ by $\cong$ in his proof. Moreover it is obvious that (1) and (2) both imply (3). We follow Radcliffe's proof stategy to show that (3) implies (2).

Step 1: Assume that $\Gamma$ and $\Omega$ are $T_0$ graphs in the sense of [Rad03], this means for every pair of vertices $v \neq w$, there exists a maximal clique which contains exactly one of the two. Moreover, for this step we only assume that the vertex groups are finite but not necessarily directly indecomposable.

Now, [Rad03, Thm. 4.2, 4.3] tell us how greatest lower bounds and least upper bounds in the posets $\mathcal{CF}(G_\Gamma)$ and $\mathcal{CF}(G_\Omega)$ behave for conjugacy classes of parabolic subgroups. More precisely, if $A, B \subseteq V(\Gamma)$ and $\langle A\rangle$ and $\langle B\rangle$ are finite, we obtain $[\langle A \cap B\rangle] = [\langle A\rangle] \bigwedge [\langle B\rangle]$. If additionally $\langle A \cup B\rangle$ is finite, then the least upper bound exists and we have $[\langle A \cup B\rangle] = [A] \bigvee [B]$. Since $\psi$ maps greatest lower bounds and least upper bounds to such bounds, we may deploy an argument indentical to the proof of [Rad03, Thm. 5.3] to construct a bijection between the vertices.

Step 2: By [Rad03, Thm. 5.3], given a graph product of finite groups with a $T_0$ graph, there is a unique graph (up to isomorphism). Starting with such a graph of finite groups, there is exactly one way to turn this into a graph with finite directly indecomposable vertex groups (see [Rad03, Thm. 5.4]). Thus, $\Gamma$ and $\Omega$ have to be isomorphic. □

## 3. Profinite rigidity

3.A. **Background on profinite completions.** Let $G$ be a group and $\mathcal{N}$ be the set of all finite index normal subgroups of $G$. We equip each $G/N$, $N \in \mathcal{N}$ with the discrete topology and endow $\prod_{N\in\mathcal{N}} G/N$ with the product topology. We define a map $i\colon G \to \prod_{N\in\mathcal{N}} G/N$ by $g \mapsto (gN)_{N\in\mathcal{N}}$. Note, that if $G$ is residually finite, then $i$ is injective. The *profinite completion* of $G$, denoted by $\widehat{G}$, is defined as $\widehat{G} := \overline{i(G)}$. Equivalently, $\widehat{G}$ may be constructed as the inverse limit $\varprojlim_{N\in\mathcal{N}} G/N$.

The homomorphism $i$ has the following universal property: Let $\mathbf{H}$ be a profinite group and $\phi\colon G \to \mathbf{H}$ be a continuous homomorphism. Then, there exists a unique continuous homomorphism $\widehat{\phi}\colon \widehat{G} \to \mathbf{H}$ making the following diagram commute

$$\begin{array}{ccc} G & \xrightarrow{\ i\ } & \widehat{G} \\ & \searrow^{\phi} & \downarrow{\scriptstyle \widehat{\phi}} \\ & & \mathbf{H}. \end{array}$$

The next theorem shows that the set $\mathcal{F}(G)$ of isomorphism classes of finite quotients of a finitely generated residually finite group $G$ encodes the same information as $\widehat{G}$.

**Theorem 3.1.** [DFPR82] *Let $G$ and $H$ be finitely generated residually finite groups. Then $\mathcal{F}(G) = \mathcal{F}(H)$ if and only if $\widehat{G} \cong \widehat{H}$.*

Note that by the work of Nikolov–Segal [NS07a, NS07b] we have that $\widehat{G}$ is isomorphic to $\widehat{H}$ as a topological group if and only if $\widehat{G}$ is isomorphic to $\widehat{H}$ as an abstract group.

For profinite groups $\mathbf{G}_1$ and $\mathbf{G}_2$ with a common closed subgroup $\mathbf{H}$, we denote the pushout $\mathbf{G}_1$ and $\mathbf{G}_2$ over $\mathbf{H}$ by $\mathbf{G} = \mathbf{G}_1 \coprod_{\mathbf{H}} \mathbf{G}_2$. If the natural maps from $\mathbf{G}_1$ and $\mathbf{G}_2$ to $\mathbf{G}$ are embeddings then we call $\mathbf{G}$ the *profinite amalgamated product* of $\mathbf{G}_1$ and $\mathbf{G}_2$ along $\mathbf{H}$.

**Theorem 3.2.** [ZM89, §5.6] *Let $\mathbf{G}_1 \coprod_{\mathbf{H}} \mathbf{G}_2$ be a profinite amalgamed product. If $F \subseteq \mathbf{G}_1 \coprod_{\mathbf{H}} \mathbf{G}_2$ is a finite subgroup, then $F$ is contained in a conjugate of $\mathbf{G}_1$ or $\mathbf{G}_2$.*

**Lemma 3.3.** *Let $G \cong A *_C B$ be a finitely generated residually finite group. If $A$, $B$ and $C$ are retracts of $G$, then $\widehat{G} \cong \widehat{A} \coprod_{\widehat{C}} \widehat{B}$.*

*Proof.* By [Rib17, Propositon 6.5.3] $\widehat{G} \cong \overline{A} \coprod_{\overline{C}} \overline{B}$ where $\overline{H}$ denotes the closure of the image of $H$ in $\widehat{G}$. Since $A$, $B$, and $C$ are retracts of $G$ each of them has the full profinite topology induced by $G$. That is, every finite index subgroup of $A$ (resp. $B$, $C$) is a closed subgroup of $G$ (with respect to the profinite topology on $G$). Indeed, (virtual) retracts are closed in the profinite topology [Min21, Lemma 2.2] and finite index subgroups of (virtual) retracts are virtual retracts [Min21, Lemma 3.2(iv)]. Hence, $\overline{A} \cong \widehat{A}$ (and similarly for $B$ and $C$). □

3.B. **Profinite rigidity and graph products.**

**Lemma 3.4.** *Let $G$ be a finitely generated residually finite group. If $G$ is finite subgroup conjugacy separable, then the canonical inclusion $G \hookrightarrow \widehat{G}$ induces an injective map : $\mathcal{CF}(G) \to \mathcal{CF}(\widehat{G})$.*

*Proof.* Let $H$ and $K$ be non-conjugate finite subgroups of $G$. We will show that they remain non-conjugate in $\widehat{G}$. Suppose not, then there exists $g \in \widehat{G}$ such that $gHg^{-1} = K$. Since $G$ is finite subgroup conjugacy separable there exists a finite quotient $\alpha\colon G \twoheadrightarrow Q$ where $\alpha(H)$ is not conjugate to $\alpha(K)$. By the universal property of the profinite completion we obtain a map $\widehat{\alpha}\colon \widehat{G} \twoheadrightarrow Q$ which restricts to $\alpha$ on $G$. But now $\widehat{\alpha}(g)\alpha(H)\widehat{\alpha}(g^{-1}) = \alpha(K)$ which is a contradiction. Hence, the lemma. □

**Proposition 3.5.** *Let $G_\Gamma$ be a graph product of finite directly indecomposable groups. Then, the canonical inclusion $G_\Gamma \hookrightarrow \widehat{G_\Gamma}$ induces an order isomorphism $\varphi\colon \mathcal{CF}(G_\Gamma) \to \mathcal{CF}(\widehat{G_\Gamma})$ such that for any $[A] \in \mathcal{CF}(G_\Gamma)$ and $B \in \varphi([A])$ we have $A \cong B$.*

*Proof.* By Lemma 2.3 the graph product $G_\Gamma$ is finite subgroup conjugacy separable. Since $G_\Gamma$ is a finitely generated residually finite group, the injectivity of $\varphi$ follows by Lemma 3.4. It remains to show surjectivity and that $\varphi$ is order preserving.

If $\Gamma$ is a clique, then $G_\Gamma$ is finite and the conclusion of the proposition follows immediately since $\widehat{G_\Gamma} = G_\Gamma$. Thus, we may assume that $\Gamma$ is not a clique. By Lemma 2.4 the group $G_\Gamma$ is an amalgamated product of special subgroups $G_\Gamma = A *_C B$. Special parabolic subgroups of $G_\Gamma$ are retracts of $G_\Gamma$, therefore we can apply Lemma 3.3 to obtain $\widehat{G_\Gamma} = \widehat{A} \coprod_{\widehat{C}} \widehat{B}$.

Let $[F] \in \mathcal{CF}(\widehat{G_\Gamma})$. Then $F$ is contained in a conjugate of $\widehat{A}$ or $\widehat{B}$ by Theorem 3.2. If $A$ is not a clique, then we decompose $A$ again into an amalgamated product. Repeating this process finitely many times we obtain that $F$ is contained in a conjugate of $\widehat{A'}$ where $A'$ is a clique and hence a finite subgroup. Thus, $\widehat{A'} = A'$ and $F \subseteq gA'g^{-1}$. In particular, there exists a finite subgroup $A'' \subseteq G_\Gamma$ such that $F = gA''g^{-1}$. Hence, $\varphi([A'']) = [F]$, which shows the surjectivity of $\varphi$. Clearly, by construction $\varphi$ is order preserving. □

We now prove our first result from the introduction.

**Theorem 3.6.** *Let $G_\Gamma$ and $G_\Omega$ be graph products of finite groups. Then $G_\Gamma \cong G_\Omega$ if and only if $\widehat{G_\Gamma} \cong \widehat{G_\Omega}$.*

*Proof.* If $G_\Gamma \cong G_\Omega$, then $\widehat{G_\Gamma} \cong \widehat{G_\Omega}$ by Theorem 3.1.

We note that if a vertex group splits as a direct product, we can replace the corresponding vertex with a clique corresponding to the direct decomposition to obtain an isomorphic graph product of finite directly indecomposable vertex groups. Thus, we can assume without loss of generality that all vertex groups are directly indecomposable.

If $\widehat{G_\Gamma} \cong \widehat{G_\Omega}$, then by Proposition 3.5 we get an order isomorphism

$$\mathcal{CF}(G_\Gamma) \to \mathcal{CF}(\widehat{G_\Gamma}) \to \mathcal{CF}(\widehat{G_\Omega}) \to \mathcal{CF}(G_\Omega).$$

Finally, by Theorem 2.5 we obtain $G_\Gamma \cong G_\Omega$. □

### 3.C. **Profinite rigidity and RACGs.**

**Lemma 3.7.** *Let $W_\Gamma$ and $W_\Omega$ be Coxeter groups. Assume that $W_\Gamma$ is a right-angled Coxeter group. If $\widehat{W_\Gamma} \cong \widehat{W_\Omega}$, then $W_\Omega$ is also a right-angled Coxeter group.*

*Proof.* Since Coxeter groups are residually finite we have $W_\Omega \hookrightarrow \widehat{W_\Gamma}$. By Proposition 3.5 we know that maximal finite subgroups in $\widehat{W_\Gamma}$ are direct products of $C_2$'s. Hence, finite subgroups in $W_\Omega$ are also direct products of $C_2$'s. It follows that the Coxeter group $W_\Omega$ is right-angled. □

We now prove our second result from the introduction. As explained earlier the result generalises [KW16, Theorem 6] from RACGs to the class of all Coxeter groups.

**Theorem 3.8.** *Let $W_\Gamma$ be a right-angled Coxeter group. Then, $W_\Gamma$ is profinitely rigid relative to the class of Coxeter groups.*

*Proof.* Let $W_\Gamma$ be a RACG and let $W$ be a Coxeter group such that $\widehat{W_\Gamma} \cong \widehat{W}$. By Lemma 3.7 it follows that $W$ is a RACG and so by Theorem 3.6 we obtain $W_\Gamma \cong W$. □

3.D. **Coxeter groups and profinite free products.**

**Proposition 3.9.** *Let $W_\Gamma$ be a Coxeter group. Then $\widehat{W_\Gamma}$ splits as a non-trivial profinite free product if and only if $\Gamma$ is disconnected.*

*Proof.* Compare to the proof of [KW16, Theorem 11]. If $\Gamma$ is disconnected, then $W_\Gamma \cong *_{i=1}^n W_{\Gamma_i}$ where $\Gamma_1, \dots, \Gamma_n$ are the connected components of $\Gamma$. Thus, $\widehat{W_\Gamma}$ is the free profinite product of $\widehat{W_{\Gamma_1}}, \dots, \widehat{W_{\Gamma_n}}$ by Lemma 3.3.

Assume now that $\Gamma$ is connected and $\widehat{W_\Gamma}$ splits as a profinite free product $G \coprod H$. This splitting induces an action of $\widehat{W_\Gamma}$ on the profinite tree $T$ associated to this splitting, in particular the edge stabilisers are trivial and the vertex stabilisers are conjugates of $G$ or $H$, see [ZM88, Proposition 3.8]. Let $\{v, w\} \in E(\Gamma)$. Then the subgroups $\langle v\rangle$, $\langle w\rangle$ and $\langle v, w\rangle$ fix the same vertex of $T$. Since $\Gamma$ is connected, it follows that $W_\Gamma$ fixes a vertex of $T$. Hence, $\widehat{W_\Gamma}$ fixes a vertex of $T$ and therefore at least one of $G$ and $H$ is trivial. □

**Theorem 3.10.** *Let $W_\Gamma$ be a Coxeter group and suppose $W_\Gamma \cong *_{i=1}^n W_{\Gamma_i}$ where $\Gamma_1, \dots, \Gamma_n$ are the connected components of $\Gamma$. If each $W_{\Gamma_i}$ is profinitely rigid amongst Coxeter groups, then so is $W_\Gamma$.*

*Proof.* Let $W_\Omega$ be a Coxeter group. We have $W_\Omega = *_{j=1}^m W_{\Omega_i}$ where $\Omega_1, \dots, \Omega_m$ are the connected components of $\Omega$. Assume that $\widehat{W_\Gamma} \cong \widehat{W_\Omega}$. By Lemma 3.3 we have $\coprod_{i=1}^n \widehat{W_{\Gamma_i}} \cong \coprod_{j=1}^m \widehat{W_{\Omega_j}}$.

Let

$$\varphi\colon \coprod_{i=1}^{n} \widehat{W_{\Gamma_i}} \to \coprod_{j=1}^{m} \widehat{W_{\Omega_j}}$$

be an isomorphism.

We consider the action of $\coprod_{j=1}^m \widehat{W_{\Omega_j}}$ on the associated profinite tree $T$. Note that the edge stabilisers are trivial and the vertex stabilizers are conjugates of $\widehat{W_{\Omega_j}}$, $j = 1, \dots, m$, see [ZM88, Proposition 3.8]. We denote this action by $\psi$.

The proof of Proposition 3.9 shows that the fixed point set of $\psi \circ \varphi(\widehat{W_{\Gamma_i}})$ is non-empty. Thus $\varphi(\widehat{W_{\Gamma_i}})$ is contained in a conjugate of $\widehat{W_{\Omega_j}}$. Similarly, $\varphi^{-1}(\widehat{W_{\Omega_j}})$ is contained in a conjugate of $\widehat{W_{\Gamma_k}}$. By [dBPZ22, Proposition 4.5] it follows that $n = m$ and that there exists $\pi \in \mathrm{Sym}(n)$ such that $\widehat{W_{\Gamma_k}} \cong \widehat{W_{\Omega_{\pi(k)}}}$. By assumption, for $k = 1, \dots, n$, the Coxeter group $W_{\Gamma_k}$ is profinitely rigid amongst Coxeter groups, thus $W_{\Gamma_k} \cong W_{\Omega_{\pi(k)}}$. Hence

$$W_\Gamma = \mathop{\ast}_{i=1}^{n} W_{\Gamma_i} \cong \mathop{\ast}_{j=1}^{n} W_{\Omega_j} = W_\Omega. \qquad \square$$

## 4. A diversion: Higman-Thompson groups $V_n$

We first give a brief description of Thompson's group $V$ and the Higman–Thompson groups $V_n$ (the interested reader can find more details in [CFP96, §6]), [Hig74]. Let $2 = \{0, 1\}$, let $2^*$ denote the set of finite sequences of 0s and 1s, and $\epsilon$ denote the empty sequence. For $\alpha, \beta \in 2^*$ we let $\alpha\beta$ denote the concatenation of $\alpha$ with $\beta$. Write $\alpha \preceq \beta$ if $\alpha$ is a prefix of $\beta$ (i.e. there exists $\gamma \in 2^*$ such that $\beta = \alpha\gamma$), so for example, $101 \preceq 10100$. Write $\alpha 2^*$ for the set of elements of $2^*$ which have $\alpha$ as a prefix.

The set $2^*$ together with the relation $\preceq$ is the complete binary rooted tree, with $\epsilon$ being the root. We will call a subset $\tau \subseteq 2^*$ a *rooted subtree* if $\tau$ is closed under taking prefixes. An element $\alpha$ in rooted subtree $\tau$ is a *leaf* if there does not exist $\beta \in \tau\backslash\{\alpha\}$ with $\alpha \preceq \beta$. Let $\mathfrak{T}_2$ denote the set of all rooted subtrees $\tau$ of $2^*$ such that

(1) $|\tau| < \infty$;
(2) $\epsilon \in \tau$ (so $\tau \neq \varnothing$);
(3) $\alpha 0 \in \tau$ if and only if $\alpha 1 \in \tau$.

For example, $\{\epsilon, 0, 1, 00, 01\}$ is in $\mathfrak{T}_2$ and each of 00, 01, and 1 is a leaf. Note that an element $\tau$ of $\mathfrak{T}_2$ is completely determined by its set $\mathrm{Lea}(\tau)$ of leaves. Supposing that $\tau_0, \tau_1 \in \mathfrak{T}_2$ have the same number of leaves, a bijection $b : \mathrm{Lea}(\tau_0) \to \mathrm{Lea}(\tau_1)$ extends to a bijection $f_b : \bigcup_{\alpha\in\mathrm{Lea}(\tau_0)} \alpha 2^* \to \bigcup_{\beta\in\mathrm{Lea}(\tau_1)} \beta 2^*$ by letting $f_b(\alpha\gamma) = b(\alpha)\gamma$. Clearly $2^*\backslash \bigcup_{\alpha\in\mathrm{Lea}(\tau)} \alpha 2^* = \tau\backslash \mathrm{Lea}(\tau)$ for each $\tau \in \mathfrak{T}_2$, and so the bijections $f_b$ and $f_b^{-1}$ are each defined on a cofinite subset of $2^*$.

If $f : X \to Y$ and $g : Z \to U$ are bijections with $X, Y, Z, U$ cofinite subsets of $2^*$, we consider $f$ and $g$ to be equivalent provided they are equal on a cofinite set in $2^*$, and write $[f]$ for the equivalence class of such a bijection $f$. The set of all such equivalence classes is easily seen to form a group, with binary operation being given by compositing appropriately restricted domains and with inverses given by $[f]^{-1} = [f^{-1}]$. Thompson's group $V$ is the subgroup consisting of those $[f_b]$ where $b : \mathrm{Lea}(\tau_0) \to \mathrm{Lea}(\tau_1)$ is a bijection; one easily checks that $[f_b][f_c]$ is indeed equal to $[f_d]$ for some bijection $d$. Moreover, each element $[f_b] \in V$ has a unique canonical representative $f_b \in [f_b]$ such that the trees associated with the bijection $b$ are minimal (under $\subseteq$). Thus, we will favor the less cumbersome notation $b$ instead of $[f_b]$ when writing elements of $V$. *We use the convention in this section that actions are on the right, so the product $b \circ c$ is understood to first apply $b$ and then to apply $c$.*

One can generalize this group by changing the parameter 2 used in describing $V$. More particularly for $n \geqslant 2$ we write $n = \{0, 1, \ldots, n-1\}$, take $n^*$ to be the set of finite sequences of elements in $\{0, 1, \ldots, n-1\}$, define $\alpha \preceq \beta$, and (rooted) subtrees of $n^*$ in the analogous way. The set $\mathfrak{T}_n$ is the set of those rooted trees $\tau$ which are finite, nonempty and such that $\alpha 0 \in \tau$ if and only if $\alpha 1 \in \tau$ if and only if $\ldots$ if and only if $\alpha(n-1) \in \tau$. Leaves of a

$\tau \in \mathfrak{T}_n$ are defined as before, and finally the group $V_n$ is defined analogously as those equivalence classes of bijections between cofinite subsets of the tree $n^*$ which are produced by a bijection between the leaves of a pair of trees in $\mathfrak{T}_n$. Particularly $V = V_2$ under this notation and $V_n = G_{n,1}$ under the notation of Higman [Hig74] (we shall not vary the parameter 1 as is done in defining general Higman-Thompson groups).

We point out that when the trees associated with the domain and codomain of a canonical $b \in V_n$ are equal, say $\tau$, then $b$ is an element of the subgroup $\mathrm{Sym}(\mathrm{Lea}(\tau))$ of $V_n$. It is also worth pointing out that an element in $\mathrm{Sym}(\mathrm{Lea}(\tau))$ might not be the canonical representative (for example, if $b : \mathrm{Lea}(\tau) \to \mathrm{Lea}(\tau)$, $\mathrm{Lea}(\tau) \supseteq \{0i\}_{i<n}$ and $b$ fixes all elements of $\{0i\}_{i<n}$ then $b$ can be represented by a bijection on the leaves of the strictly smaller tree $\tau\backslash\{0i\}_{i<n}$.

When $n \geqslant 2$ is even the group $V_n$ is simple. When $n \geqslant 2$ is odd the group $V_n$ has a simple subgroup of index 2 which we will describe [Hig74, §5] (see [Hig74, Theorem 5.4] for simplicity in both cases). Fixing $n \geqslant 3$ odd, an element of $V_n$ given by bijection $b : \mathrm{Lea}(\tau_0) \to \mathrm{Lea}(\tau_1)$ is *even* (respectively *odd*) if the number of pairs $\alpha_1, \alpha_2 \in \mathrm{Lea}(\tau_0)$ such that $\alpha_1$ is lexicographically below $\alpha_2$ and $(\alpha_1)b$ is lexicographically above $(\alpha_2)b$ is even (resp. odd). When $\tau_0 = \tau_1$ this corresponds to the standard terminology for the symmetric group $\mathrm{Sym}(\mathrm{Lea}(\tau_0))$. Using the oddness of $n$ one can show that the even elements form a subgroup, which we will denote $V_n^+$.

Graham Higman has pointed out that the group $V$ can be generated by four involutions [Hig74, page 49]. We generalise this as follows.

**Theorem 4.1.** *The group $V_n$, with $n \geqslant 2$, is generated by four involutions.*

*Proof.* As Higman has already considered the case $n = 2$, we fix $n \geqslant 3$. The group $V_n$ has a non-trivial involution, for example the one represented canonically by taking the tree $\{\epsilon\} \cup \{i\}_{i<n}$ and letting $b$ swap the leaf 0 with the leaf 1 and fixing all other leaves. If $n$ is even, the group $V_n$ is simple and the set of non-trivial involutions in $V_n$ is nonempty and closed under conjugation so we see that $V_n$ is generated by its set of involutions. If $n$ is odd, the involution described above is in $V_n\backslash V_n^+$ (it is a transposition) and it is easy to see that there are involutions in $V_n^+$ (say, a product of two disjoint transpositions of the leaves of a tree). As $V_n^+$ is simple we have $V_n^+$ is generated by the set of involutions in $V_n^+$, and so $V_n$ is generated by its set of involutions since $V_n^+$ has index 2 in $V_n$. In either case, $V_n$ is generated by the set of involutions.

Next, we notice that if bijection $b : \mathrm{Lea}(\tau_0) \to \mathrm{Lea}(\tau_1)$ is a canonical representative of an involution in $V_n$ then $b = b^{-1}$. To see this we note that the inverse of this group element is induced canonically by the map $b^{-1} : \mathrm{Lea}(\tau_1) \to \mathrm{Lea}(\tau_0)$. This latter claim is true since if $b^{-1}$ could be induced using strictly smaller trees, say $\tau_1' \subsetneq \tau_1$ and $\tau_0' \subsetneq \tau_0$, then the bijection $b' : \mathrm{Lea}(\tau_1) \to \mathrm{Lea}(\tau_0)$ is equal on a cofinite set to $b^{-1}$. But now it is easy to see that $(b')^{-1} : \mathrm{Lea}(\tau_0') \to \mathrm{Lea}(\tau_1')$ is cofinitely equal to $b$, witnessing that $b$ was not canonical. Thus, $b^{-1}$ canonically gives the inverse of $b$ in $V_n$. As $b$ was assumed to give an involution in $V_n$, we have $b = b^{-1}$ since a canonical representative is unique.

Since an involution in $V_n$ which is canonically represented by $b$ has an inverse which is canonically represented again by $b = b^{-1}$, we know in particular that the tree associated with the domain of $b$ is the same as that associated with the codomain, and so we have a tree $\tau$ such that $b$ is an involution in the symmetric group $\mathrm{Sym}(\mathrm{Lea}(\tau))$ on the leaves of $\tau$. An involution in the symmetric group on a set is given as a product of commuting transpositions. Thus $V_n$ is generated by the set of all transpositions of subsets of the form $\mathrm{Lea}(\tau))$, with $\tau \in \mathfrak{T}_n$ (a presentation for $V$ using such generators is given in [BQ17, Theorem 1.1]).

We now give four involutions in $V_n$ which generate $V_n$. Let $\overline{\tau}_2 \in \mathfrak{T}_n$ be the rooted subtree consisting of those sequences of length at most 2, so $\mathrm{Lea}(\overline{\tau}_2)$ is the set of those sequences in $n^*$ of length 2. We take $b_1, b_2, b_3 \in \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_2))$ to be three involutions which generate $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_2))$. For example, we label the vertices on a regular $n^2$-gon using the elements of $\mathrm{Lea}(\overline{\tau})$ so that the labels increase as one goes counterclockwise around the $n^2$-gon, and of course the vertex labeled $(n-1)(n-1)$ is adjacent to those labeled $(n-1)(n-2)$ and 00.

In the following we will always exemplify the construction in the case $n = 3$.

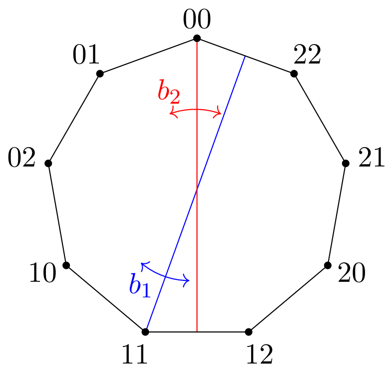


FIGURE 2. Labeled 9-gon.

Take $b_1$ to be the involution in the dihedral group which swaps the vertices 00 and $(n-1)(n-1)$, that means $b_1$ is the reflection across the axis which intersects the edge between the vertices 00 and $(n-1)(n-1)$ orthogonally. Let $b_2$ to be the involution in the dihedral group which swaps 01 with $(n-1)(n-1)$ and leaves 00 fixed, that is $b_2$ corresponds to the reflection across the axis passing through 00 and the unique opposite point in the regular $n^2$-gon (note that this is the midpoint of an edge if $n$ is odd and a vertex if $n$ is even). More generally speaking, we take two neighbouring reflection axis whose reflections generate the dihedral group with $2n^2$ elements.

Then the product $b_1b_2$ is the $n^2$ cycle which moves through the leaves of $\overline{\tau}$ in lexicographic order (recall our convention that $b_1b_2$ is acting on the right, so $b_1$ is applied first and then $b_2$ is applied second).

Taking $b_3 \in \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_2))$ to be the transposition of 00 and 01, it is a standard exercise to show that the two elements $b_1b_2$ and $b_3$ generate $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_2))$.

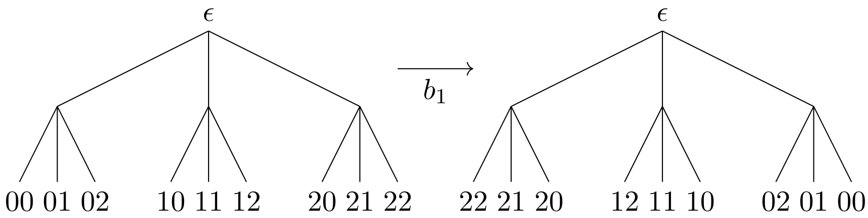


FIGURE 3. Reduced tree diagram for $b_1 \in V_3$.

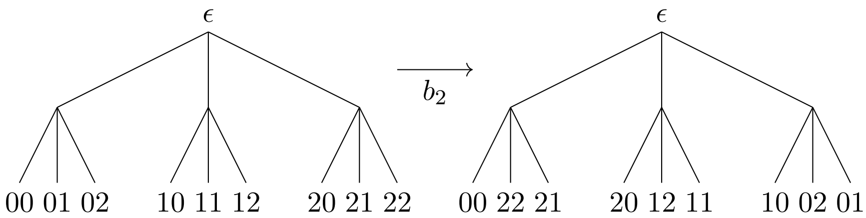


FIGURE 4. Reduced tree diagram for $b_2 \in V_3$.

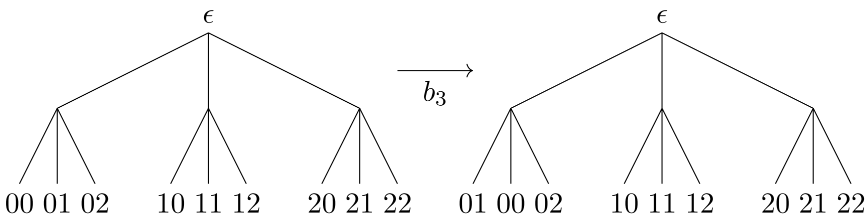


FIGURE 5. Reduced tree diagram for $b_3 \in V_3$.

We define $b_4 \in V_n \backslash \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_2))$ by taking $\underline{\tau}$ to be the tree whose leaves are $\{0i\}_{i<n} \cup \{j\}_{0<j<n}$ and having $b_4$ swap the leaf 00 with 1 and fixing all others.

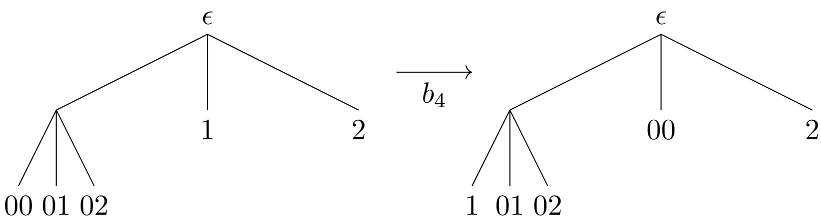


FIGURE 6. Reduced tree diagram for $b_4 \in V_3$.

Now we check that the set $\{b_1, b_2, b_3, b_4\}$ generates $V_n$. Let $H$ denote the subgroup of $V_n$ generated by this set. For a natural number $k$ we let $\overline{\tau}_k \in \mathfrak{T}_n$ be the rooted tree consisting of sequences of length $\leqslant k$.

**Claim 4.2.** *The inclusion* $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_1)) \leqslant H$ *holds.*

*Proof.* Given an element $b \in \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_1))$ we define $b' \in \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_2))$ by $b'(ij) = b(i)j$ for $0 \leqslant i, j < n$. It is clear that $b$ and $b'$ represent the same element in $V_n$ and so $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_1)) \leqslant \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_2)) \leqslant H$. ∎

**Claim 4.3.** *The bijection defined by*

$$b^{\uparrow} : 00 \mapsto 0; 01 \mapsto 10; 0i \mapsto 1i \ \forall 2 \leqslant i < n; 1 \mapsto 11; j \mapsto j \ \forall 1 < j < n$$

*has* $b^{\uparrow} \in H$.

*Proof.* Notice that this map is obtained by first applying $b_4$; then the map $0 \mapsto 1$, $1 \mapsto 0$; and then the map $0 \mapsto 0$, $10 \mapsto 11$, $11 \mapsto 10$. (The reader can check where each of the leaves in the domain of $b$ is mapped under this composition.) Of course $b_4 \in H$ and the other two maps are in the symmetric groups $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_1))$ and $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_2))$ respectively, so $b^{\uparrow} \in H$ as well. ∎

**Claim 4.4.** *For each natural number $k$ we have* $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_k)) \leqslant H$.

*Proof.* We prove this claim by induction. Certainly the claim is true for $k = 0$ (this symmetric group is the trivial subgroup), and $k = 2$ holds by how $b_1$, $b_2$, and $b_3$ were selected, and $k = 1$ is Claim 4.2. Supposing that the claim is true for some $k \geqslant 2$, we show that the claim holds for $k + 1$. Given leaves $\alpha, \beta \in \mathrm{Lea}(\overline{\tau}_{k+1})$, it is sufficient to prove that the transposition of $\alpha$ and $\beta$ is in $H$, since $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_{k+1}))$ is generated by the transpositions of elements in $\mathrm{Lea}(\overline{\tau}_{k+1})$. Let $\alpha = \alpha' i$ and $\beta = \beta' j$, where $\alpha'$ and $\beta'$ are of length $k$ and $i, j \in \{0, \dots, n-1\}$. We will consider an easier-to-understand case first before the more general situation.

If $\alpha' = \beta'$ then we take $c \in \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_k))$ to be the transposition of $0^k$ and $\alpha'$ (here $0^k$ denotes the sequence of $k$-many 0s). In case $\alpha' = 0^k$ then in fact this "transposition" is simply the identity map. We know by induction that $c \in H$. We have that $(\alpha)c \circ b^{\uparrow} = 0^{k-1}i$ and $(\beta)c \circ b^{\uparrow} = 0^{k-1}j$. Take $c' \in \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_k))$ to be the transposition of $0^{k-1}i$ and $0^{k-1}j$. Then,

$$(\alpha)c \circ b^{\uparrow} \circ c' \circ (b^{\uparrow})^{-1} \circ c^{-1} = \beta$$

and

$$(\beta)c \circ b^{\uparrow} \circ c' \circ (b^{\uparrow})^{-1} \circ c^{-1} = \alpha$$

and it is easy to check that all other elements of $\mathrm{Lea}(\overline{\tau}_{k+1})$ are fixed by the composition $c \circ b^{\uparrow} \circ c' \circ (b^{\uparrow})^{-1} \circ c^{-1}$. Thus, in this case the transposition in $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_{k+1}))$ of $\alpha$ and $\beta$ is in $H$.

More generally, if say $\alpha' \neq \beta'$, then we take $c \in \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_k))$ to be such that $\{(\alpha')c, (\beta')c\} \subseteq \{0^k, 0^{k-1}1\}$. Take $c' \in \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_k))$ to be the transposition of $(\alpha)c \circ b^{\uparrow}$ with $(\beta)c \circ b^{\uparrow}$. Once again

$$(\alpha)c \circ b^{\uparrow} \circ c' \circ (b^{\uparrow})^{-1} \circ c^{-1} = \beta$$

and

$$(\beta)c \circ b^{\uparrow} \circ c' \circ (b^{\uparrow})^{-1} \circ c^{-1} = \alpha$$

and all other elements of $\mathrm{Lea}(\overline{\tau}_{k+1})$ are fixed by the composition $c \circ b^{\uparrow} \circ c' \circ (b^{\uparrow})^{-1} \circ c^{-1}$. The proof of the claim is complete. ∎

We must still consider a transposition of two leaves in an arbitrary $\tau \in \mathfrak{T}_n$ and then, by earlier observations, the lemma will immediately follow. For the next claim we will use for the first time the assumption that $n \geqslant 3$.

**Claim 4.5.** *Take $b^{\uparrow\uparrow}$ to be the bijection given by*

$$b^{\uparrow\uparrow} : 00 \mapsto 0; 02 \mapsto 20; 0i \mapsto 2i \ \forall i \neq 0,2; 2 \mapsto 22; j \mapsto j \ \forall j \neq 0,2$$

*we have $b^{\uparrow\uparrow} \in H$.*

*Proof.* Letting $c \in \mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_1))$ be the transposition of 1 with 2. Letting $\underline{\tau}$ be the tree whose leaves are $\{0i\}_{i<n} \cup \{j\}_{0<j<n}$ we see that $c \circ b_4 \circ c$ is the transposition in $\mathrm{Sym}(\mathrm{Lea}(\underline{\tau}))$ which switches 00 with 2. The proof that $b^{\uparrow\uparrow} \in H$ now follows that of Claim 4.3 with the symbol "2" now playing the role of "1" in that proof. ■

**Claim 4.6.** *Let $m \geqslant 1$ and $\tau$ be the minimal element in $\mathfrak{T}_n$ containing the elements $0^m$ and 1. Letting $d_m$ be the transposition in* $\mathrm{Sym}(\mathrm{Lea}(\tau))$ *which switches $0^m$ and 1 we have $d_m \in H$.*

*Proof.* If $m = 1$ then $d_m \in \mathrm{Sym}(\mathrm{Lea}(\tau))$ and if $m = 2$ then $d_m = b_4$. If $m \geqslant 3$ it is easy to see that $(b^{\uparrow\uparrow})^{m-2} \circ b_4 \circ (b^{\uparrow\uparrow})^{2-m} = d_m$ and by Claim 4.5 we have $d_m \in H$ in this case as well. ■

**Claim 4.7.** *Let $p \geqslant 0$, $m \geqslant 1$, and $\tau$ be the minimal element in $\mathfrak{T}_n$ containing $0^{p+m}$ and $0^p1$. Letting $d_{m,p}$ be the transposition in* $\mathrm{Sym}(\mathrm{Lea}(\tau))$ *which switches $0^{p+m}$ with $0^p1$ we have $d_{m,p} \in H$.*

*Proof.* Note that $d_{m,p} = (b^{\uparrow})^p \circ d_m \circ (b^{\uparrow})^{-p} \in H$. ■

**Claim 4.8.** *Let $\alpha$ and $\beta$ be elements of $n^*$ such that neither is a prefix of the other (i.e. neither $\alpha \preceq \beta$ nor $\beta \preceq \alpha$, in the earlier notation). Let $\tau$ be the minimal element of $\mathfrak{T}_n$ containing $\alpha$ and $\beta$. The transposition in* $\mathrm{Sym}(\mathrm{Lea}(\tau))$ *which switches $\alpha$ and $\beta$ is an element in $H$.*

*Proof.* Let $L_\alpha$ and $L_\beta$ be the lengths of $\alpha$ and $\beta$ respectively. By assumption, $0 < L_\alpha, L_\beta$. Without loss of generality $L_\alpha \geqslant L_\beta$. Let $c_1$ be the transposition in $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_{L_\beta}))$ which switches $0^{L_\beta-1}1$ with $\beta$. Let $c_2$ be the transposition in $\mathrm{Sym}(\mathrm{Lea}(\overline{\tau}_{L_\alpha}))$ which switches $0^{L_\alpha}$ with $(\alpha)c_1$. Let $m = L_\alpha - L_\beta + 1$ and $p = L_\beta - 1$. Now

$$(\alpha)c_1 \circ c_2 \circ d_{m,p} \circ c_2^{-1} \circ c_1^{-1} = \beta$$

and

$$(\beta)c_1 \circ c_2 \circ d_{m,p} \circ c_2^{-1} \circ c_1^{-1} = \alpha$$

and moreover it is easy to see that the map $c_1 \circ c_2 \circ d_{m,p} \circ c_2^{-1} \circ c_1^{-1}$ fixes all other leaves in the smallest element of $\mathfrak{T}_n$ which includes each of $\{\alpha, (\alpha)c_1, 0^{L_\alpha}, \beta, 0^{L_\beta-1}1\}$, so this composition is equivalent to the desired transposition as an element of $V_n$. ■

Now the proof of the theorem is complete, since any involution in $V_n$ can be written as a product of elements as in Claim 4.8. □

## 5. Grothendieck pairs and flexibility

Long and Reid introduced in [LR11] the following definition.

**Definition 5.1.** A finitely generated residually finite group $G$ is *Grothendieck rigid* if for every finitely generated proper subgroup $H \subset G$ the inclusion induced map $\widehat{H} \to \widehat{G}$ is not an isomorphism.

We require the following definition.

**Definition 5.2.** For $i = 1, \dots, d$, let $\varphi_i \colon G_i \to Q$ be a homomorphism. The *fibre product* of this family of maps is defined as

$$P_d := \{(g_1, \dots, g_d) \in G_1 \times \dots \times G_d \mid \varphi_i(g_i) = \varphi_j(g_j), i, j = 1, \dots, d\}.$$

The following lemma gives us a criterion for the finite generation of fibre products.

**Lemma 5.3.** [Bri23, Lemma 4.3] *For $i = 1, \dots, d$, let $\pi_i \colon G_i \twoheadrightarrow Q$ be an epimorphism. If the groups $G_i$ are finitely generated and $Q$ is finitely presented, then the fibre product $P_d \subseteq G_1 \times \dots \times G_d$ is finitely generated.*

Before we turn to the proof of profinite flexibility of some graph products we need the following result.

**Theorem 5.4.** [Bri23, Theorem 4.6] *For $i = 1, \dots, d$, let $G_i \twoheadrightarrow Q$ be an epimorphism of finitely generated groups, and let $P_d \subseteq G_1 \times \dots \times G_d$ be the associated fibre product. If $Q$ is finitely presented, $\widehat{Q} = 1$ and $H_2(Q; \mathbb{Z}) = 0$, then the inclusion $\iota \colon P_d \hookrightarrow G_1 \times \dots \times G_d$ induces an isomorphism $\widehat{\iota} \colon \widehat{P_d} \to \widehat{G_1} \times \dots \times \widehat{G_d}$.*

**Theorem 5.5.** *Let $d \geqslant 2$. For $n \in \mathbb{N}$, $n \geqslant 2$, there exists $\ell_n \in \mathbb{N}$ such that*

$$\prod_{i=1}^{d} \left( \mathop{\ast}_{i=1}^{\ell_n} C_n \right)$$

*is not Grothendieck rigid and not profinitely rigid.*

*Proof.* Let $V$ denote Richard Thompson's simple group $V$. By [Hig74], $V$ is finitely presented and by [Kap02, Theorem 0.1] $H_2(V; \mathbb{Z}) = 0$ (in fact $V$ is acyclic [SW19]). Let $X_n = \{g \in V \mid \operatorname{ord}(g) = n\}$. Now, $\langle X_n \rangle \trianglelefteq V$ but every finite group embeds into $V$, see [CFP96, page 241]. So $\langle X_n \rangle$ is a non-trivial normal subgroup of a simple group. Hence, $\langle X_n \rangle = V$. Thus, there is a finite set $Y_n \subseteq X_n$ of cardinality $\ell_n$ such that $Y_n$ generates $V$.

Let $H_n = \ast_{i=1}^{\ell_n} C_n$. We have a surjection $\pi_n \colon H_n \twoheadrightarrow V$ sending a generator of each factor $C_n$ to a distinct element of $Y_n$. Since $V$ is infinite $N_n := \ker \pi_n$ has infinite index. Now, a classical result of Baumslag [Bau66, §6] implies that $\ker \pi_n$ is not finitely generated.

For $d \geqslant 2$ we denote by $P_{n,d}$ the $d$-fold fibre product of the map $\pi_n$. By Lemma 5.3 we see that $P_{n,d}$ is finitely generated. Now, Theorem 5.4 implies that the inclusion $P_{n,d} \hookrightarrow H_n^d$ induces an isomorphism $\widehat{P}_{n,d} \to \widehat{H}_n^d$.

To conclude we need to show $P_{n,d}$ is not isomorphic to $H_n^d$. Suppose for a contradiction that they are. By the Kurosh subgroup theorem [Kur34] centralizers in $H_n$ are cyclic. It follows that centralizers of non-cyclic subgroups in $H_n^d$ are (up to relabelling factors) of the form $H_n^{d-1} \times \{1\}$. Similarly, centralizers of non-cyclic subgroups in $P_n^d$ are (up to relabelling factors) of the form $N_n^{d-1} \times \{1\}$. But these latter groups are not finitely generated. A contradiction. We conclude $P_{n,d}$ is not isomorphic to $H_n^d$ and so $H_n^d$ is neither Grothendieck rigid nor profinitely rigid in the absolute sense. □

**Theorem 5.6.** *Let $G_1, \dots, G_\ell$ be finitely generated, residually finite groups such that for all $1 \leqslant j \leqslant \ell$ the centralizer of a nonabelian subgroup of $G_j$ is* *trivial. Assume also that at least four of the $G_j$ have a subgroup of index 2. Let $d \geqslant 2$. The genus of*

$$\prod_{i=1}^{d} \left( \mathop{*}_{j=1}^{\ell} G_j \right)$$

*is infinite.*

*Proof.* Fix $d \geqslant 2$. Letting $H = *_{j=1}^{\ell} G_j$ we know that $H$ is residually finite and finitely generated, that the centralizer of a nonabelian subgroup of $H$ is trivial (by the Kurosh subgroup theorem [Kur34]), and that for each even $n \geqslant 2$ we have a surjective homomorphism $\phi_n : H \to V_n$ (by Theorem 4.1). For each even $n \geqslant 2$ let $P_n$ denote the $d$-fold fibre product of the map $\phi_n$ (since we fix $d$, we omit it from the notation here). Further, $V_n$ is finitely presented by [Hig74] and $H_2(V_n; \mathbb{Z}) = 0$ by [Kap02, Theorem 0.1]. Hence, by Theorem 5.4 we see that $\widehat{P_n}$ is isomorphic to $\widehat{\prod_{i=1}^{d} H}$ and by Lemma 5.3 each $P_n$ is finitely generated.

It will be sufficient to prove that if $P_n$ is isomorphic to $P_m$, where $n, m \geqslant 2$ are even, then $n = m$. Assume such $P_n$ and $P_m$ are isomorphic and let $N_n = \ker(\phi_n)$ and $N_m = \ker(\phi_m)$. We know that $\prod_{i=1}^{d} N_n \leqslant P_n \leqslant H^d$. For $1 \leqslant i \leqslant d$ let $p_i : H^d \to H$ denote projection to the $i$ coordinate. If $K \leqslant P_n$ is nonabelian then for some $i$ we have $p_i(K) \leqslant H$ is nonabelian, without loss of generality $i = 1$. The centralizer in $H$ of $p_1(K)$, denoted $C_H(p_1(K))$, is trivial and this implies that $C_{H^d}(K)$ is a subgroup of $\{1\} \times \prod_{i=2}^{d} H$. Therefore $C_{P_n}(K) = P_n \cap C_{H^d}(K)$ is a subgroup of $\{1\} \times \prod_{i=2}^{d} N_n$. What we have just shown is that if $K \leqslant P_n$ is nonabelian then there is some $1 \leqslant i_0 \leqslant d$ for which $C_{P_n}(K) \leqslant (\prod_{i \neq i_0} N_n) \times \{1\}$.

We know $N_n$ is a subgroup of the free product $H$, so by the Kurosh subgroup theorem [Kur34] if $N_n$ were abelian then it would be either infinite cyclic or conjugate to a subgroup of one of the factors of $H$. However $N_n$ cannot be conjugate to a subgroup of one of the factors of $H$ because $H$ has at least 2 nontrivial factors which are residually finite, this would imply that the infinite simple group $H/N_n \cong V_n$ has a proper finite index subgroup, a contradiction. If on the other hand $N_n$ is infinite cyclic then by normality its centralizer has index either 1 or 2 in $H$; index 1 is impossible since the free product $H$ has trivial center and index 2 is impossible for it would give an index 2 subgroup of the infinite simple group $H/N_n \cong V_n$. Thus $N_n$ is not abelian. We therefore have

$$C_{P_n}\left( N_n \times \prod_{i \neq i_0} \{1\} \right) = \left( \prod_{i \neq i_0} N_n \right) \times \{1\}.$$

Thus, each subgroup $(\prod_{i_0 \neq 1} N_n) \times \{1\}$ is an element in the set $\mathcal{X} = \{C_{P_n}(K) \mid K \leqslant P_n \text{ nonabelian}\}$ and each element in $\mathcal{X}$ includes into such a subgroup. Therefore the subgroup of $P_n$ generated by maximal subgroups which centralize a nonabelian subgroup is precisely $\prod_{i=1}^{d} N_n$, and by the same reasoning the comparably defined subgroup in $P_m$ is $\prod_{i=1}^{d} N_m$. As $P_n$ is isomorphic to $P_m$, $\prod_{i=1}^{d} N_n$ corresponds to $\prod_{i=1}^{d} N_m$ under the isomorphism. Therefore $P_n / \prod_{i=1}^{d} N_n \cong V_n$ is isomorphic to $P_m / \prod_{i=1}^{d} N_m \cong V_m$, and $V_n \simeq V_m$ implies that $n = m$ [Hig74, Theorem 6.4].

□

Recall that a group is *indicable* if it has a surjective homomorphism onto the integers. A group $J$ is *3/2-generated* if for every nontrivial $g \in J$ there exists a $g' \in J$ such that $\{g, g'\}$ generates $J$.

**Theorem 5.7.** *Let $G$ and $L$ be finitely generated, residually finite groups in which the centralizer of a nonabelian subgroup is trivial. Assume further that $G$ is nontrivial and $L$ is indicable. Then the product $\prod_{i=1}^{d}(G * L)$, where $d \geqslant 2$, has infinite genus.*

*Proof.* Fix $d \geqslant 2$. Let $H$ denote the free product $G * L$. We know that $V_n$ is 3/2-generated for each natural number $n \geqslant 2$ [DH20, Theorem 1] and includes every finite group. Since $G$ is nontrivial and residually finite, and $L$ is indicable we have for each even $n \geqslant 2$ a surjective homomorphism $\phi_n : H \to V_n$. Take $P_n \leqslant \prod_{i=1}^{d} H$ to be the $d$-fold fibre product of $\phi_n$ and $N_n$ to be its kernel. As before, the inclusion map $\iota : P_n \hookrightarrow \prod_{i=1}^{d} H$ induces an isomorphism $\widehat{\iota}\colon \widehat{P_n} \to \prod_{i=1}^{d} \widehat{H}$.

The proof that $P_n \cong P_m$ implies $n = m$ is the same as in the proof of Theorem 5.6, word for word. □

For a straightforward application of Theorem 5.7, one can take $L = \mathbb{Z}$ and take $G$ to be a nontrivial finitely generated abelian group (say, a nontrivial cyclic group).

For another application one can take $L = \mathbb{Z}$ and $G$ to be a nontrivial torsion-free group with finite $C'(1/6)$ presentation. Such a $G$ is famously residually finite since it is hyperbolic and by combining [Wis04, Theorem 1.2] with [Ago13, Corollary 1.2]. Furthermore, the centralizer in a torsion-free hyperbolic group of a nontrivial element is cyclic, so $G$ satisfies the hypotheses of Theorem 5.7. Random groups in the few-relator sense are torsion-free and $C'(1/6)$. More concretely $G$ can be the fundamental group of an orientable surface of genus at least 2.

Our last application is a generalisation of [BG04, Proposition 9.2].

**Corollary 5.8.** *Let $d \in \mathbb{N}$, $d \geqslant 2$. The product $\prod_{i=1}^{d} F_n$ has infinite genus if and only if $n \geqslant 2$.*

*Proof.* If $n = 1$, then it is known that $\mathbb{Z}^d$ has genus 1. Assume that $n \geqslant 2$. We take $G = F_{n-1}$ and $L = \mathbb{Z}$. The group $G$ is a torsion free hyperbolic group, hence the centralizer of a nonabelian subgroup is trivial. Thus by Theorem 5.7 the genus of $\prod_{i=1}^{d} F_n$ is infinite. □

The following definitions follow [Rad03]. Let $\Gamma$ be a graph and let $\Omega$ be a subgraph. We say $\Omega$ is a *module* if $\Omega$ is a full subgraph and if for each edge $\{v, k\} \in E(\Gamma)$ with $v \in V(\Gamma)\backslash V(\Omega)$ and $k \in \Omega$, and for each $k' \in \Omega$, there exists an edge $\{v, k'\} \in E(\Gamma)$. We form the *collapsed graph* $\Gamma'$ by taking the vertex set to be $V(\Gamma)\backslash V(\Omega) \cup \{*\}$ and the edge set to be

$$\{\{v, w\} \in E(\Gamma) \mid v, w \in V(\Gamma)\backslash V(\Omega)\} \cup \{(v, *) \mid v \in V(\Gamma),\ \{v, k\} \in E(\Gamma),\ k \in \Omega\}.$$

See Figure 7 for an example.

There is an isomorphism of the graph product $G_\Gamma$ with the graph product $G_{\Gamma'}$ when the group $G_*$ is isomorphic to $G_\Omega$. Suppose there exists a module

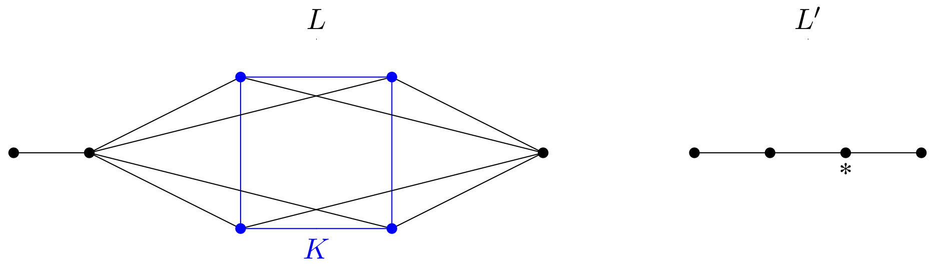


Figure 7

$\Omega$ of $\Gamma$ and let $G_{\Gamma'}$ denote the collapsed graph. We say $G_\Omega$ is $G_\Gamma$-*stably isomorphic* to another group $P$ (not isomorphic to $G_\Omega$) if $G_\Gamma$ is isomorphic to the graph product $G_{\Gamma'}$ where the group $G_*$ is isomorphic to $P$.

We say a group $G$ is *profinitely flexible with pair* $P$ if $P$ is a finitely generated residually finite group such that $G \not\cong P$ but $\widehat{G} \cong \widehat{P}$.

**Proposition 5.9.** *Let $\Gamma$ be a graph and let $\Omega$ be a module. Let $G_\Gamma$ be a graph product on $\Gamma$ and suppose $G_\Omega$ is profinitely flexible with pair $P$. If $P$ is not stably $G_\Gamma$-isomorphic to $H$, then $G$ is profinitely flexible.*

*Proof.* A model for $\widehat{G}_\Gamma$ is given by taking the graph product of $G_\Gamma$ in the category of profinite groups. Namely, each vertex group $G_v$ is replaced by $\widehat{G}_v$, we form the profinite free product $\ast_{v \in V(L)} \widehat{G}_v$, and then quotient out by the set of relations $[g_v, g_w]$ for $g_v \in \widehat{G}_v$ and $g_w \in \widehat{G}_w$, whenever $\{v, w\}$ is an edge of $\Gamma$.

Now, $\Omega$ is collapsible in $\Gamma$. It follows there is an isomorphism of $G_\Gamma$ with the graph product $G_{\Gamma'}$ when the group $G_*$ is isomorphic to $G_\Omega$. On the level of profinite completions this means that $\widehat{G}_\Gamma$ is isomorphic to $\widehat{G}_{\Gamma'}$. Replacing the group $G_*$ with $P$ we obtain a graph product $G_{\Gamma'}$ not isomorphic to $G_\Gamma$. But, on the level of profinite completions $\widehat{P}$ and $\widehat{G}_\Omega$ are isomorphic. Hence, $\widehat{G}_\Gamma$ and $\widehat{G}_{\Gamma'}$ are isomorphic. So $G_\Gamma$ is profinitely flexible. $\square$

**Remark 5.10.** The stability hypothesis is indeed necessary. Upon dropping the hypothesis one can easily construct counter examples by considering graph products with vertex groups isomorphic to $\mathbb{Z}$ or the metacyclic groups considered by Baumslag [Bau74].

**Corollary 5.11.** *Let $\Gamma$ be a graph and let $\Omega$ be a $(4,4)$-bipartite module. Then, $W_\Gamma$ is profinitely flexible.*

*Proof.* In Theorem 5.5 we showed that $W_\Omega$ is profinitely flexible with pair $P$. Thus, by Proposition 5.9 it suffices to show that $P$ is not stably $W_\Gamma$-isomorphic to $W_\Omega$. But this is immediate since $P$ was not finitely presented. $\square$

**Corollary 5.12.** *Let $\Gamma$ be a graph and let $\Omega$ be a $(2,2)$-bipartite module. Then $A_\Gamma$ is profinitely flexible.*

*Proof.* The proof is analogous. The key difference is profinite flexibility of $F_2 \times F_2$ was shown in [PT86]. $\square$

**Corollary 5.13.** *There exist right-angled Coxeter and Artin groups that are profinitely flexible but which are neither freely nor directly reducible.*

*Proof.* We apply the previous two corollaries to the graph in Figure 7. In the RACG case we take the graph product with vertex groups equal to $D_\infty = C_2 \ast C_2$ and for the RAAG case we take the graph product with vertex groups equal to $\mathbb{Z}$. □

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