BNSR INVARIANTS AND ℓ^2-HOMOLOGY

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Dedicated to the memory of Peter A. Linnell.

Abstract. We prove that if the nth ℓ^2-Betti number of a group is non-zero then its nth BNSR invariant over $\mathbb{Q}$ is empty, under suitable finiteness conditions. We apply this to answer questions of Friedl–Vidussi and Llosa Isenrich–Py about aspherical Kähler manifolds, to verify some cases of the Singer Conjecture, and to compute certain BNSR invariants of poly-free and poly-surface groups.

1. Introduction

It was shown by Lück [Lüc94] that non-vanishing ℓ^2-Betti numbers $b_n^{(2)}$ obstruct finite CW complexes being homotopic to mapping tori. In the case of the classifying space of a group G, this can be restated in terms of finiteness properties of the kernel of a homomorphism $G \twoheadrightarrow \mathbb{Z}$ (see [Lüc02, Theorem 6.63] for the statement regarding connected CW complexes).

Theorem 1.1 (Lück)**.** *Let G be a group of type F_n, that is admitting a classifying space with finite n-skeleton, and let $\varphi\colon G \twoheadrightarrow \mathbb{Z}$ be an epimorphism. If $b_n^{(2)}(G) \neq 0$, then $\ker\varphi$ is not of type F_n.*

Bieri, Neumann, Renz, and Strebel [BNS87, BR88] introduced a family of group invariants which are intimately related to the finiteness properties of kernels of homomorphisms $G \to \mathbb{R}$. In spite of this, the precise relation between the nth ℓ^2-Betti number and the nth BNSR invariants Σ^n has remained mysterious. Indeed, the question is rather subtle: for $n = 1$, Brown [Bro87] gave a characterisation of the integral characters $G \twoheadrightarrow \mathbb{Z}$ in $\Sigma^1(G)$ as those that are induced by G splitting as an ascending HNN extension over a finitely generated group; such a splitting gives G the structure of an algebraic mapping torus. There is however no such characterisation for $n > 1$; the best result in this direction seems to be a theorem of Renz for $n = 2$ [Ren89]. Our main theorems clarify the link between the BNSR invariants and ℓ^2-Betti numbers in both the cases of groups and spaces.

Theorem A. *Let G be a group of type $\mathsf{FP}_n(\mathbb{Q})$. If $b_n^{(2)}(G) \neq 0$, then $\Sigma^n(G;\mathbb{Q}) = \varnothing$.*

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Recall that type FP_n is a homological version of type F_n; we give a formal definition in Section 2.G.

The theorem was already known for RFRS groups (see [HK22, Theorem 5.10] building on [Kie20] and [Fis21]). It is also significantly easier to establish this theorem when G is torsion free and known to satisfy the Atiyah conjecture. We have an analogous statement for CW complexes.

Theorem B. *Let X be a connected CW complex with finite n-skeleton. If $b_n^{(2)}(\widetilde{X};\pi_1 X) \neq 0$, then $\Sigma^n(X;\mathbb{Q}) = \varnothing$.*

A special case of the above result with X being a closed smooth manifold can be deduced from work of Farber [Far00, Corollary 2] (based on [NS86]). The methods used here are of a very different nature: they are ring-theoretic rather than analytic, as is the case of Farber.

For a commutative ring R, recall that an *n-dimensional Poincaré duality complex M over R* (or a *$\mathsf{PD}_n(R)$-complex*) is a finitely dominated connected CW complex with a distinguished class $[M]$ in the nth homology group $H_n(M;D)$, where D is an RG module isomorphic to R as a group, such that the cap product

$$[M] \frown -\colon H^k(M;A) \to H_{n-k}(M;A\otimes_R D)$$

is an isomorphism for all choices of local coefficients A over R, where the action on $A\otimes_R D$ is diagonal. Similarly, a *$\mathsf{PD}_n(R)$-group* is a group of type $\mathsf{FP}(R)$ with cohomological dimension $\operatorname{cd}_R(G) = n$ for which $D = H^n(G;RG)$ is isomorphic to R as an abelian group and $H^i(G;M) \cong H_{n-i}(G;M\otimes_R D)$ for all RG-modules M (here $M\otimes_R D$ is equipped with the diagonal action).

Note that the fundamental group of an aspherical $\mathsf{PD}_n(R)$-complex is a finitely presented $\mathsf{PD}_n(R)$-group, but for $n \geqslant 4$ there exist examples of $\mathsf{PD}_n(R)$-groups which are not finitely presented [Dav98a, Dav98b] (in fact there are uncountably many such groups [Lea18, KLS20]).

The motivation for the authors to begin investigating the connection between vanishing of the nth BNSR invariants (over $\mathbb{Q}$) and non-vanishing of the nth ℓ^2-Betti number was provided by the following question which appeared in [FV21, remark after Proposition 3.4] and [LP22, Question 7].

Question 1.2 (Friedl–Vidussi, Llosa Isenrich–Py)**.** *Let M be a closed aspherical Kähler $2n$-manifold. If $\chi(M) \neq 0$, is $\Sigma^n(\pi_1 M)$ empty?*

In fact we are able to give an affirmative answer to the more general question obtained by dropping the hypotheses 'aspherical' and 'Kähler' and instead studying the BNSR invariants of the manifold.

Corollary C. *Let M be a closed connected $2n$-manifold or (more generally) a finite $\mathsf{PD}_{2n}(\mathbb{Q})$-complex. If $\chi(M) \neq 0$, then $\Sigma^n(M) = \Sigma^n(M;\mathbb{Q}) = \varnothing$. In particular, if M is additionally aspherical, then $\Sigma^n(\pi_1 M) = \Sigma^n(\pi_1 M;\mathbb{Q}) = \varnothing$.*

Remark 1.3. A near-identical proof yields the following result: If G is a $\mathsf{PD}_{2n}(\mathbb{Q})$-group such that $\chi(G) \neq 0$, then $\Sigma^n(G;\mathbb{Q}) = \varnothing$.

The *Singer Conjecture* is one of the major unresolved problems regarding ℓ^2-Betti numbers and the topology of manifolds. For more information the reader is directed to [Lüc02, Chapter 11].

The Singer Conjecture. *If M is a closed aspherical n-manifold, then $b_p^{(2)}(\widetilde{M};\pi_1 M) = 0$ for all $p \neq n/2$.*

The Singer Conjecture can clearly be generalised to $\mathsf{PD}_n(\mathbb{Q})$-groups; in this setting we have the following corollary.

Corollary D. *Let G be a $\mathsf{PD}_n(\mathbb{Q})$-group and let $k = \lceil n/2 \rceil - 1$. If $\Sigma^k(G;\mathbb{Q}) \neq \varnothing$, then the Singer Conjecture holds for G.*

The idea behind the proofs. The main technical result is Proposition 5.1, in which we prove that if a chain complex over a group ring $\mathbb{Q}G$ has vanishing Novikov homology up to some dimension, then its ℓ^2-homology vanishes as well, up to the same dimension. Novikov homology has been shown by Sikorav to encode the BNSR invariants, and ℓ^2-homology can also be understood algebraically, as homology with coefficients in the algebra $\mathcal{U}G$ of operators affiliated to the group von Neumann algebra of G. Hence, we are trying to show that if we have a partial chain contraction defined over the Novikov ring, then we also have one over $\mathcal{U}G$. We do this in two steps: first, in Proposition 3.6 we show that one does not need to work with the whole Novikov ring, but instead it is enough to use the division closure of $\mathbb{Q}G$ inside the Novikov ring. Then, in Proposition 4.24, we show that there is a ring that we call the ring of *weakly rational elements* of $\mathcal{U}G$, that is simultaneously a subring of $\mathcal{U}G$, and an overring of the division closure of $\mathbb{Q}G$ in the Novikov ring.

The construction of the ring of weakly rational elements is where the blood, sweat, tears, and toil went. In an ideal world, for example when G is torsion free and satisfies the Atiyah conjecture, one argues as follows: the Novikov ring corresponding to $\phi\colon G \to \mathbb{Z}$ consists naturally of twisted Laurent power series with coefficients in $\mathbb{Q}\ker\phi$, and so in particular in the Linnell skew field $\mathcal{D}\ker\phi$, a particularly useful subring of $\mathcal{U}\ker\phi$. Since twisted Laurent polynomials over $\mathcal{D}\ker\phi$ satisfy the Ore condition, one can look at the rational functions, and it is not hard to show that, on the one hand, such rational functions contain the division closure of $\mathbb{Q}G$ inside the Novikov ring, and on the other, are naturally contained in $\mathcal{U}G$, even though the ring of Laurent power series is not. Unfortunately, the authors were unable to show that the twisted Laurent polynomial ring satisfies the Ore condition – this remains an open problem. Instead, we show that this ring satisfies an approximate version of the Ore condition, which allows us to form a ring of 'approximate rational functions', that is precisely the ring of weakly rational elements.

Outline of the paper. In Section 2, we give the relevant background on ℓ^2-homology, BNSR theory, some ring theoretic tools, and several rings and algebras related to a countable discrete group that we will need in our proofs, with particular emphasis on group von Neumann algebras.

In Section 3, we prove that vanishing of Novikov homology can be passed to a division-closed subring. We highlight two results of independent interest: The first is Proposition 3.1 that states that for a discrete group G the division closure of RG in a Novikov ring $\widehat{RG}^{\varphi}$ is equal to its rational closure. The second is Proposition 3.3 that gives a method for taking chain contractions over $\widehat{RG}^{\varphi}$ and rebuilding them over the division closure of RG.

Section 4 is the technical heart of the paper. Here we introduce the ring of weakly rational elements of $\mathcal{U}G$, and prove that it contains the division closure of $\mathbb{Q}G$ inside of the Novikov ring.

In Section 5, we prove each of the results A - D.

In Section 6, we prove a characteristic p version of Theorem A and Theorem B (see Theorem 6.4) for groups whose group rings admit embeddings into certain universal skew fields. We also give a simplified proof of A - D when G is torsion free and satisfies the Atiyah Conjecture.

In Section 7, we detail a number of example applications of our results. In Theorem 7.2 we prove that $\Sigma^n(G;\mathbb{Q}) = \varnothing$ when G is a poly-elementarily-free group of length n. This class includes poly-free and poly-surface groups. The result generalises [KV22, Proposition 1.5] where the authors compute $\Sigma^2(G)$ for groups isomorphic to $F_n \rtimes F_m$ or $\pi_1\Sigma_g \rtimes \pi_1\Sigma_h$ with $n, m, g, h > 1$ (see the first part of [KW19, Theorem 6.1] for the case $F_2 \rtimes F_m$); here Σ_g is the closed orientable connected surface of genus g. We also highlight how our Theorem A applies to real and complex hyperbolic lattices.

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2. Preliminaries

Throughout, all rings are assumed to be associative and unital. All groups will be discrete, and modules will be right modules, unless stated otherwise.

2.A. Group von Neumann algebras.

Let G be a countable group and let $\ell^2 G$ denote the Hilbert space of square summable formal sums of elements of G with complex coefficients, that is the space of expressions

$$\sum_{g\in G} \lambda_g g \text{ such that } \sum_{g\in G} |\lambda_g|^2 < \infty \text{ where } \lambda_g \in \mathbb{C}.$$

The group G acts on $\ell^2 G$ by right multiplication.

Definition 2.1 (Group von Neumann algebra)**.** We define the *group von Neumann algebra* $\mathcal{N}G$ of G to be the algebra of G-equivariant bounded operators $\ell^2 G \to \ell^2 G$.

Since G acts on $\ell^2 G$ from the right, it is natural to view $\mathcal{N}G$ as operating from the left on $\ell^2 G$. In particular, the copy of $\mathbb{C}G$ in $\mathcal{N}G$ acts on $\ell^2 G$ from the left. Since $\mathcal{N}G$ contains a copy of $\mathbb{C}G$, multiplication turns it into a $\mathbb{C}G$-bimodule.

One of the key features of group von Neumann algebras is that for every closed G-invariant subspace V of $\ell^2 G$, there is an associated projection $\pi_V \in \mathcal{N}G$; it is a self-adjoint idempotent with $\operatorname{im} \pi_V = V$ and $\ker \pi_V = V^\perp$.

Another important class of operators in $\mathcal{N}G$ are *partial isometries*, that is operators u such that $u^*u = \pi_{(\ker u)^\perp}$. It is easy to see that this implies that $\operatorname{im} u$ is closed and that $uu^* = \pi_{\operatorname{im} u}$.

Definition 2.2 (Polar decomposition)**.** The *polar decomposition* of $s \in \mathcal{N}G$ is a canonical factorization $s = vp$ where v is a partial isometry and p is a non-negative operator, and both lie in $\mathcal{N}G$. Furthermore, v^*v is the projection onto $\ker s^\perp$, and vv^* is the projection onto the closure of $\operatorname{im} s$, whence it easily follows that $\ker v = \ker s$ and $\operatorname{im} v$ is the closure of $\operatorname{im} s$.

The group von Neumann algebra comes equipped with the *von Neumann trace* tr. There is a canonical dimension function $\dim_{\mathcal{N}G}$, taking values in $[0, \infty]$, defined on $\mathcal{N}G$-modules called the *von Neumann dimension* - the precise definition and basic properties we use can be found in [Lüc02, Chapter 6]. One can also define such a dimension for closed G-invariant subspaces of $\ell^2 G$, in which case it is equal to the von Neumann trace of the corresponding projection. Since the only projection with zero trace is 0, the only G-invariant subspace of $\ell^2 G$ of von Neumann dimension zero is the trivial subspace.

We will also use the following theorem.

Theorem 2.3 (Linnell)**.** [Lin92, Theorem 4] *Let $H \trianglelefteq G$ be such that G/H is right orderable with total order $\leqslant$ and let $\varphi\colon G \twoheadrightarrow G/H$ be the natural epimorphism. Let T be a transveral for H in G, let $x \in \ell^2 G$, and write $x = \sum_{t \in T} x_t t$ where $x_t \in \ell^2 H$ for all $t \in T$. Suppose that there exists $t_0 \in T$ such that $x_t = 0$ for t with $\varphi(t) < \varphi(t_0)$. If $x_{t_0} y' \neq 0$ for all non-zero $y' \in \mathcal{N}H$, then $xy \neq 0$ for all non-zero $y \in \ell^2 G$.*

2.B. **ℓ^2-homology and Betti numbers.**

Definition 2.4 (ℓ^2-homology and Betti numbers)**.** Let X be a G-CW complex. We define the *ℓ^2-homology* of X with respect to G as

$$H_p^G(X; \mathcal{N}G) := H_p\big(C_\bullet(X; \mathbb{Q}) \otimes_{\mathbb{Q}G} \mathcal{N}G\big),$$

where $C_\bullet(X; \mathbb{Q})$ is the cellular chain complex of X with rational coefficients considered as a complex of free $\mathbb{Q}G$-modules.

We define the *ℓ^2-Betti numbers* of X with respect to G to be

$$b_p^{(2)}(X; G) := \dim_{\mathcal{N}G} H_p^G(X; \mathcal{N}G).$$

The ℓ^2-Betti numbers of a group G are defined to be the ℓ^2-Betti numbers of the universal free G-space EG. A number of properties of ℓ^2-Betti numbers can be found in [Lüc02, Theorem 6.54].

The $\mathcal{N}G$-dimension does not change when one qoutients an $\mathcal{N}G$-module by its torsion.

There are two other ways of defining ℓ^2-homology; for details, see [Lüc98]. One can look at the reduced homology of the complex

$$C_\bullet(X; \mathbb{Q}) \otimes_{\mathbb{Q}G} \ell^2 G,$$

where *reduced* means that we divide kernels by closures of images. This way the homology groups are actually closed subspaces of powers of $\ell^2 G$, and one can look at the von Neumann dimension of such a subspace. The dimension coincides with the ℓ^2-Betti number, as defined above.

The third way of defining ℓ^2-homology involves the algebra of affiliated operators $\mathcal{U}G$, which we will define in a moment. One can look at the homology of $C_\bullet(X; \mathbb{Q}) \otimes_{\mathbb{Q}G} \mathcal{U}G$, which ends up being equal to the homology of $C_\bullet(X; \mathbb{Q}) \otimes_{\mathbb{Q}G} \mathcal{N}G$ tensored with $\mathcal{U}G$. This has essentially the effect of quotienting the ℓ^2-homology as we defined it by the torsion. There is again a notion of a dimension for $\mathcal{U}G$-modules, and again one obtains the same Betti numbers.

2.C. **Ore localisation.** In this section we will describe an analogue of localisation for non-commutative rings.

Definition 2.5. Let R be a ring. An element $x \in R$ is a *zero-divisor* if $x \neq 0$, and $xy = 0$ or $yx = 0$ for some non-zero $y \in R$. A non-zero element that is not a zero-divisor will be called *regular*.

Definition 2.6 (Right Ore condition)**.** Let R be a ring and $S \subseteq R$ a multiplicatively closed subset consisting of regular elements. The pair (R, S) satisfies the *right Ore condition* if for every $r \in R$ and $s \in S$ there are elements $r' \in R$ and $s' \in S$ satisfying $rs' = sr'$.

Definition 2.7 (Right Ore localisation)**.** If (R, S) satisfies the right Ore condition we may define the *right Ore localisation*, denoted RS^{-1}, to be the following ring. Elements are represented by pairs $(r, s) \in R \times S$ up to the following equivalence relation: $(r, s) \sim (r', s')$ if and only if there exists $u, u' \in R$ such that the equations $ru = r'u'$ and $su = s'u'$ hold, and $su = s'u'$ belongs to S. The addition is given by

$$(r, s) + (r', s') = (rc + r'd, t), \text{ where } t = sc = s'd \in S,$$

and the multiplication is given by

$$(r, s)(r', s') = (rc, s't), \text{ where } sc = r't \text{ with } t \in S.$$

There is a natural ring homomorphism $R \to RS^{-1}$ defined by $r \mapsto (r, 1)$. For more information on this construction the reader is referred to [Pas85, Section 4.4]. Note that there is also an analogously defined *left* Ore condition.

2.D. **The algebra of affiliated operators.**

Definition 2.8 (Affiliated operators)**.** We say that an operator

$$f\colon \operatorname{dom}(f) \to \ell^2 G$$

with $\operatorname{dom}(f) \subseteq \ell^2 G$ is *affiliated* (to $\mathcal{N}G$) if f is densely defined with domain $\operatorname{dom}(f)$, is closed, and is a G-operator, that is, $\operatorname{dom}(f)$ is a linear G-invariant subspace and $f(x).g = f(x.g)$ for all $g \in G$ (recall that G acts on $\ell^2 G$ on the right).

The set of all operators affiliated to $\mathcal{N}G$ forms the *algebra of affiliated operators* $\mathcal{U}G$ of G.

Since an adjoint of a densely defined closed operator is densely defined and closed, every $x \in \mathcal{U}G$ has a well-defined adjoint $x^* \in \mathcal{U}G$.

Note that we have inclusions of $\mathbb{Q}G$-modules

$$\mathbb{Q}G \rightarrowtail \mathbb{C}G \rightarrowtail \mathcal{N}G \rightarrowtail \mathcal{U}G.$$

For more information on these constructions the reader is referred to [Lüc02] – specifically Theorem 8.22 and more generally Chapter 8. We highlight one theorem of special importance to us.

Theorem 2.9. [Lüc02, Theorem 8.22(1)] *The set S of regular elements of $\mathcal{N}G$ forms a right Ore set. Moreover, $\mathcal{U}G$ is canonically isomorphic to $(\mathcal{N}G)S^{-1}$.*

The algebra $\mathcal{U}G$ is von Neumann regular, with explicit control on what the partial inverses look like. In particular, for every $x \in \mathcal{U}G$ there exists a canonical $x^\dagger \in \mathcal{U}G$ such that $xx^\dagger = \pi_{\overline{\operatorname{im} x}}$ as an affiliated operator. Note that in reality the composition $x \circ x^\dagger$ is defined only on $\operatorname{im} x \oplus (\operatorname{im} x)^\perp$, and coincides with $\pi_{\overline{\operatorname{im} x}}$ on this subspace. However, the graph of this composition is not closed, and hence the composition is not an affiliated operator. It can be extended to one, and this extension is precisely $\pi_{\overline{\operatorname{im} x}}$. Similarly, we have $x^\dagger x = \pi_{(\ker x)^\perp}$. We also have $\ker x^\dagger = (\operatorname{im} x)^\perp$ and $\overline{\operatorname{im} x^\dagger} = (\ker x)^\perp$.

The partial inverse can be constructed directly, as in the proof of [Lüc94, Lemma 8.3(2)], or its existence can be deduced algebraically, as in [JZ19, Proposition 3.2].

2.E. Division and rational closures.

Definition 2.10 (Division and rational closure)**.** Let R be a ring and S a subring. We say that S is *division closed* if every element of S invertible over R is invertible over S. We say that S is *rationally closed* if every finite square matrix over S invertible over R is invertible over S.

Define the *division closure of S in R*, denoted by $\mathcal{D}(S \subset R)$, to be the smallest division-closed subring of R containing S. Define the *rational closure of S in R*, denoted by $\mathcal{R}(S \subset R)$, to be the smallest rationally closed subring of R containing S.

2.F. Twisted polynomial rings.

Definition 2.11 (Twisted polynomial ring)**.** Let R be a ring and let $R[t^{\pm 1}]$ be the abelian group of Laurent polynomials over R. Given a homomorphism $\nu\colon \mathbb{Z} \to \operatorname{Aut}(R)$ we may endow $R[t^{\pm 1}]$ with a non-commutative multiplication given by

$$xt^m \cdot yt^n = x\nu(t^m)(y)t^{m+n},$$

and extended linearly. We call this new ring the *ring of twisted Laurent polynomials over R with respect to ν* or simply a *twisted Laurent polynomial ring.*

Remark 2.12. Suppose that G splits as $H \rtimes \mathbb{Z}$. The group ring RG is isomorphic to a ring of twisted Laurent polynomials $RH[t^{\pm 1}]$ in a natural way.

2.G. **BNSR invariants.** In this section we follow the treatment of Farber–Geoghegan–Schütz [FGS10, Section 2]. Note that the authors only give statements and proofs for BNSR invariants over the ring $\mathbb{Z}$, however the arguments easily generalise to arbitrary rings R.

Let R be a ring. A monoid M is of *type* $\mathsf{FP}_n(R)$ if there exists a projective resolution $P_\bullet \to R$ of the trivial RM-module R with P_i finitely generated for $i \leqslant n$. Let G be a finitely generated group. Define $S(G) = \operatorname{Hom}(G;\mathbb{R}) \smallsetminus \{0\}$. Given $\varphi \in S(G)$, define a submonoid of G by

$$G_\varphi := \{g \in G : \varphi(g) \geqslant 0\}.$$

Definition 2.13 (Homological BNSR invariants of groups)**.** Let R be a ring and G be a group of type $\mathsf{FP}_\mathsf{n}(R)$ for $n \in \mathbb{N} \cup \{\infty\}$. For $\varphi \in S(G)$ we write $\varphi \in \Sigma^n(G; R)$ if G_φ is of type $\mathsf{FP}_n(R)$.

For R a ring, $n \in \mathbb{N}$, and $C_\bullet$ a chain complex over R, we say that $C_\bullet$ has *finite n-type* if there exists a finitely generated chain complex $P_\bullet$ of projective R-modules and a chain map $f\colon C_\bullet \to P_\bullet$ such that $f_i\colon H_i(C_\bullet) \to H_i(P_\bullet)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$. In this case we call f an *n-equivalence*.

Let $C_\bullet$ be a chain complex of RG-modules and let $k \in \mathbb{N}$. We define

$$\Sigma^k(C_\bullet; R) := \{\varphi \in S(G) : C_\bullet \text{ is of finite } k\text{-type over } RG_\varphi\}.$$

Definition 2.14 (Homological BNSR invariants of spaces)**.** Let X be a connected CW complex with finite n-skeleton. For $k \leqslant n$ the *k-th (homological) BNSR invariant of X over R* is defined to be

$$\Sigma^k(X; R) := \Sigma^k(C_\bullet(\widetilde{X}; R); R),$$

where $\widetilde{X}$ is the universal cover of X, and $C_\bullet(-; R)$ denotes the cellular chain complex with coefficients in R.

Definition 2.15 (Homotopic BNSR invariants of groups and spaces via closed 1-forms)**.** Let X be a connected CW complex with finite n-skeleton and with $\pi_1 X = G$. Let ω be a closed 1-form on X representing $\varphi \in S(G)$ and let $0 \leqslant k \leqslant n$. We define the *$k$-th homotopic BNSR invariant of X* to be the set $\Sigma^k(X) \subseteq S(G)$ defined as follows: $\varphi \in \Sigma^k(X)$ if there exists an $\epsilon > 0$ and a cellular homotopy $H\colon X^{(k)} \times I \to X$ such that $H(x, 0) = x$ and

$$\int_{\gamma_x} \omega \geqslant \epsilon$$

for all $x \in X^{(k)}$, where $\gamma_x\colon [0,1] \to X$ is given by $\gamma_x(t) = H(x,t)$. For a group G we take $\Sigma^n(G) := \Sigma^n(BG)$.

The following theorem essentially combines Theorem 4, Proposition 3, and Corollaries 1 and 2 of [FGS10]; we have taken the liberty to state these results over a general ring R.

Theorem 2.16 (Basic properties of BNSR invariants)**.** *Let X be a connected CW complex with finite n-skeleton and with $\pi_1 X = G$, and let R be a ring. Then,*

(1) $\Sigma^k(X)$ and $\Sigma^k(X; R)$ are open subsets of $S(G)$;
(2) $\Sigma^k(X) \subseteq \Sigma^k(X; R)$;
(3) if $\widetilde{X}$ is k-connected then

$$\Sigma^k(X) = \Sigma^k(G) \text{ and } \Sigma^k(X; R) = \Sigma^k(G; R)$$

and

$$\Sigma^{k+1}(X) \subseteq \Sigma^{k+1}(G) \text{ and } \Sigma^{k+1}(X; R) \subseteq \Sigma^{k+1}(G; R);$$

(4) if X is finite, then for all $k \geqslant \dim X$ we have

$$\Sigma^k(X) = \Sigma^{\dim X}(X) \text{ and } \Sigma^k(X; R) = \Sigma^{\dim X}(X; R).$$

2.H. **Novikov–Sikorav homology.**

Definition 2.17 (Novikov–Sikorav ring)**.** Let G be a finitely generated group, let $\varphi \in S(G)$, and define the *Novikov–Sikorav ring* of G with respect to φ to be

$$\widehat{RG}^{\varphi} := \left\{ \sum_{g \in G} n_g g \;:\; \left|\{g : n_g \neq 0 \text{ and } \varphi(g) < t\}\right| < \infty \text{ for all } t \in \mathbb{R} \right\}.$$

Definition 2.18 (Truncation)**.** Given an element $x = \sum_{g\in G} n_g g \in \widehat{RG}^{\varphi}$ and a real number r, we define the *truncation* of x at r to be the element $\sum_{g\in G} n'_g g$ where

$$n'_g = \begin{cases} n_g & \text{if} \quad \varphi(g) \leqslant r \\ 0 & \text{if} \quad \varphi(g) > r. \end{cases}$$

Note that the truncation lies in RG.

Definition 2.19 (Novikov–Sikorav homology)**.** Let X be a CW complex with $\pi_1 X = G$ and let $\varphi \in S(G)$. The *Novikov–Sikorav homology*

$$H^G_\bullet(\widetilde{X}; \widehat{RG}^{\varphi})$$

of X is the G-equivariant homology of $\widetilde{X}$, the universal cover of X, with non-trivial coefficients $\widehat{RG}^{\varphi}$, that is, the homology of the chain complex $C_\bullet(\widetilde{X}; R) \otimes_{RG} \widehat{RG}^{\varphi}$.

Remark 2.20. Straight from the definition, it is easy to see that if H is a finite index subgroup of G and $H^G_i(\widetilde{X}; \widehat{RG}^{\varphi}) = 0$, then $H^H_i(\widetilde{X}; \widehat{RH}^{\varphi|_H}) = 0$ as well, since $\widehat{RG}^{\varphi} = RG \otimes_{RH} \widehat{RH}^{\varphi|_H}$.

The following result has an essentially identical proof as [Fis21, Theorem 5.3] (see also [FGS10, Proposition 5], [Bie07, Appendix], [Kie20, Theorem 3.11], and [Sik87]).

Theorem 2.21 (Sikorav's Theorem for spaces)**.** *Let X be a connected CW complex with finite n-skeleton and $\pi_1 X = G$. Let $\varphi \in S(G)$ and let $k \leqslant n$. The following are equivalent:*

(1) $\varphi \in \Sigma^k(X; R)$;
(2) $H^G_i(\widetilde{X}; \widehat{RG}^{\varphi}) = 0$ for $i \leqslant k$.

Theorem 2.22 (Sikorav's Theorem for groups [Fis21, Theorem 5.3])**.** *Let G be a group of type $\mathsf{FP}_n(R)$. Let $\varphi \in S(G)$ and let $k \leqslant n$. The following are equivalent:*

(1) $\varphi \in \Sigma^k(G; R)$;
(2) $H_i(G; \widehat{RG}^{\varphi}) = 0$ for $i \leqslant k$.

3. Passing vanishing of Novikov homology to a division-closed subring

In this section we will show that vanishing of Novikov homology up to a given degree can be determined by computing homology with coefficients being the divison closure of RG in $\widehat{RG}^{\varphi}$. First, we show that the division and rational closures are equal.

Proposition 3.1. *Let R be a ring and let G be a finitely generated group. If $\varphi \in S(G)$, then $\mathcal{D}(RG \subset \widehat{RG}^{\varphi}) = \mathcal{R}(RG \subset \widehat{RG}^{\varphi})$.*

Proof. Let $\mathcal{D} = \mathcal{D}(RG \subset \widehat{RG}^{\varphi})$ and $\mathcal{R} = \mathcal{R}(RG \subset \widehat{RG}^{\varphi})$; clearly, $\mathcal{D} \subseteq \mathcal{R}$. Consider a square matrix $A \in \mathbf{M}_n(\mathcal{D})$ that is invertible over $\widehat{RG}^{\varphi}$. Let $B \in \mathbf{M}_n(\widehat{RG}^{\varphi})$ be such that $AB = I$ where I is the identity matrix. We need to show that A is invertible over $\mathcal{D}$, which will show that $\mathcal{D}$ is rationally closed, and hence that $\mathcal{D} = \mathcal{R}$.

Since A is in particular a finite matrix over $\widehat{RG}^{\varphi}$, there exists $r \in \mathbb{R}$ such that all entries of A are supported on $\phi^{-1}\big((-r, \infty)\big)$. We truncate the entries of B at r, and obtain a matrix $\bar{B} \in \mathbf{M}_n(RG)$. Now, $A\bar{B} = I - P$ where the elements of the supports of the entries of P are all positive with respect to φ. Indeed, let $Q = B - \bar{B}$. Then we have

$$I = AB = A(Q + \bar{B}) = AQ + A\bar{B}.$$

In particular, $A\bar{B} = I - AQ = I - P$. Moreover, since $A\bar{B} \in \mathbf{M}_n(\mathcal{D})$ we have $P \in \mathbf{M}_n(\mathcal{D})$.

Claim 3.2. *$I - P$ is invertible over $\mathcal{D}$.*

Proof of Claim 3.2. The first diagonal entry of $I - P$ is of the form $1 - u \in \mathcal{D}$, where every element in the support of u has positive value under φ. In particular, $1 - u$ is invertible in $\widehat{RG}^{\varphi}$, and hence in $\mathcal{D}$. It follows that we may multiply $I - P$ with a diagonal matrix D over $\mathcal{D}$, that is invertible over $\mathcal{D}$, to obtain $I - P'$ such that the first diagonal entry of P' is 0 and the remaining entries are either zero or have strictly positive support with respect to φ, in the same sense as P did.

The entries of $I - P'$ all lie in $\mathcal{D}$. We may now use elementary matrices over $\mathcal{D}$ to clear the non-diagonal entries in the first row and column of $I - P'$. Repeating the process finitely many times for each diagonal entry of $I - P$ constructs a matrix $(I - P)^{-1} \in \mathbf{M}_n(\mathcal{D})$. ■

To finish the proof we observe that $A\bar{B}(I-P)^{-1} = I$ and that $\bar{B}(I-P)^{-1} \in \mathbf{M}_n(\mathcal{D})$ since $\bar{B} \in \mathbf{M}_n(RG)$ and $(I - P)^{-1} \in \mathbf{M}_n(\mathcal{D})$ by Claim 3.2. □

Proposition 3.3. *Let R be a ring, let G be a finitely generated group, and let $\varphi \in S(G)$. Let*

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

be a chain complex of based free $\widehat{RG}^{\varphi}$-modules with C_n and C_{n-1} finitely generated and ∂_i defined over $\mathcal{D} := \mathcal{D}(RG \subset \widehat{RG}^{\varphi})$. Suppose that we are given $\widehat{RG}^{\varphi}$-module homomorphisms $H_{n-1}\colon C_{n-1} \to C_n$ defined over $\mathcal{D}$ and $H_n\colon C_n \to C_{n+1}$ defined over $\widehat{RG}^{\varphi}$ satisfying $H_{n-1}\partial_n + \partial_{n+1}H_n = \mathrm{id}_{C_n}$. Then, we may take H_n to be defined over $\mathcal{D}$.

Proof. We have the following diagram (which does not in general commute since the vertical map is equal to the sum of the two diagonal composites with codomain C_n):

$$\begin{tikzcd} C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "H_n"'] \arrow[d, "\mathrm{id}_{C_n}"] & C_{n-1} \arrow[dl, "H_{n-1}"] \\ C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1}. \end{tikzcd}$$

We have

$$\partial_n = \partial_n(\partial_{n+1}H_n + H_{n-1}\partial_n) = \partial_n H_{n-1}\partial_n$$

and similarly $\partial_{n+1} = \partial_{n+1}H_n\partial_{n+1}$. Using these two equalities we obtain

$$\begin{aligned} H_{n-1}\partial_n + \partial_{n+1}H_n &= \mathrm{id}_{C_n} \\ &= \mathrm{id}_{C_n} \circ \mathrm{id}_{C_n} \\ &= (H_{n-1}\partial_n + \partial_{n+1}H_n)^2 \\ &= H_{n-1}\partial_n + \partial_{n+1}H_n + \partial_{n+1}H_nH_{n-1}\partial_n \end{aligned}$$

and so $\partial_{n+1}H_nH_{n-1}\partial_n = 0$.

Since the modules are based, H_n is naturally a matrix over $\widehat{RG}^{\varphi}$ with potentially infinitely many rows, finitely many columns, and finitely many nonzero entries. We will truncate the entries to obtain a matrix $\bar{H}_n$ over RG, and then construct a matrix A such that $\partial_{n+1}\bar{H}_nA + H_{n-1}\partial_n = \mathrm{id}_{C_n}$ and $\bar{H}_nA$ is defined over

$$\mathcal{R} := \mathcal{R}(RG \subset \widehat{RG}^{\varphi}).$$

From here we may apply Proposition 3.1. Of course nothing in life is that straight-forward so we spell out the details.

Claim 3.4. *We have $C_n = \ker \partial_n \oplus \operatorname{im} H_{n-1}\partial_n$.*

Proof of Claim 3.4. Consider $w \in \ker \partial_n \cap \operatorname{im} H_{n-1}\partial_n$. We have

$$w = (H_{n-1}\partial_n + \partial_{n+1}H_n)w = \partial_{n+1}H_nw = 0,$$

where the second equality follows from $\partial_n w = 0$, and the third from

$$\partial_{n+1}H_nH_{n-1}\partial_n = 0$$

and $w \in \operatorname{im} H_{n-1}\partial_n$. This shows that the sum in the claim is direct.

Note that by the equation $\mathrm{id}_{C_n} = \partial_{n+1}H_n + H_{n-1}\partial_n$, it suffices to argue that $\operatorname{im} \partial_{n+1}H_n = \ker \partial_n$. We have $\operatorname{im} \partial_{n+1}H_n \leqslant \ker \partial_n$. For the other inclusion let $v \in \ker \partial_n$. Then,

$$v = \partial_{n+1}H_nv + H_{n-1}\partial_nv = \partial_{n+1}H_nv,$$

which completes the proof of the claim. ∎

Note that H_n, when viewed as a matrix, has finitely many columns, and only finitely many non-zero entries (even though the number of rows might be infinite). We truncate the entries of H_n at a very large real number r with respect to φ to obtain a matrix $\bar{H}_n$ over RG. We set

$$P = \partial_{n+1}(H_n - \bar{H}_n).$$

By choosing r to be sufficiently large, we arrange for the minimum of the support of every entry of P with respect to φ to be strictly greater than 0.

Claim 3.5. *We have $P(C_n) \leqslant \ker \partial_n$.*

Proof of Claim 3.5. First, suppose that $v \in \ker \partial_n$. Then,

$$v = H_{n-1}\partial_n v + \partial_{n+1} H_n v = \partial_{n+1} H_n v = Pv + \partial_{n+1}\bar{H}_n v.$$

So, $Pv = v - \partial_{n+1}\bar{H}_n v$ where $\partial_{n+1}\bar{H}_n v \in \operatorname{im} \partial_{n+1} \leqslant \ker \partial_n$. In particular, $Pv \in \ker \partial_n$.

Now, suppose that $v \in \operatorname{im} H_{n-1}\partial_n$. Then,

$$Pv = \partial_{n+1} H_n v - \partial_{n+1}\bar{H}_n v = -\partial_{n+1}\bar{H}_n v$$

which is contained in $\operatorname{im} \partial_{n+1} \leqslant \ker \partial_n$. We are now done by applying the direct sum decomposition of Claim 3.4. ■

Write $I := \mathrm{id}_{C_n}$. The map $I - P\colon C_n \to C_n$ is invertible over $\widehat{RG}^{\varphi}$ with inverse $(I-P)^{-1} = I + \sum_{i=1}^{\infty} P^i$. Moreover, we have

$$H_{n-1}\partial_n (I-P)^{-1} = H_{n-1}\partial_n, \tag{3.5a}$$

because $H_{n-1}\partial_n P^i = 0$ for $i \geqslant 1$ by Claim 3.5. We have

$$I - P = \partial_{n+1}\bar{H}_n + H_{n-1}\partial_n,$$

the right hand side of which is defined over $\mathcal{D}$. We apply Proposition 3.1 to see that $I - P$ is invertible over $\mathcal{D}$. Finally,

$$I = (\partial_{n+1}\bar{H}_n + H_{n-1}\partial_n)(I-P)^{-1} = \partial_{n+1}\bar{H}_n(I-P)^{-1} + H_{n-1}\partial_n,$$

where the second equality uses (3.5a). Thus, we may take our new H_n to be $\bar{H}_n(I-P)^{-1}$ which is defined over $\mathcal{D}$. □

We are now ready to prove the main result of this section.

Proposition 3.6. *Let G be a finitely generated group and let $\varphi \in S(G)$. Let $C_\bullet$ be a chain complex of free RG-modules finitely generated up to dimension n, with $C_k = 0$ for $k < 0$. Then,*

$$H_j(C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}) = 0 \textit{ for } j \leqslant n$$

if and only if

$$H_j\big(C_\bullet \otimes_{RG} \mathcal{D}(RG \subset \widehat{RG}^{\varphi})\big) = 0 \textit{ for } j \leqslant n.$$

Proof. First, suppose that $H_j(C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}) = 0$ for $j \leqslant n$. Let

$$H_i\colon C_i \otimes_{RG} \widehat{RG}^{\varphi} \to C_{i+1} \otimes_{RG} \widehat{RG}^{\varphi}$$

be a chain homotopy over $\widehat{RG}^{\varphi}$ between 0 and $\mathrm{id}_{C_\bullet}$ for $i \leqslant n$.

Let $\mathcal{D} := \mathcal{D}(RG \subset \widehat{RG}^{\varphi})$. We now inductively apply Proposition 3.3 to H and parts of the resolution $C_\bullet$, with the base case clearly given by the chain complex $C_0 \to 0 \to 0$, to obtain a chain homotopy between 0 and $\mathrm{id}_{C_\bullet}$ defined over $\mathcal{D}$ for $i \leqslant n$.

Consider the chain complex $C_\bullet \otimes_{RG} \mathcal{D}$. We have that

$$C_\bullet \otimes_{RG} \mathcal{D} \subset C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}$$

and the larger chain complex admits a chain homotopy between 0 and id_{C_n} in degrees less than or equal to n defined over $\mathcal{D}$. In particular, $H_j(C_\bullet \otimes_{RG} \mathcal{D}) = 0$ for $j \leqslant n$.

The converse follows easily by arguing with chain contractions. □

4. Approximate Ore condition

Throughout this section, G denotes a group endowed with an epimorphism $\phi\colon G \to \mathbb{Z}$, with $K = \ker \phi$ and $t \in G$ such that $\phi(t)$ generates $\operatorname{im} \phi$. Various polynomials and power series with t are always twisted, and the action of t is always the conjugation action inside $\mathbb{Z}G$, $\mathcal{N}G$, and $\mathcal{U}G$, depending on context.

Definition 4.1. Given a non-zero $x = \sum_{i=k}^{\infty} t^i x_i \in \mathcal{U}K[\![t^{\pm 1}]\!]$ with $x_i \in \mathcal{U}K$ for all i, and with $x_k \neq 0$, we define its *initial term* $\operatorname{init} x$ to be $t^k x_k$. Note that k might be negative. We will refer to x_k as the *pure part* of $\operatorname{init} x$, and to k as the *associated power*. We also set $\operatorname{init} 0 = 0$, with pure part 0 and associated power 0.

The *nullity* $\operatorname{null} x$ is defined to be the $\mathcal{N}K$-dimension of the kernel of the pure part of $\operatorname{init} x$.

The kernel of an affiliated operator in $\mathcal{U}K$ is always a closed subspace of $\ell^2 K$, since the operator is closed, and hence it makes sense to talk about the $\mathcal{N}K$-dimension of the kernel.

The definitions in particular apply to Laurent polynomials in $\mathcal{U}K[t^{\pm 1}]$ and in $\mathcal{N}K[t^{\pm 1}]$.

Definition 4.2. We say that a sequence $(q_n)_n$ in $\mathcal{U}K[\![t^{\pm 1}]\!]$ is *admissible* if the sequence of powers associated to $\operatorname{init} q_n$ is bounded from below.

The sequence is *asymptotically injective* if it is admissible and

$$\sum_n \operatorname{null} q_n < \infty.$$

Remark 4.3. It is not hard to see that a term-wise product of asymptotically injective sequences is also asymptotically injective.

Lemma 4.4. *For every $x \in \mathcal{U}K[t^{\pm 1}]$ we have*

$$\dim_{\mathcal{N}G} \ker x \leqslant \operatorname{null} x.$$

Proof. The result is clear when $x = 0$. Let us assume it is not.

Without loss of generality we may assume that the power associated to $\operatorname{init} x$ is 0. Since $\mathcal{U}K$ is the Ore localisation of $\mathcal{N}K$, there exist an injective operator $z \in \mathcal{N}K$ such that $zx \in \mathcal{N}G$ and $\ker zx = \ker x$. Let $V = \ker x$. The polar decomposition gives us a partial isometry $v \in \mathcal{N}K$ such that $\ker v = \ker \operatorname{init} zx$ and $\operatorname{im} v = \overline{\operatorname{im} \operatorname{init} zx}$. Taking $u = 1 - v$ we get a partial isometry u with $\ker u = (\ker \operatorname{init} zx)^\perp$ and $\operatorname{im} u = (\operatorname{im} \operatorname{init} zx)^\perp$. Then

$$\operatorname{init}(u + zx) = u + \operatorname{init} zx$$

is injective, and therefore $u + zx$ is injective by Theorem 2.3. This means that $u|_V$ is injective as well, and hence $\dim_{\mathcal{N}G} V \leqslant \dim_{\mathcal{N}G} \operatorname{im} u$. Now, the latter dimension is equal to the $\mathcal{N}G$-trace of uu^*. Since $uu^* \in \mathcal{N}K$, this is equal to the $\mathcal{N}K$-trace, and hence to $\dim_{\mathcal{N}K} \operatorname{im} u = \operatorname{null} zx = \operatorname{null} x$. $\square$

Remark 4.5. The proof above also shows that if $x \in \mathcal{U}K$ then $\dim_{\mathcal{N}G} \ker x = \dim_{\mathcal{N}K} \ker x$.

4.A. **Approximate Ore condition.** We now introduce the main technical tool of this section.

Proposition 4.6 (Approximate Ore condition)**.** *For every $q, q' \in \mathcal{N}K[t^{\pm 1}]$ and every $\epsilon > 0$ there exist $r, r' \in \mathcal{N}K[t^{\pm 1}]$ such that* $\operatorname{null} r < \operatorname{null} q' + \epsilon$ *and* $\operatorname{null} r' < \operatorname{null} q + \operatorname{null} q' + \epsilon$, *and*

$$qr = q'r'.$$

Proof. The proof is inspired by Tamari's argument [Tam57].

It is clear that we may assume that the powers associated to $\operatorname{init} q$ and $\operatorname{init} q'$ are both 0. Let N denote the maximum of the degrees of q and q'. We write

$$q = \sum_{i=0}^{N} t^i q_i, \qquad q' = \sum_{i=0}^{N} t^i q_i'$$

with $q_i, q_i' \in \mathcal{N}K$. For a natural number k, consider the right $\mathcal{N}K$-linear map

$$\lambda_k \colon \mathcal{N}K^{2k} \to \mathcal{N}K^{k+N}$$
$$(x_0, y_0, \ldots, x_{k-1}, y_{k-1}) \mapsto \left(\sum_{i+j=l} t^{-j}(q_i t^j x_j - q_i' t^j y_j) \right)_l .$$

(Secretly, we think of $(x_0, y_0, \ldots, x_{k-1}, y_{k-1})$ as representing two twisted polynomials $r = \sum_{i=0}^{k-1} t^i x_i$ and $r' = \sum_{i=0}^{k-1} t^i y_i$, and lying in $\ker \lambda_k$ translates directly into $qr = q'r'$, since the right-hand side above collects the terms of $qr - q'r'$ according to the power of t.)

Let $d_k = \dim_{\mathcal{N}K} \ker \lambda_k$, and note that $d_k \geqslant k - N$. We may embed $\mathcal{N}K^{2k} \to \mathcal{N}K^{2+2k}$ and $\mathcal{N}K^{k+N} \to \mathcal{N}K^{1+k+N}$ by augmenting vectors with zeroes in initial positions. These embeddings form commutative squares with the maps λ_k and λ_{k+1}. The image of $\ker \lambda_k$ under the first map will be denoted by $t \ker \lambda_k$; it is a subspace of $\ker \lambda_{k+1}$ of dimension d_k.

Let $p, p' \colon \mathcal{N}K^{2k} \to \mathcal{N}K$ denote the projections onto, respectively, the first and the second factor.

Claim 4.7. *For some k we have*

$$\dim_{\mathcal{N}K} p(\ker \lambda_k) > 1 - \operatorname{null} q' - \epsilon.$$

Proof of claim. Without loss of generality, we will assume that $\epsilon < 1$.

Consider $(x_0, y_0, \ldots, x_{k-1}, y_{k-1}) \in \ker p|_{\ker \lambda_k}$. We immediately see that $x_0 = 0$, and $q_0' y_0 = 0$. Hence p' sends $\ker p|_{\ker \lambda_k}$ to $\ker q_0'$. Therefore,

$$\dim_{\mathcal{N}K} \left(\ker p|_{\ker \lambda_k}\right) - \operatorname{null} q' \leqslant \dim_{\mathcal{N}K} \left(\ker p|_{\ker \lambda_k} \cap \ker p'|_{\ker \lambda_k}\right).$$

The intersection of the kernels on the right-hand side is precisely $t \ker \lambda_{k-1}$. Hence,

$$d_k - \dim_{\mathcal{N}K} p(\ker \lambda_k) - \operatorname{null} q' \leqslant d_{k-1}.$$

Rearranging, we obtain

$$\dim_{\mathcal{N}K} p(\ker \lambda_k) \geqslant d_k - d_{k-1} - \operatorname{null} q'.$$

If the left-hand side is bounded above by $1 - \operatorname{null} q' - \epsilon$ for all k, then

$$1 - \epsilon \geqslant d_k - d_{k-1}$$

and so adding such terms together gives

$$(1-\epsilon)k \geqslant d_k \geqslant k - N$$

for all k, which is a contradiction. We conclude that for some k the $\mathcal{N}K$-dimension of $p(\ker \lambda_k)$ is greater than $1 - \operatorname{null} q' - \epsilon$, as claimed. ∎

It follows that for k as above, $p(\ker \lambda_k)$ is dense inside of $\pi_V \mathcal{N}K$ for some closed subspace $V \leqslant \ell^2(K)$ of $\mathcal{N}K$-dimension greater than $1 - \operatorname{null} q' - \epsilon$. Take $x \in \ker \lambda_k$ such that $\pi_V - p(x)$ has norm less than 1. Then

$$V \cap \ker p(x) = \{0\},$$

and so $\ker p(x)$ has dimension less than $\operatorname{null} q' + \epsilon$. Write

$$x = (x_0, y_0, \ldots, x_{k-1}, y_{k-1});$$

let $r = \sum_{i=0}^{k-1} t^i x_i$ and $r' = \sum_{i=0}^{k-1} t^i y_i$. We have shown that $\ker p(x) = \ker x_0$ has dimension less than $\operatorname{null} q' + \epsilon$. Also, $x \in \ker \lambda_k$ means precisely that

$$qr = q'r'.$$

Finally, the last equality implies that $q_0 \operatorname{init} r = q_0' \operatorname{init} r'$, and hence

$$\operatorname{null} r' \leqslant \operatorname{null} q + \operatorname{null} r < \operatorname{null} q + \operatorname{null} q' + \epsilon.$$ □

Remark 4.8. Clearly, if q and q' are Laurent polynomials over $\mathcal{U}K$, we may multiply them by a suitable injective operator in $\mathcal{N}K$ and obtain Laurent polynomials over $\mathcal{N}K$ with the same nullities. Hence the conclusion of the above statement holds verbatim if q and q' lie in $\mathcal{U}K[t^{\pm 1}]$, and we will use it in this generality.

Corollary 4.9. *Every two asymptotically injective sequences $(q_n)_n$ and $(q_n')_n$ over $\mathcal{U}K[t^{\pm 1}]$ admit an* asymptotically injective common multiple*, that is, an asymptotically injective sequence $(x_n)_n$ over $\mathcal{N}K[t^{\pm 1}]$ such that there exist two asymptotically injective sequences $(y_n)_n$ and $(z_n)_n$ over $\mathcal{N}K[t^{\pm 1}]$ with $x_n = q_n y_n = q_n' z_n$ for every n.*

Proof. For every n we obtain x_n, y_n, and z_n with $x_n = q_n y_n = q_n' z_n$ from Proposition 4.6, setting $\epsilon = 2^{-n}$. This way

$$\sum_n \operatorname{null} x_n \leqslant \sum_n (\operatorname{null} q_n + \operatorname{null} y_n) \leqslant \sum_n \left(\operatorname{null} q_n + \operatorname{null} q_n' + 2^{-n}\right) < \infty$$

and similarly for $(y_n)_n$ and $(z_n)_n$. □

4.B. **Asymptotic agreement.** In this section we are dealing with sequences, but in reality we think of them as proxies, and we are really interested in their limits (which we will define later). Hence it is natural to introduce an equivalence relation on sequences.

Definition 4.10. We say that two sequences $(x_n)_n$ and $(y_n)_n$ in $\mathcal{U}G$ *asymptotically agree as operators*, written $(x_n)_n \approx (y_n)_n$, if

$$\sum_n \dim_{\mathcal{N}G}(\ker(x_n - y_n)^{\perp}) < \infty.$$

A sequence $(x_n)_n$ in $\mathcal{U}G$ *stabilises* if $(x_n)_n \approx (x_{n+1})_n$.

Note that if $(x_n)_n$ and $(y_n)_n$ are sequences in $\mathcal{U}K$, then

$$\dim_{\mathcal{N}G} \ker(x_n - y_n) = \dim_{\mathcal{N}K} \ker(x_n - y_n)$$

by Remark 4.5, and hence such sequences asymptotically agree as sequences in $\mathcal{U}K$ if and only if they asymptotically agree as sequences in $\mathcal{U}G$. Therefore, there is no need to specify over which group we are working.

It is very easy to see that being in asymptotic agreement is an equivalence relation.

Lemma 4.11. *Let $(x_n)_n$, $(y_n)_n$, and $(z_n)_n$ be sequences in $\mathcal{U}G$. If $(x_n)_n \approx (y_n)_n$, then all of the following hold:*

(1) $(x_n + z_n)_n \approx (y_n + z_n)_n$,
(2) $(x_n z_n)_n \approx (y_n z_n)_n$,
(3) $(z_n x_n)_n \approx (z_n y_n)_n$.

Proof. This is immediate. For (2) and (3), it is enough to observe that the dimension of the kernel of a product of operators is bounded from below by the dimension of the kernel of either factor. □

We now extend the relation $\approx$ to power series.

Definition 4.12. Two sequences $(x_n)_n$ and $(y_n)_n$ over $\mathcal{U}K[\![t^{\pm 1}]\!]$ *asymptotically agree as power series*, written $(x_n)_n \approx_K (y_n)_n$, if for every fixed degree d, the sequence (over $\mathcal{U}K$) of coefficients of x_n by t^d and the sequence of coefficients of y_n by t^d asymptotically agree as operators.

A sequence $(x_n)_n$ over $\mathcal{U}K[\![t^{\pm 1}]\!]$ is *K-stabilising* if $(x_n)_n \approx_K (x_{n+1})_n$.

For power series, we add K as a subscript to $\approx$ due to the potential confusion for sequences of Laurent polynomials in $\mathcal{U}K[t^{\pm 1}]$. Such Laurent polynomials are at the same time elements of $\mathcal{U}G$, in which case the definition of $\approx$ applies, and Laurent power series, in which context we use $\approx_K$. It is clear that $\approx_K$ is again an equivalence relation.

We now collect basic arithmetic properties of the equivalence relation $\approx_K$.

Lemma 4.13. *Let $(x_n)_n$, $(y_n)_n$, and $(z_n)_n$ be sequences in $\mathcal{U}K[\![t^{\pm 1}]\!]$. If $(x_n)_n \approx_K (y_n)_n$, then all of the following hold:*

(1) $(x_n + z_n)_n \approx_K (y_n + z_n)_n$,
(2) $(x_n z_n)_n \approx_K (y_n z_n)_n$,
(3) $(z_n x_n)_n \approx_K (z_n y_n)_n$.

Proof. This follows from Lemma 4.11. □

4.C. **Partial inverse.** We are now approaching the main technical onslaught. We will introduce two constructions that play the role of inverses of elements in $\mathcal{U}K[t^{\pm 1}]$, one in $\mathcal{U}G$, and one in $\mathcal{U}K[\![t^{\pm 1}]\!]$. They will share many properties.

Recall that in $\mathcal{U}G$ we have the notion of a partial inverse $x \mapsto x^\dagger$. The partial inverse of an element $x \in \mathcal{U}K$ lies in $\mathcal{U}K$, and for such an x we have $(t^i x)^\dagger = x^\dagger t^{-i}$.

Lemma 4.14. *Let $(p_n)_n$ be a sequence in $\mathcal{U}K[t^{\pm 1}]$, and let $(s_n)_n$ and $(q_n)_n$ be asymptotically injective sequences in $\mathcal{U}K[t^{\pm 1}]$. All of the following hold:*

(1) $\left((q_n s_n)^\dagger\right)_n \approx \left({s_n}^\dagger {q_n}^\dagger\right)_n$.

(2) $(s_n {s_n}^\dagger)_n \approx (1)_n$.
(3) $(p_n {q_n}^\dagger)_n \approx (p_n s_n (q_n s_n)^\dagger)_n$.

Proof. We prove the items in turn.

(1) We have
$$(q_n s_n)(q_n s_n)^\dagger - q_n s_n {s_n}^\dagger {q_n}^\dagger = \pi_{\overline{\operatorname{im} q_n s_n}} - q_n \pi_{\overline{\operatorname{im} s_n}} {q_n}^\dagger.$$
The right-hand side is 0 on $(\operatorname{im} q_n)^\perp$, since both summands are 0 there, and on $q_n(\operatorname{im} s_n \cap (\ker q_n)^\perp)$, since the summands restrict to the identity there. Lemma 4.4 tells us that $\dim_{\mathcal{N}G} \ker s_n \leqslant \operatorname{null} s_n$. Hence
$$\dim_{\mathcal{N}G} \ker((q_n s_n)(q_n s_n)^\dagger - q_n s_n {s_n}^\dagger {q_n}^\dagger) \geqslant 1 - \operatorname{null} s_n.$$
Since, using the same argument as above,
$$\dim_{\mathcal{N}G} \ker q_n s_n \leqslant \operatorname{null} q_n + \operatorname{null} s_n,$$
we conclude that
$$\dim_{\mathcal{N}G} \ker((q_n s_n)^\dagger - {s_n}^\dagger {q_n}^\dagger) \geqslant 1 - 2 \operatorname{null} s_n - \operatorname{null} q_n.$$
Thus,
$$\sum_n \dim_{\mathcal{N}G} \left(\ker((q_n s_n)^\dagger - {s_n}^\dagger {q_n}^\dagger)^\perp \right) \leqslant \sum_n (2 \operatorname{null} s_n + \operatorname{null} q_n) < \infty.$$

(2) The operator
$$1 - s_n {s_n}^\dagger = 1 - \pi_{\overline{\operatorname{im} s_n}}$$
has kernel of dimension at least $1 - \operatorname{null} s_n$, and we finish the argument as above.

(3) This follows immediately from the two items above and Lemma 4.11(2) and (3). ☐

Lemma 4.15. *Let $(x_n)_n$ and $(y_n)_n$ be sequences in $\mathcal{U}G$. If $(x_n)_n$ and $(y_n)_n$ asymptotically agree as operators, then so do the sequences of adjoints $({x_n}^*)_n$ and $({y_n}^*)_n$, and the sequences of partial inverses $({x_n}^\dagger)_n$ and $({y_n}^\dagger)_n$.*

Proof. This is clear for adjoints, since $\ker({x_n}^* - {y_n}^*) = (\operatorname{im}(x_n - y_n))^\perp$ has the same $\mathcal{N}G$-dimension as $\ker(x_n - y_n)$.

For partial inverses, we need to introduce some notation. Let
$$1 - d_n = \dim_{\mathcal{N}G} \ker(x_n - y_n),$$
and note that $\sum_n d_n < \infty$.

The subspace $\ker({x_n}^\dagger - {y_n}^\dagger)$ contains $(\operatorname{im} x_n)^\perp \cap (\operatorname{im} y_n)^\perp$, since this is the intersection of the kernels of ${x_n}^\dagger$ and ${y_n}^\dagger$, and it contains
$$\overline{x_n(\ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp)},$$
since this is the closure of a subspace on which ${x_n}^\dagger$ and ${y_n}^\dagger$ act as the identity – here, we are using the fact that the action of x_n and y_n are the same on $\ker(x_n - y_n)$. Observe that $(\operatorname{im} x_n)^\perp \cap (\operatorname{im} y_n)^\perp$ and
$$\overline{x_n(\ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp)}$$
are perpendicular. We will now bound the dimensions of these two spaces from below.

We have

$$\operatorname{im} x_n \geqslant x_n(\ker(x_n - y_n)) = y_n(\ker(x_n - y_n)) \leqslant \operatorname{im} y_n.$$

Moreover, the definition of d_n tells us that the $\mathcal{N}G$-dimension of the perpendicular complement of $x_n(\ker(x_n - y_n))$ in $\overline{\operatorname{im} x_n}$ is bounded above by d_n, and so is the complement in $\overline{\operatorname{im} y_n}$. We conclude that

$$\begin{aligned}\dim_{\mathcal{N}G}\left((\operatorname{im} x_n)^\perp \cap (\operatorname{im} y_n)^\perp\right) &\geqslant \dim_{\mathcal{N}G}\left(x_n(\ker(x_n - y_n))\right)^\perp - 2d_n \\ &\geqslant \dim_{\mathcal{N}G}(\operatorname{im} x_n)^\perp - 3d_n.\end{aligned}$$

We will now focus on $x_n(\ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp)$. We have

$$(\ker x_n)^\perp \cap (\ker y_n)^\perp = \operatorname{im} {x_n}^* \cap \operatorname{im} {y_n}^* \geqslant {x_n}^*(\ker({x_n}^* - {y_n}^*)).$$

The $\mathcal{N}G$-codimension of this last subspace in $\operatorname{im} {x_n}^* = (\ker x_n)^\perp$ is bounded from above by d_n, and therefore

$$\begin{aligned}&\dim_{\mathcal{N}G} \overline{x_n(\ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp)} \\ &\geqslant \dim_{\mathcal{N}G} \overline{x_n(\ker(x_n - y_n) \cap (\ker x_n)^\perp)} - d_n \\ &\geqslant \dim_{\mathcal{N}G} \overline{x_n((\ker x_n)^\perp)} - 2d_n \\ &= \dim_{\mathcal{N}G} \overline{\operatorname{im} x_n} - 2d_n.\end{aligned}$$

Combining the last two inequalities gives $\dim_{\mathcal{N}G} \ker({x_n}^\dagger - {y_n}^\dagger) \geqslant 1 - 5d_n$, and hence $\dim_{\mathcal{N}G} \ker(x_n^\dagger - y_n^\dagger)^\perp \leqslant 5d_n$, and the result follows. □

In particular, if $(x_n)_n$ stabilises, then so do $({x_n}^*)_n$ and $({x_n}^\dagger)_n$.

We will now introduce the first limit – it will later allow us to view (equivalence classes of) our sequences as elements of $\mathcal{U}G$.

Lemma 4.16. *For every stabilising sequence $(x_n)_n$ over $\mathcal{U}G$, there exists a unique element $x_\infty \in \mathcal{U}G$ such that $(x_n)_n$ and $(x_\infty)_n$ asymptotically agree.*

Proof. Let $V_n = \ker(x_{n+1} - x_n)$. By assumption, $\sum_n (1 - \dim_{\mathcal{N}G} V_n) < \infty$. Let $U_n = \overline{\sum_{m \geqslant n} {V_m}^\perp}$. The subspaces U_n are closed and G-invariant, and form a nested sequence with $\lim_n \dim_{\mathcal{N}G} U_n = 0$. Therefore $\bigcap_n U_n = \{0\}$.

Observe that the affiliated operators x_m with $m \geqslant n$ all agree on L_n, where L_n is defined to be ${U_n}^\perp$ intersected with all of their domains. In fact, L_n is equal to ${U_n}^\perp$ intersected with the domain of x_n, since the domains are essentially dense and therefore their intersections with ${U_n}^\perp$ are dense therein, and since the operators are closed. Now, the subspaces L_n are G-invariant, form an ascending chain, and $\overline{L} = \ell^2 G$ with $L = \bigcup L_n$. We define $x'_\infty \colon L \to \ell^2 G$ by $x'_\infty|_{L_n} = x_n|_{L_n}$. It is clear that x'_∞ is densely defined and G-equivariant.

We now apply the same procedure to the stabilising sequence $({x_n}^*)_n$, and obtain a densely defined G-invariant operator x^*_∞. It is easy to see that x^*_∞ is the adjoint of x'_∞. Since x'_∞ is densely defined, the adjoint $x_\infty = (x^*_\infty)^*$ is defined on a superspace of L, and on L agrees with x'_∞. Moreover, since x^*_∞ is densely defined, x_∞ is closed. Hence x_∞ is the desired affiliated operator.

To prove uniqueness, suppose that we have another affiliated operator y_∞ such that$(x_n)_n$ and $(y_\infty)_n$ asymptotically agree. Then the sequences $(x_\infty)_n$ and $(y_\infty)_n$ asymptotically agree, forcing $\ker(x_\infty - y_\infty) = \ell^2 G$. This means that $x_\infty = y_\infty$ as affiliated operators. □

We will refer to the element x_∞ as the *limit* of the sequence $(x_n)_n$. The map $(x_n)_n \mapsto x_\infty$ will be denoted by $\lambda_{\mathcal{U}G}$. Note that if $x_n \in \mathcal{U}K$ for every n, then $x_\infty \in \mathcal{U}K$ as well.

4.D. **Expansion.** We are now ready to construct the second function that will serve as an inverse, this time in $\mathcal{U}K[\![t^{\pm 1}]\!]$.

Definition 4.17 (Expansion)**.** We define the *expansion map*

$$\mathcal{U}K[\![t^{\pm 1}]\!] \smallsetminus \{0\} \to \mathcal{U}K[\![t^{\pm 1}]\!]$$

by

$$q \mapsto \overline{q} = (\operatorname{init} q)^\dagger \sum_{k=0}^{\infty} \left((\operatorname{init} q - q)(\operatorname{init} q)^\dagger\right)^k.$$

In the unlikely event of the reader not recognising the construction immediately, it is instructive to consider the case in which $\operatorname{init} q$ is invertible in $\mathcal{U}K$.

Lemma 4.18. *For $q \in \mathcal{U}K[\![t^{\pm 1}]\!]$, suppose that* $\operatorname{init} q$ *is invertible in $\mathcal{U}K$. Then $\overline{q}$ is precisely the inverse of q in $\mathcal{U}K[\![t^{\pm 1}]\!]$.*

Proof. Observe that $(\operatorname{init} q)^\dagger$ is the inverse of $\operatorname{init} q$. Let $r = q(\operatorname{init} q)^\dagger$, and observe that we have $r = 1 + \sum_{i>0} t^i r_i$ with $r_i \in \mathcal{U}K$. Hence

$$\overline{r} = \sum_{k\geqslant 0}(1-r)^k$$

is the inverse of r. Thus, $(\operatorname{init} q)^\dagger \sum_{k\geqslant 0}(1-r)^k$ is the inverse of q. But

$$(\operatorname{init} q)^\dagger \sum_{k\geqslant 0}(1-r)^k = (\operatorname{init} q)^\dagger \sum_{k\geqslant 0}(\operatorname{init} q(\operatorname{init} q)^\dagger - q(\operatorname{init} q)^\dagger)^k = \overline{q}. \qquad \square$$

Lemma 4.19. *Let $(q_n)_n$ and $(r_n)_n$ be asymptotically injective sequences over $\mathcal{U}K[\![t^{\pm 1}]\!]$. All of the following hold:*

1. *If $(q_n)_n \approx_K (r_n)_n$ then $(\overline{q_n})_n \approx_K (\overline{r_n})_n$;*
2. *$(\overline{\overline{q_n}})_n \approx_K (q_n)_n$;*
3. *$(q_n\overline{q_n})_n \approx_K (\overline{q_n}q_n)_n \approx_K (1)_n$;*
4. *$(\overline{q_n r_n})_n \approx_K (\overline{r_n} \cdot \overline{q_n})_n$.*

Proof. It is easy to see what happens when we multiply the terms of our sequences by powers of t, and hence we will assume that $\operatorname{init} q_n$ and $\operatorname{init} r_n$ lie in $\mathcal{U}K$ for all n.

1. For every fixed d, the terms of $\overline{q_n}$ and $\overline{r_n}$ appearing next to t^d are obtained from finitely many corresponding terms in q_n and r_n via the same arithmetic operation. We now apply Lemma 4.11 for every degree d separately.
2. Recall that the polar decomposition gives us partial isometries u_n and v_n in $\mathcal{N}K$ mapping $\ker \operatorname{init} q_n$ onto $(\operatorname{im} \operatorname{init} q_n)^\perp$ and $\ker \operatorname{init} r_n$ onto $(\operatorname{im} \operatorname{init} r_n)^\perp$ and being trivial on $(\ker \operatorname{init} q_n)^\perp$ and $(\ker \operatorname{init} r_n)^\perp$, respectively. The operators $\operatorname{init} q_n + u_n$ and $\operatorname{init} r_n + v_n$ are then invertible in $\mathcal{U}K$. Crucially, $(u_n)_n \approx_K (0)_n \approx_K (v_n)_n$, since $(q_n)_n$ and (r_n) are asymptotically injective. Lemma 4.13 tells us that
$$(q_n + u_n)_n \approx_K (q_n)_n \quad \text{and} \quad (r_n + v_n)_n \approx_K (r_n)_n.$$

Now, Lemma 4.18 yields $\overline{\overline{q_n + u_n}} = q_n + u_n$ for all n, and combining this with the previous item finishes this part of the proof.

(3) By Lemma 4.18, for all n we have

$$(q_n + u_n)\overline{(q_n + u_n)} = \overline{(q_n + u_n)}(q_n + u_n) = 1.$$

We are now done thanks to the first item and Lemma 4.13.

(4) Finally, by Lemma 4.18 and uniqueness of inverses we have

$$\overline{(q_n + u_n)(r_n + v_n)} = \overline{r_n + v_n} \cdot \overline{q_n + u_n}.$$

By Lemma 4.13, $(q_n r_n)_n \approx_K (q_n + u_n)(r_n + v_n)_n$, and we finish the proof using the first item. $\square$

4.E. **Weakly rational elements.**

Definition 4.20. Let $\mathrm{WRat}_0(K, t)$ be the set of sequences $(p_n, q_n)_n$ such that all of the following hold:

- $(p_n)_n$ is an admissible sequence in $\mathcal{U}K[t^{\pm 1}]$,
- $(q_n)_n$ is an asymptotically injective sequence in $\mathcal{U}K[t^{\pm 1}]$,
- the sequence $(p_n(q_n)^\dagger)_n$ stabilises,
- the sequence $(p_n\overline{q_n})_n$ K-stabilises.

We let $\sim$ be a relation on $\mathrm{WRat}_0(K, t)$ given by $(p_n, q_n)_n \sim (p'_n, q'_n)_n$ if there exist asymptotically injective sequences $(r_n)_n$ and $(s_n)_n$ in $\mathcal{U}K[t^{\pm 1}]$ such that

$$p_n r_n = p'_n s_n, \qquad q_n r_n = q'_n s_n$$

for all n.

Lemma 4.21. *The relation $\sim$ is an equivalence relation.*

Proof. Reflexivity is clear by taking $r_n = 1 = s_n$ for every n. Symmetry is also clear by exchanging the sequences $(r_n)_n$ and $(s_n)_n$. Transitivity follows from Corollary 4.9. Here are the details.

Consider three sequences $(p_n, q_n)_n$, $(p'_n, q'_n)_n$ and $(p''_n, q''_n)_n$ in $\mathrm{WRat}_0(K, t)$, and suppose that $(p_n, q_n)_n \sim (p'_n, q'_n)_n$ and $(p'_n, q'_n)_n \sim (p''_n, q''_n)_n$. The definition gives us asymptotically injective sequences $(r_n)_n$, $(s_n)_n$, $(r'_n)_n$, and $(s'_n)_n$ such that

$$\begin{aligned} p_n r_n &= p'_n s_n, & q_n r_n &= q'_n s_n, \\ p'_n r'_n &= p''_n s'_n, & q'_n r'_n &= q''_n s'_n, \end{aligned}$$

for all n. By Corollary 4.9, there exists an asymptotically injective right common multiple $(x_n)_n$ of $(s_n)_n$ and $(r'_n)_n$. Let y_n and z_n be such that $x_n = s_n y_n = r'_n z_n$. Set $r''_n = r_n y_n$ and $s''_n = s'_n z_n$, and note that $(r''_n)_n$ is asymptotically injective. Then

$$p_n r''_n = p_n r_n y_n = p'_n s_n y_n = p'_n x_n = p'_n r'_n z_n = p''_n s'_n z_n = p''_n s''_n. \qquad \square$$

Let $\mathrm{WRat}(K, t)$ denote the set of equivalence classes in $\mathrm{WRat}_0(K, t)$ under this equivalence relation. We will abuse notation by not differentiating between the elements of $\mathrm{WRat}_0(K, t)$ and the equivalence classes they lie in. The elements of $\mathrm{WRat}(K, t)$ will be called *weakly rational sequences.*

Lemma 4.22. *For every $(p_n, q_n)_n \in \mathrm{WRat}_0(K,t)$ and every asymptotically injective sequence $(x_n)_n$ in $\mathcal{U}K[t^{\pm 1}]$, the sequence $(p_n x_n, q_n x_n)_n$ lies in $\mathrm{WRat}_0(K,t)$ in the same equivalence class as $(p_n, q_n)_n$.*

Proof. The only conditions that are non-trivial to check are the stability of $(p_n x_n (q_n x_n)^\dagger)_n$ and $(p_n x_n \overline{q_n x_n})_n$. For the first sequence, this follows immediately from Lemma 4.14(3); for the second, we use Lemmata 4.13 and 4.19. □

Proposition 4.23. *The map $\iota\colon p \mapsto (p,1)_n$ embeds $\mathcal{U}K[t^{\pm 1}]$ into $\mathrm{WRat}(K,t)$, and $\mathrm{WRat}(K,t)$ admits a ring structure making this embedding into a ring homomorphism.*

Proof. We will break the proof into three parts.

Embedding. We start with the first claim. Suppose that $(p,1)_n \sim (p',1)_n$. The definition gives us an asymptotically injective sequence $(r_n)_n$ with $pr_n = p'r_n$. Let $x = \mathrm{init}(p-p')$. We then have

$$x \,\mathrm{init}\, r_n = 0$$

for all n. Since $\lim_n \mathrm{null}\, r_n = 0$, the images of the operators $\mathrm{init}\, r_n$ have closures with dimensions tending to 1. Hence $x = 0$, and so $p = p'$. ◆

Ring structure: addition. Now we need to define the ring structure. We start with addition. Let $(p_n, q_n)_n$ and $(p_n', q_n')_n$ in $\mathrm{WRat}(K,t)$ be given. We are first going to bring them to a common denominator: Corollary 4.9 gives us asymptotically injective sequences $(x_n)_n, (y_n)_n$, and $(z_n)_n$ with $x_n = q_n y_n = q_n' z_n$ for every n. Lemma 4.22 tells us that $(p_n y_n, x_n)_n$ and $(p_n' z_n, x_n)_n$ lie in $\mathrm{WRat}(K,t)$; we define their sum to be $(p_n y_n + p_n' z_n, x_n)_n$. We now need to check that this sequence lies in $\mathrm{WRat}(K,t)$, and that the sum is independent of the choices of representatives.

To check that $(p_n y_n + p_n' z_n, x_n)_n$ lies in $\mathrm{WRat}(K,t)$, we need to verify the stability conditions of the definition. This is easy, since

$$(p_n y_n + p_n' z_n) {x_n}^\dagger = p_n y_n {x_n}^\dagger + p_n' z_n {x_n}^\dagger,$$

and

$$(p_n y_n + p_n' z_n)\overline{x_n} = p_n y_n \overline{x_n} + p_n' z_n \overline{x_n}.$$

Now suppose that we have picked different sequences $(x_n')_n, (y_n')_n$, and $(z_n')_n$ with properties analogous to the ones of $(x_n)_n, (y_n)_n$, and $(z_n)_n$. Thanks to Corollary 4.9 and Lemma 4.22 we may assume that $(x_n)_n = (x_n')_n$, which in particular implies that $q_n y_n = q_n y_n'$ and $q_n' z_n = q_n' z_n'$. We now need to show that $(p_n y_n + p_n' z_n, x_n)_n \sim (p_n y_n' + p_n' z_n', x_n)_n$. Proposition 4.6 and Remark 4.8 give us sequences $(a_n)_n$ and $(b_n)_n$ in $\mathcal{N}K[t^{\pm 1}]$ such that for all n

$$(y_n - y_n') a_n = \pi_{(\ker \mathrm{init}\, q_n)^\perp} b_n,$$

and such that $(a_n)_n$ is asymptotically injective. We claim that

$$\pi_{(\ker \mathrm{init}\, q_n)^\perp} b_n = 0.$$

We have

$$0 = 0 \cdot a_n = q_n (y_n - y_n') a_n = q_n \pi_{(\ker \mathrm{init}\, q_n)^\perp} b_n.$$

Hence,

$$\operatorname{init} q_n \operatorname{init}(\pi_{(\ker \operatorname{init} q_n)^\perp} b_n) = 0.$$

But $\operatorname{init}(\pi_{(\ker \operatorname{init} q_n)^\perp} b_n)$ is of the form $\pi_{(\ker \operatorname{init} q_n)^\perp} ct^j$ for some j and $c \in \mathcal{N}K$, and $\operatorname{init} q_n$ is clearly injective on $(\ker \operatorname{init} q_n)^\perp$. Hence $\pi_{(\ker \operatorname{init} q_n)^\perp} ct^j = 0$ and the claim follows. We deduce that $(y_n - y'_n)a_n = 0$. We then have

$$(p_n y_n + p'_n z_n, x_n)_n \sim (p_n y_n a_n + p'_n z_n a_n, x_n a_n)_n$$

and $p_n y_n a_n = p_n y'_n a_n$. We repeat the argument for $z_n a_n$ and $z'_n a_n$. This shows that addition is well defined.

It is clear that addition has a neutral element $(0,1)_n$, and that

$$(p_n, q_n)_n + (-p_n, q_n)_n = (0, q_n)_n \sim (0,1)_n. \qquad \blacklozenge$$

Ring structure: multiplication. We are left with defining multiplication. Let $(p_n, q_n)_n$ and $(p'_n, q'_n)_n$ be given elements of $\operatorname{WRat}(K,t)$. Using Proposition 4.6, we get an asymptotically injective sequence $(s_n)_n$ and a sequence $(r_n)_n$ such that

$$p'_n s_n = q_n r_n$$

for every n. We define $(p_n, q_n)_n \cdot (p'_n, q'_n)_n = (p_n r_n, q'_n s_n)$. We first need to check that the output is an element of $\operatorname{WRat}(K,t)$. By Lemma 4.14(1), $\left((q'_n s_n)^\dagger\right)_n \approx ({s_n}^\dagger {q'_n}^\dagger)_n$; by (2), $({q_n}^\dagger q_n)_n \approx ({s_n}^\dagger s_n)_n \approx (1)_n$. We therefore have

$$\begin{aligned}
\left(p_n r_n (q'_n s_n)^\dagger\right)_n &\approx \left(p_n {q_n}^\dagger q_n r_n {s_n}^\dagger {q'_n}^\dagger\right)_n \\
&= \left(p_n {q_n}^\dagger p'_n s_n {s_n}^\dagger {q'_n}^\dagger\right)_n \\
&\approx \left(p_n {q_n}^\dagger p'_n {q'_n}^\dagger\right)_n .
\end{aligned}$$

Since $\left(p_n {q_n}^\dagger\right)_n$ and $\left(p'_n {q'_n}^\dagger\right)_n$ stabilise, we have

$$\begin{aligned}
\left(p_n {q_n}^\dagger p'_n {q'_n}^\dagger\right)_n &\approx \left(p_n {q_n}^\dagger p'_{n+1} {q'_{n+1}}^\dagger\right)_n \\
&\approx \left(p_{n+1} {q_{n+1}}^\dagger p'_{n+1} {q'_{n+1}}^\dagger\right)_n \\
&\approx \left(p_{n+1} r_{n+1} (q'_{n+1} s_{n+1})^\dagger\right)_n .
\end{aligned}$$

The argument for

$$\left(p_n r_n \overline{q'_n s_n}\right)_n \approx_K \left(p_{n+1} r_{n+1} \overline{q'_{n+1} s_{n+1}}\right)_n$$

is analogous.

The next step is to check that our multiplication is well defined. There are three parts to this. First, for fixed sequences $(p_n, q_n)_n$ and $(p'_n, q'_n)_n$, we could have chosen different sequences $(r_n)_n$ and $(s_n)_n$. By Corollary 4.9, it is enough to check what happens if we multiply $(s_n)_n$ and $(r_n)_n$ on the right by the same asymptotically injective sequence. But then it is clear that the output of our multiplication is in the same $\sim$-equivalence class. Second, we could have used different sequences to represent the equivalence class of $(p_n, q_n)_n$. Again, it is enough to check what happens if we multiply $(p_n)_n$ and $(q_n)_n$ by the same asymptotically injective sequence $(x_n)_n$. Using

Proposition 4.6, we get an asymptotically injective sequence $(s'_n)_n$ and a sequence $(r'_n)_n$ such that

$$p'_n s'_n = q_n x_n r'_n.$$

This is the same as choosing $(x_n r'_n)_n$ and $(s'_n)_n$ instead of $(r_n)_n$ and $(s_n)_n$, and we already know that such a change does not affect the final output. The last case, in which we multiply $(p'_n)_n$ and $(q'_n)_n$ by the same asymptotically injective sequence, is analogous.

Finally, $(1,1)_n$ is obviously a neutral element of the multiplication. We are left with distributivity. Consider three elements $(p_n,q_n)_n$, $(p'_n,q'_n)_n$, and $(p''_n,q''_n)_n$ of $\mathrm{WRat}(K,t)$. We need to verify that

$$\big((p_n,q_n)_n + (p'_n,q'_n)_n\big)\cdot(p''_n,q''_n)_n = (p_n,q_n)_n\cdot(p''_n,q''_n)_n + (p'_n,q'_n)_n\cdot(p''_n,q''_n)_n.$$

We have already checked that we may move freely within the equivalence classes, and so we may assume that the first two sequences are already brought to common denominators, that is, $q_n = q'_n$ for every n. Now let $(r_n)_n$ be a sequence, and $(s_n)_n$ an asympotically injective sequence, such that $p''_n s_n = q_n r_n$. Then the left-hand side of our equation becomes

$$\big((p_n + p'_n)r_n, q''_n s_n\big)_n$$

and the right-hand side becomes

$$\big(p_n r_n, q''_n s_n\big)_n + \big(p'_n r_n, q''_n s_n\big)_n .$$

These two expressions yield equal elements in $\mathrm{WRat}(K,t)$. ◆

As we have verified all of the claims, the proof is complete. □

Henceforth, we will always endow $\mathrm{WRat}(K,t)$ with the above ring structure.

Recall that if we are given a stabilising sequence in $\mathcal{U}G$, we have the limit map $\lambda_{\mathcal{U}G}$ described in Lemma 4.16 returning a single element in $\mathcal{U}G$. Similarly, given an admissible K-stabilising sequence in $\mathcal{U}K[\![t^{\pm1}]\!]$, we may apply the limit map over $\mathcal{U}K$ in every degree separately, and obtain a map $\lambda_{\mathcal{U}K[\![t^{\pm1}]\!]}$ returning an element of $\mathcal{U}K[\![t^{\pm1}]\!]$.

Proposition 4.24. *The map $\Lambda_{\mathcal{U}G}\colon \mathrm{WRat}(K,t) \to \mathcal{U}G$ obtained by composing*

$$(p_n,q_n)_n \mapsto (p_n {q_n}^\dagger)_n$$

with $\lambda_{\mathcal{U}G}$ is an injective ring homomorphism.

Similarly, the map $\Lambda_{\mathcal{U}K[\![t^{\pm1}]\!]}\colon \mathrm{WRat}(K,t) \to \mathcal{U}K[\![t^{\pm1}]\!]$ obtained by composing

$$(p_n,q_n)_n \mapsto (p_n\overline{q_n})_n$$

with $\lambda_{\mathcal{U}K[\![t^{\pm1}]\!]}$ is an injective ring homomorphism.

Proof. Let us start with $\Lambda_{\mathcal{U}G}$. Take $(p_n,q_n)_n \sim (p'_n,q'_n) \in \mathrm{WRat}(K,t)$. By Lemma 4.14, we have $(p_n{q_n}^\dagger)_n \approx (p'_n{q'_n}^\dagger)_n$, and so $\lambda_{\mathcal{U}G}$ sends both to the same element of $\mathcal{U}G$ by Lemma 4.16. Hence $\Lambda_{\mathcal{U}G}$ is well defined.

To check that $\Lambda_{\mathcal{U}G}$ is additive and multiplicative, we use Lemma 4.11 and the uniqueness part of Lemma 4.16. Thus, $\Lambda_{\mathcal{U}G}$ is a ring homomorphism.

Finally, suppose that $\Lambda_{\mathcal{U}G}((p_n,q_n)_n) = 0$. Unravelling the definitions, we get $(p_n {q_n}^\dagger)_n \approx (0)_n$. Since $(q_n)_n$ is asymptotically injective, Lemmata 4.11 and 4.14 give

$$(0)_n \approx (0 \cdot q_n)_n \approx (p_n {q_n}^\dagger q_n)_n \approx (p_n)_n$$

and so $(p_n,q_n) = (0,1)$ in $\mathrm{WRat}(K,t)$.

The situation for the map $\Lambda_{\mathcal{U}K[t^{\pm 1}]\!]}$ is completely analogous: we use Lemmata 4.13 and 4.19 instead of Lemmata 4.11 and 4.14, respectively. □

Using the two maps above, we may view $\mathrm{WRat}(K,t)$ as a subring of both $\mathcal{U}G$ and $\mathcal{U}K[t^{\pm 1}]\!]$.

Lemma 4.25. *If $(p_n)_n$ and $(q_n)_n$ are asymptotically injective sequences over $\mathcal{U}K[t^{\pm 1}]$ and $(p_n,q_n)_n \in \mathrm{WRat}(K,t)$, then $(q_n,p_n)_n \in \mathrm{WRat}(K,t)$ as well, and $(p_n,q_n)_n \cdot (q_n,p_n)_n = (1,1)_n$.*

Proof. To check that $(q_n,p_n)_n \in \mathrm{WRat}(K,t)$, we need to show that $(q_n {p_n}^\dagger)_n$ stabilises and that $(q_n\overline{p_n})_n$ K-stabilises. The first follows immediately from the fact that $(p_n {q_n}^\dagger)_n$ stabilises, Lemma 4.15, and the observation that ${q_n}^{\dagger\dagger} = q_n$.

For the second, we first observe that $\mathrm{init}(\overline{p_n}) = (\mathrm{init}\, p_n)^\dagger$, and hence $(\overline{p_n})_n$ is an asymptotically injective sequence. Therefore so is $(q_n\overline{p_n})_n$. We then use Lemma 4.19.

The last statement follows directly from the definition of multiplication. □

Proposition 4.26. *The subring*

$$\widehat{\mathbb{Q}G}^{\phi} \cap \mathrm{WRat}(K,t)$$

of $\mathcal{U}K[t^{\pm 1}]\!]$ contains $\mathbb{Q}G$ and is division-closed inside $\widehat{\mathbb{Q}G}^{\phi}$.

Proof. Proposition 4.23 tells us that $\mathrm{WRat}(K,t)$ contains $\mathcal{U}K[t^{\pm 1}]$, which in turn contains $\mathbb{Q}K[t^{\pm 1}] = \mathbb{Q}G$.

Now take a Laurent power series $x = \sum_{i=k}^{\infty} t^i x_i \in \widehat{\mathbb{Q}G}^{\phi} \cap \mathrm{WRat}(K,t)$ with $x_i \in \mathbb{Q}K$ for all i, and suppose that it is invertible in $\widehat{\mathbb{Q}G}^{\phi}$. Truncating the inverse we find $y \in \mathbb{Q}G$ such that $\mathrm{init}(xy) = 1$. Since x and y lie in $\mathrm{WRat}(K,t)$, and since the latter object is a ring, we conclude that $xy \in \mathrm{WRat}(K,t)$ as well. We thus have $(p_n,q_n)_n \in \mathrm{WRat}(K,t)$ with $\Lambda_{\mathcal{U}K[t^{\pm 1}]\!]}((p_n,q_n)_n) = xy$. By multiplying by suitable powers of t, we easily arrange for the associated power of $\mathrm{init}\, q_n$ to be 0, for every n.

We now need to worry about a pathological situation in which the powers associated to $\mathrm{init}\, p_n$ are not 0. It is easy to see that the power cannot be positive infinitely often, since then the sequence of terms next to t^0 in $p_n\overline{q_n}$ could not be equal to 1 in the limit. So the powers are eventually non-positive. This remains true for every other pair of sequences $\sim$-equivalent to $(p_n,q_n)_n$.

Since p_n is admissible, these powers are bounded from below. We are going to modify p_n, staying in the same $\sim$-equivalence class, so that the $\liminf$ of the powers becomes zero.

Suppose that the lowest power that appears infinitely often is k. Since the terms next to t^k in $p_n\overline{q_n}$ tend to 0, we easily produce an asymptotically injective sequence $(r_n)_n$ of projections in $\mathcal{N}K$ such that $p_n r_n$ has initial term with associated power greater than k for every n. We also replace q_n with $q_n r_n$, and obtain $(p_n, q_n)_n \sim (p_n r_n, q_n r_n)_n$. We repeat this process $|k|$-times, and arrive at $(p_n, q_n)_n \sim (p'_n, q'_n)_n$ with the initial terms of q'_n all having associated power 0, and with the corresponding power for $\operatorname{init} p'_n$ being eventually 0.

We claim that $(p'_n)_n$ is asymptotically injective. Once this is shown, Lemma 4.25 will tell us that $(p'_n, q'_n)_n$ is invertible in $\mathrm{WRat}(K, t)$, and hence so is xy, finishing the proof.

To prove the claim, observe that for large enough n we have

$$\operatorname{init}(p'_n \overline{q'_n}) = \operatorname{init} p'_n (\operatorname{init} q'_n)^\dagger,$$

since $(q'_n)_n$ is asymptotically injective. If $(p'_n)_n$ is not asymptotically injective, then neither is

$$(\operatorname{init} p'_n (\operatorname{init} q'_n)^\dagger)_n \approx (1)_n.$$

This is a contradiction. □

5. The main results

Proposition 5.1. *Let G be a countable discrete group and let*

$$\varphi \colon G \to \mathbb{Z}$$

be a character. Let $C_\bullet$ be a complex of free $\mathbb{Q}G$-modules which is finitely generated up to degree n and such that $C_k = 0$ for $k < 0$. If

$$H_j(C_\bullet \otimes_{\mathbb{Q}G} \widehat{\mathbb{Q}G}^{\varphi}) = 0$$

for $j \leqslant n$, then $H_j(C_\bullet \otimes_{\mathbb{Q}G} \mathcal{U}G) = 0$ for $j \leqslant n$.

Proof. By Proposition 3.6, we see that

$$H_j(C_\bullet \otimes_{\mathbb{Q}G} \mathcal{D}(\mathbb{Q}G \subset \widehat{\mathbb{Q}G}^{\phi})) = 0$$

for $j \leqslant n$. By Proposition 4.26, the ring $\mathcal{D}(\mathbb{Q}G \subset \widehat{\mathbb{Q}G}^{\phi})$ is a subring of $\mathrm{WRat}(K, t)$, where $K = \ker \phi$ and $t \in G$ denotes an element such that $\phi(t)$ generates $\operatorname{im} \phi = \mathbb{Z}$. Using a chain contraction up to dimension n we conclude that

$$H_j(C_\bullet \otimes_{\mathbb{Q}G} \mathrm{WRat}(K, t)) = 0$$

for $j \leqslant n$. Now, Proposition 4.24 tells us that we may view $\mathrm{WRat}(K, t)$ as a subring of $\mathcal{U}G$, and hence again using chain contractions we see that

$$H_j(C_\bullet \otimes_{\mathbb{Q}G} \mathcal{U}G) = 0$$

for $j \leqslant n$, and the proof is finished. □

We are now ready to prove the remaining results from the introduction.

Theorem A. *Let G be a group of type $\mathsf{FP}_n(\mathbb{Q})$. If $b_n^{(2)}(G) \neq 0$, then $\Sigma^n(G; \mathbb{Q}) = \varnothing$.*

Proof. As $\Sigma^k(G;\mathbb{Q}) \subseteq \Sigma^\ell(G;\mathbb{Q})$ for all $k \geqslant \ell$, we may assume n is the lowest dimension for which $b_n^{(2)}(G) \neq 0$. In particular, $H_n(G;\mathcal{U}G) \neq 0$. Since G is of type $\mathsf{FP}_n(\mathbb{Q})$, there is a resolution $C_\bullet$ of $\mathbb{Q}$ by free $\mathbb{Q}G$-modules that is finitely generated up to dimension n. We may compute $H_p(G;\mathcal{U}G)$ as $H_p(C_\bullet \otimes_{\mathbb{Q}G} \mathcal{U}G)$. It follows that $H_n(C_\bullet \otimes_{\mathbb{Q}G} \mathcal{U}G) \neq 0$. Thus, by Proposition 5.1, for every character $\varphi\colon G \twoheadrightarrow \mathbb{Z}$, we have $H_n(G;\widehat{\mathbb{Q}G}^{\varphi}) = H_n(C_\bullet \otimes_{\mathbb{Q}G} \widehat{\mathbb{Q}G}^{\varphi}) \neq 0$. In particular, by openness of the BNSR invariants (Theorem 2.16(1)) and Sikorav's Theorem, $\Sigma^n(G;\mathbb{Q}) = \varnothing$. □

The proof of Theorem B is entirely analogous once one replaces $C_\bullet$ with $C_\bullet(\widetilde{X};\mathbb{Q})$, the chain complex of the universal cover of X, viewed as a chain complex of free $\mathbb{Q}\pi_1 X$ modules.

Corollary C. *Let M be a closed connected $2n$-manifold or (more generally) a finite $\mathsf{PD}_{2n}(\mathbb{Q})$-complex. If $\chi(M) \neq 0$, then $\Sigma^n(M) = \Sigma^n(M;\mathbb{Q}) = \varnothing$. In particular, if M is additionally aspherical, then $\Sigma^n(\pi_1 M) = \Sigma^n(\pi_1 M;\mathbb{Q}) = \varnothing$.*

Proof. Let us start with M being a manifold. After passing to a finite cover we may assume that M is orientable and by [KS69, Theorem III] we may assume that M has the homotopy type of a finite CW complex. By [Lüc02, Remark 6.81] we have

$$\sum_{p \geqslant 0} (-1)^p b_p^{(2)}(\widetilde{M};\pi_1 M) = \chi(M) \neq 0.$$

In particular, there is some p where $b_p^{(2)}(\widetilde{M};\pi_1 M) \neq 0$. By Poincaré duality [Lüc02, Theorem 1.35(3)] we have $b_{2n-k}^{(2)}(M) = b_k^{(2)}(M)$. In particular, we may assume that $p \leqslant n$. Now, Theorem B implies that $\Sigma^p(M;\mathbb{Q}) = \varnothing$, of which $\Sigma^n(M)$ is a subset by Theorem 2.16(2). The argument for a $\mathsf{PD}_{2n}(\mathbb{Q})$-complex is identical.

Suppose in addition that M is aspherical. Then, M is a model for $K(\pi_1 M, 1)$ and so $b_p^{(2)}(\pi_1 M) = b_p^{(2)}(\widetilde{M};\pi_1 M)$. Now, by Theorem A we have $\Sigma^p(\pi_1 M;\mathbb{Q}) = \varnothing$. The result follows from Theorem 2.16(2). □

Corollary D. *Let G be a $\mathsf{PD}_n(\mathbb{Q})$-group and let $k = \lceil n/2 \rceil - 1$. If $\Sigma^k(G;\mathbb{Q})$ is non-empty, then the Singer Conjecture holds for G.*

Proof. Arguing by Poincaré duality as in the proof of Corollary C it suffices to show that $b_p^{(2)}(G) = 0$ for $p \leqslant k$. Now, by hypothesis there exists $\varphi \in \Sigma^k(G,\mathbb{Q})$. So, by Theorem A, $b_p^{(2)}(G) = 0$ for $p \leqslant k$. □

6. The Atiyah Conjecture and locally-indicable groups

In this section we prove versions of Theorem A and Theorem B in positive characteristic. This relies on the existence of certain Hughes-free[1] skew fields.

Let R be a skew field and let G be a group. When it exists, we denote by $\mathcal{D}_{RG}$ the *Hughes-free skew field* of RG. We omit the technical definition of a Hughes-free skew field as we do not require it. However, we note that if it exists it is unique up to an RG-algebra isomorphism.

[1]No relation to the first author.

Remark 6.1. It is conjectured that $\mathcal{D}_{RG}$ exists for any skew field R for all locally indicable groups [JZ21, Conjecture 1]; it is known to exist for residually {locally indicable amenable} groups [JZ21, Corollary 1.3]. In particular, $\mathcal{D}_{RG}$ exists for RFRS groups.

Definition 6.2. A group G is *agrarian* over a ring R if there exists a skew-field $\mathcal{D}$ and a monomorphism $\psi\colon RG \to \mathcal{D}$ of rings. If G is agrarian over R and X is a space with $\pi_1 X = G$, then we define the *agrarian $\mathcal{D}$-homology of* X to be

$$H_p^{\mathcal{D}}(X) = H_p(C_\bullet(\widetilde{X}; R) \otimes_{RG} \mathcal{D})$$

where $\mathcal{D}$ is viewed as an RG-$\mathcal{D}$-bimodule via the embedding $RG \to \mathcal{D}$. We also define the *agrarian $\mathcal{D}$-homology of* G to be

$$H_p^{\mathcal{D}}(G) = \operatorname{Tor}_p^{RG}(R, \mathcal{D}).$$

Since modules over a skew field have a canonical dimension function taking values in $\mathbb{N} \cup \{\infty\}$ we may define

$$b_p^{\mathcal{D}}(X) = \dim_{\mathcal{D}} H_p^{\mathcal{D}}(X) \quad \text{and} \quad b_p^{\mathcal{D}}(G) = \dim_{\mathcal{D}} H_p^{\mathcal{D}}(G).$$

If $\mathcal{D}_{RG}$ exists, then (up to RG-isomorphism) we have a canonical choice of $\mathcal{D}$ for each skew-field R.

The Atiyah Conjecture. *Let G be a torsion-free countable group. Then, the ring $\mathcal{D}(\mathbb{C}G \subset \mathcal{U}G)$ is a skew field.*

Remark 6.3. If G satisfies the Atiyah Conjecture, then $\mathcal{D}_{\mathbb{C}G}$ exists and is isomorphic to $\mathcal{D}(\mathbb{C}G \subset \mathcal{U}G)$. This applies for instance to torsion-free subgroups of right-angled Artin and Coxeter groups [LOS12], torsion-free virtually special groups [Sch14], locally indicable groups [JZLÁ20], and more [Lin93, FL06, JZ19].

Theorem 6.4. *Let R be a skew-field and let G be a group such that $\mathcal{D}_{RG}$ exists. Let $\varphi \in S(G)$.*

(1) *Suppose that G is of type $\mathsf{FP}_n(R)$. If $b_n^{\mathcal{D}_{RG}}(G) \neq 0$, then $\Sigma^n(G; R) = \varnothing$.*
(2) *Let X be a connected CW complex with finite n-skeleton. If $b_n^{\mathcal{D}_{RG}}(X) \neq 0$, then $\Sigma^n(X; R) = \varnothing$.*

In particular, if G satisfies the Atiyah Conjecture, then statements (1) *and* (2) *hold with $R = \mathbb{C}$ and with ℓ^2-Betti numbers replacing $\mathcal{D}_{\mathbb{C}G}$-agrarian Betti numbers.*

Proof. We proceed as in [HK22, Theorem 5.10]. Let $K = \ker \varphi$. Let $\mathbb{K}$ be the skew-field of twisted Laurent series with variable t and coefficients in the skew field $\mathcal{D}_{RK}$; here t is an element of G with $\varphi(t) = 1$ and the twisting extends the conjugation action of t on K. This is possible since $\mathcal{D}_{RK}$ is Hughes-free (see [JZ21] for an explanation).

We have two embeddings, firstly $\widehat{RG}^{\varphi}$ embeds into $\mathbb{K}$, and secondly $\mathcal{D}_{RG}$ embeds into $\mathbb{K}$. To see the first embedding, $\widehat{RG}^{\varphi}$ may be viewed as a ring of twisted Laurent series in t with coefficients in RH. The second embedding exists because $\mathcal{D}_{RG}$ is Hughes-free and so isomorphic as an RG-module to $\mathcal{D}(RK[t^{\pm 1}] \subset \mathbb{K})$ where we identify RG with $RK[t^{\pm 1}]$. In particular, we may view $\mathbb{K}$ as a $\mathcal{D}_{RG}$-module.

Claim 6.5. *Let $C_\bullet$ be a chain complex of finitely generated free RG-modules such that $C_i = 0$ for $i < 0$. If $H_j(C_\bullet \otimes_{RG} \widehat{RG}^\varphi) = 0$ for $j \leqslant n$, then $H_j(C_\bullet \otimes_{RG} \mathcal{D}_{RG}) = 0$.*

Proof of Claim 6.5. Since $H_j(C_\bullet \otimes_{RG} \widehat{RG}^\varphi) = 0$ for $j \leqslant n$ and $\widehat{RG}^\varphi \subset \mathbb{K}$ it follows using chain contractions that $H_j(C_\bullet \otimes_{RG} \mathbb{K}) = 0$ for $j \leqslant n$. Now, $\mathbb{K}$ and $\mathcal{D}_{RG}$ are skew-fields, and so

$$H_j(C_\bullet \otimes_{RG} \mathcal{D}_{RG}) \otimes_{\mathbb{K}} \mathbb{K} = H_j(C_\bullet \otimes_{RG} \mathbb{K}) = 0,$$

forcing $H_j(C_\bullet \otimes_{RG} \mathcal{D}_{RG}) = 0$ for $j \leqslant n$, as claimed. ■

The theorem is now proved either by taking $C_\bullet$ in the claim to be a free resolution of G, finitely generated up to degree n, in the case of (1); or taking $C_\bullet$ to be $C_\bullet(\widetilde{X}; R)$, viewed as a chain complex of free RG-modules, in the case of (2). Finally, the result follows from the appropriate version of Sikorav's Theorem and openness of the BNSR invariants (Theorem 2.16(1)). □

Finally, we remark that characteristic p versions of Corollary C and Corollary D can be formulated and proved for groups G where $\mathcal{D}_{\mathbb{F}_pG}$ exists by almost verbatim arguments – with the exception of substituting Theorems A and B with Theorem 6.4.

7. Some examples

In this section we detail a number of examples that both complement results already in the literature and might be of independent interest.

An *elementarily free group* is a group with the same first order theory as a free group. Every finitely generated such group is of type F (see [BTW07]). A *poly-elementarily-free group* of length n is a group G which admits a subnormal filtration $\{1\} = N_0 \lhd N_1 \lhd \cdots \lhd N_n = G$ with N_i/N_{i-1} isomorphic to a finitely generated elementarily free group. Note that poly-{finitely generated free or surface} groups are poly-elementarily-free.

Lemma 7.1. *Let G be a poly-elementarily-free group of length n. If $\chi(G) \neq 0$, then $b_p^{(2)}(G) = 0$ for $p \neq n$ and $b_n^{(2)}(G) = |\chi(G)|$.*

Proof. The case $n = 0$ is easily dealt with, and so we may assume that $n > 0$.

Let $(N_i)_i$ be a subnormal chain with every $N_i/N_{i-1} = G_i$ non-trivial and finitely generated elementarily free, and with $N_0 = \{1\}$ and $N_n = G$. Note that each N_i is a group of type F since it is an extension of such groups. Since $\chi(G) \neq 0$ and Euler characteristic is multiplicative over short exact sequences of groups of type F, it follows that $\chi(N_i) \neq 0$ and $\chi(G_i) \neq 0$ for every i. By [BK17] we have $b_p^{(2}(G_i) = 0$ unless $p = 1$, in which case the first ℓ^2-Betti number of G_i may be positive. Since $\chi(G_i) \neq 0$ we have that $b_1^{(2)}(G_i) > 0$.

An inductive application of [Lüc02, Theorem 6.67] yields that for every i and every $p < i$,

$$b_p^{(2)}(N_i) = 0.$$

By [BTW07, Theorem B], an elementarily free group is measure equivalent to a free group. Now, inductively applying [ST10, Theorem 1.10] shows that for every i and every $p > i$ we have $b_i^{(2)}(N_i) = 0$. Thus, $b_i^{(2)}(N_i) = |\chi(N_i)|$, and hence taking $i = n$ we obtain $b_n^{(2)}(G) = |\chi(G)|$. □

The following result generalises [KV22, Proposition 1.5], dealing with free-by-free or surface-by-surface groups, and the first part of [KW19, Theorem 6.1], dealing with {free group of rank 2}-by-free groups.

Theorem 7.2. *Let G be a poly-elementarily-free group of length n. If $\chi(G) \neq 0$, then $\Sigma^n(G;\mathbb{Q}) = \Sigma^n(G) = \varnothing$.*

Proof. The result now follows from Lemma 7.1 and Theorem A. □

We remark that the conclusions of Lemma 7.1 and Theorem 7.2 remain valid if G is a poly-$\mathcal{X}$ group where $\mathcal{X}$ is the class of groups of type FP that are measure equivalent to a free group.

Example 7.3 (Pure mapping class group of a punctured sphere)**.** Let $m \geqslant 3$, and let S_m denote the m-punctured 2-sphere. Recall that the *pure mapping class group* $\Gamma_m := \mathrm{PMCG}(S_m)$ of S_m is the group of mapping classes of S_m which fix the m-punctures pointwise. It is well known (see e.g. [FM11, Section 9.3]) that Γ_m is poly-free of length $m-2$ and each subnormal quotient in the poly-free filtration is non-abelian. Hence, $\chi(\Gamma_m) \neq 0$. We have verified the hypotheses of Theorem 7.2 and conclude that $\Sigma^{m-2}(\Gamma_m;\mathbb{Q}) = \Sigma^{m-2}(\Gamma_m) = \varnothing$.

Example 7.4 (Real and complex hyperbolic lattices)**.** Let Γ be a lattice in $\mathrm{SO}(2n,1)$ or $\mathrm{SU}(n,1)$. By [Dod79] (see also [Bor85]), we have $b_p^{(2)}(\Gamma) = 0$ except when $p = n$, in which case $b_n^{(2)}(\Gamma) \neq 0$. By Theorem A, we have $\Sigma^n(\Gamma) = \Sigma^n(\Gamma;\mathbb{Q}) = \varnothing$. This result was already known for 'simplest type lattices' in $\mathrm{SU}(n,1)$, see [LP22].

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