SIMPLE GROUPS AND COMPLEMENTS OF SMOOTH SURFACES IN SIMPLY CONNECTED 4-MANIFOLDS

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Abstract. For each integer n we construct a simply connected 4-manifold X admitting a smoothly embedded surface Σ of self intersection number n such that the complement of the surface has non-trivial fundamental group. This answers a question of Kronheimer in Kirby's 1997 problem list. The proof combines a topological construction with homological properties of simple groups such as Thompson's group V and certain sporadic finite simple groups.

1. Introduction

A basic invariant of an embedding of a surface Σ in a 4-manifold X is the fundamental group of its complement, and it is reasonable to ask what groups can occur, especially if X is simply connected. With no further restrictions on the embedding, there is a simple characterisation of such groups [KR08]: A finitely presented group G is $\pi_1(X - \Sigma)$ for some $\Sigma \subset X$ if and only if $H_1(G)$ is cyclic and G is the normal closure of a single element; a question of Wiegold [KM23, Problem 5.52] asks if the second condition follows automatically from the first.

The construction in [KR08] provides a surface (in fact a symplectically embedded surface in a symplectic 4-manifold) with self-intersection 0, but the problem is more challenging if one requires that the self-intersection is non-zero. Indeed, a question of Kronheimer in Kirby's 1997 problem list [Kir97, Problem 4.109] asks whether $\pi_1(X - \Sigma)$ is trivial if Σ has non-zero self-intersection that is square-free (all prime divisors appear once). The motivation is an observation of Kronheimer–Mrowka [KM93, Proposition 5.7] that in this situation, $\pi_1(X - \Sigma)$ would have no non-trivial representations in SO(3).

In this note, we construct surfaces with non-simply connected complements and provide a negative answer to Kronheimer's question. The result applies to all non-zero self-intersections, not just square-free ones.

Theorem A. *For any non-zero n, there is a simply connected 4-manifold X and surface Σ smoothly embedded in X where $\Sigma \cdot \Sigma = n$ and $\pi_1(X - \Sigma) \neq \{1\}$.*

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Remark 1.1. We have not been particularly careful about signs and orientations. So the reader may find it reassuring to note that by reversing the orientation of X, the statement of Theorem A for n is equivalent to the same statement for $-n$.

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2. From group theory to surface complements

Choose an orientation for Σ, and let N be the disk bundle over Σ of Euler class n. Its boundary, a circle bundle over Σ, will be denoted by Y. We note that the for an embedding of Σ with normal bundle N, the complement $X-\Sigma$ of Σ deformation retracts onto the exterior of Σ, defined as $X-\mathrm{int}(N)$; it is more convenient henceforth to work with the exterior. The idea of the proof of Theorem A is to find a suitable manifold W with $\partial W = Y$ to serve as the exterior of Σ in X. In fact we will take X to be $N \cup W$, and the goal is to choose W so that X is simply connected.

To this end, we recall that for a group G, the oriented bordism group $\Omega_k(G)$ consists of pairs (M^k, φ) where M is an oriented closed k-manifold and $\varphi : \pi_1(M) \to G$ is a homomorphism, up to an obvious cobordism relation. Equivalently, it is the bordism group $\Omega_k(BG)$. One defines, similarly, the spin bordism group $\Omega_k^{\mathrm{spin}}(G)$ as spin cobordism classes of triples $(M, \mathfrak{s}, \varphi)$ where $\mathfrak{s}$ is a spin structure on M. In Section 4 we will make use of the readily proven fact that such bordism groups are functorial with respect to homomorphisms of G.

The following lemma is a straightforward consequence of the Atiyah-Hirzebruch spectral sequence for the generalised homology theory of oriented bordism [AH61, McC01] or spin bordism. The first part can be proved geometrically as in [Gor86].

Lemma 2.1. *The map $\Omega_3(G) \to H_3(G;\mathbb{Z})$ that assigns $\varphi_*([M])$ to a pair (M, φ) is an isomorphism. If $H_1(G;\mathbb{Z}) = H_2(G;\mathbb{Z}) = 0$, then the same is true for $\Omega_3^{\mathrm{spin}}(G)$.*

Recall that the fundamental group of Y is a central extension

$$\{1\} \longrightarrow \mathbb{Z} \longrightarrow \pi_1(Y) \longrightarrow \pi_1(\Sigma) \longrightarrow \{1\}$$

and that the $\mathbb{Z}$ subgroup is the kernel of the map $\pi_1(Y) \to \pi_1(N)$. Denote by μ the generator of this $\mathbb{Z}$ subgroup corresponding to the oriented meridian of Σ.

Lemma 2.2. *Suppose that there is a finitely presented simple group P and a homomorphism $\varphi : \pi_1(Y) \to P$ such that $\varphi(\mu)$ is non-trivial in P and the bordism class of (Y, φ) in $\Omega_3(P)$ is trivial. Then there is an embedding of Σ in a simply connected 4-manifold X such that the exterior $W = X - \mathrm{int}(N)$ has fundamental group P. If n is even, choose a spin structure on N with* *restriction σ_Y to Y. If the bordism class of (Y, σ_Y, φ) in $\Omega_3^{\mathrm{spin}}(P)$ is trivial, then X may be chosen to be a spin manifold.*

Proof. By hypothesis, there is a 4-manifold W_0 and the portion connected by solid arrows in the diagram below.

$$\begin{tikzcd}
\pi_1(Y) \arrow[rr, "\varphi"] \arrow[dr, "j_*"'] & & P & \\
& \pi_1(W_0) \arrow[ur, "\Phi_0"] \arrow[r, dotted, "i"] & \pi_1(W_1) \arrow[u, dotted, two heads, "\Phi_1"'] \arrow[r, dotted, "q"] & \pi_1(W) \arrow[ul, dotted, "\Phi", "\cong"']
\end{tikzcd}$$

We modify W_0 in two stages to fill in the dotted portion of the diagram so that Φ is an isomorphism. Since P is finitely presented, there is a closed 4-manifold W_P with $\pi_1(W_P) \cong P$. Replace W_0 with $W_0 \# W_P$ and note that the homomorphism $\pi_1(W_0 \# W_P) \cong \pi_1(W_0) * P \to P$ given by Φ_0 on the first factor and the identity on the second factor is a surjection. Choose finitely many generators $\{x_j\}$ for $\pi_1(W_0)$. Now do surgery on circles in $W_0 \# W_P$ representing the elements $x_j^{-1}\Phi_0(x_j)$ to obtain a manifold W_1 with a surjection Φ_1 as indicated. The homomorphism $i : \pi_1(W_0) \to \pi_1(W_1)$ is induced by the inclusion of the summand W_0 in $W_0 \# W_P$. As in [Wal66, Wal99] do surgery on finitely many circles in W_1 to kill the kernel of Φ_1, obtaining the manifold W and isomorphism Φ as indicated.

The image of μ in $\pi_1(W)$ is taken to $\varphi(\mu)$, which is assumed to be non-trivial in P. Since P is simple, μ normally generates $\pi_1(W)$. On the other hand, μ is trivial in $\pi_1(N)$, so van Kampen's theorem says that $X = W \cup N$ is simply connected.

The argument goes through in the spin case to produce a simply connected spin manifold X. By hypothesis, the initial manifold W_0 has a spin structure extending the given one on Y (which by hypothesis extends over N). But it is standard that the framing of a surgery on a circle in a spin 4-manifold can be chosen so that the new manifold inherits a spin structure. This choice of framing does not affect the fundamental group arguments. ☐

Hence, to complete the proof of Theorem A, it suffices to find groups and homomorphisms satisfying the hypotheses of Lemma 2.2; this is carried out in the next two sections of the paper. The first of these makes use of Thompson's infinite simple group, while the second uses an assortment of finite simple groups.

3. From Seifert fibred 3-manifolds to Thompson's group

Proposition 3.1. *Let Y be a Seifert fibred space and let $H = \pi_1 Y$. Let $\mu \in Y$ be a generator of the centre of Y. Then, there exists a group V and a homomorphism $\psi\colon H \to V$ such that*

(1) V is finitely presented, in fact F_∞;
(2) V is acyclic; in particular $\Omega_3(V) = 0$ and $\Omega_3^{\mathrm{spin}}(V) = 0$;
(3) $\psi(\mu)$ normally generates V.

Proof. We let V denote Thompson's group V. We recount the following facts

(1) V is a simple group [Hig74];
(2) V is acyclic [SW19];
(3) V contains every finite group as a subgroup [Hig74];
(4) V is finitely presented [Hig74];
(5) V is type FP_∞ [Bro87];
(6) V is type F_∞, which follows from the previous two items.

By [Hem87] the group H is residually finite. Thus, we can find a finite quotient $h\colon H \twoheadrightarrow Q$ such that $h(\mu)$ is non-trivial. Since every finite group is a subgroup of V we may embed Q into V via some homomorphism $i\colon Q \to V$. We define ψ to be $i \circ h$. Now, as V is simple, every non-trivial element of V normally generates it. Thus, $\psi(\mu)$ normally generates V. □

When the self-intersection n is greater than one in absolute value, we don't have to appeal to the residual finiteness of H. For in this case, H has a surjection onto $\mathbb{Z}/n$ taking μ to a generator. By item (3) above, $\mathbb{Z}/n$ is a subgroup of V, which provides the desired ψ. It is slightly more delicate to find explicit homomorphisms when $n = \pm 1$; in the next section we do this with V replaced by various finite simple groups.

Remark 3.2 (Higman–Thompson groups)**.** In fact we can build more examples out of the Higman–Thompson groups $V_{m,r}$. In [SW19], the authors show $H_*(V_{m,r};\mathbb{Z}) \cong H_*(\Omega_0^\infty \mathbf{M}_{m-1};\mathbb{Z})$, where the second object is the homology of the zeroth component of the infinite loop space of the mod $m-1$ Moore spectrum. The relevance for us is Propositions 6.1 and 6.2 of loc. cit.; there it is shown that when m is even and p is the smallest prime dividing $m-1$ we have $\widetilde{H}_d(V_{m,r};\mathbb{Z}) = 0$ for $d < 2p-3$ and $H_{2p-3}(V_{m,r};\mathbb{Z}) = \mathbb{Z}/p$. In particular, these groups also apply to the above construction with one caveat: if $3|m-1$ then one must map to a finite group Q with $|H_3(Q;\mathbb{Z})|$ coprime to 3 to ensure the bordism class vanishes.

4. Finite simple groups

In this section we provide more examples of manifolds satisfying Theorem A. The key difference is that the fundamental group of the complement will be a finite group instead of Thompson's group V.

Self intersection number one. We first suppose the surface Σ has self intersection number equal to one or minus one. In this case the associated Seifert fibre space has fundamental group H equal to

$$\left\langle a_1, b_1, \dots, a_g, b_g, z \mid \prod_{i=1}^{g}[a_i, b_i] = z,\ [a_j, z] = [b_j, z] = 1 \text{ for } 1 \leqslant j \leqslant g \right\rangle.$$

The group H admits a surjection onto $\mathrm{He}_3(\mathbb{Z})$, the 3-dimensional integer Heisenberg group (uni-triangular matrices), with presentation

$$\langle a, b, z \mid [a, b] = z,\ [a, z] = [b, z] = 1\rangle.$$

The homomorphism is given by $a_1 \mapsto a$, $b_1 \mapsto b$, $a_i, b_i \mapsto 1$ for $i \geqslant 2$, and of course $z \mapsto z$. For each prime p, the group $\mathrm{He}_3(\mathbb{Z})$ admits a surjection onto the p-group $\mathrm{He}_3(p)$ by considering the modulo p reduction of matrices in $\mathrm{He}_3(\mathbb{Z})$. Note that when $p = 2$ this group is the dihedral group on 4 points D_4 containing 8 elements.

We want to apply Lemma 2.2; the following conditions are sufficient on some finite simple group P:

(1) $\mathrm{He}_3(p) \leqslant P$
(2) $H_3(P;\mathbb{Z})$ has no p-torsion.

Indeed, we want the class (Y,φ) to be trivial in $\Omega_3(P)$, but this class is in the image of the composition

$$H_3(Y;\mathbb{Z}) \to H_3(\pi_1 Y;\mathbb{Z}) \to H_3(\mathrm{He}_3(p);\mathbb{Z}) \to H_3(P;\mathbb{Z})$$

and group $H_3(\mathrm{He}_3(p);\mathbb{Z})$ is annihilated by multiplication by p^3.

The following table details the sporadic simple groups P whose third homology is known and which ones admit a $\mathrm{He}_3(p)$ subgroup with p coprime to the order of the third homology of P.

	M_{22}	M_{23}	HS	He	McL	J_3	J_4	Ly
$H_2(G;\mathbb{Z})$	12	0	2	0	0	3	0	0
$H_3(G;\mathbb{Z})$	0	0	2+2	12	0	15	0	0
$\mathrm{He}_3(p)$	2	2	5	7	2,3,5	2	2,3	2,3,5

	Co_3	Co_2	Co_1	Suz	$O'N$	Fi_{22}	M
$H_2(G;\mathbb{Z})$	0	0	2	6	3	6	0
$H_3(G;\mathbb{Z})$	6	4	12	4	8	1	$24 + (\leqslant 4)$
$\mathrm{He}_3(p)$	5	3,5	5	3	7	2,3	5, 7

Note that for the Monster group M, it is known that $H_3(G;\mathbb{Z})$ is a subgroup of $\mathbb{Z}/24 \oplus \mathbb{Z}/4$ but the exact group is not known. The groups M_{11}, M_{12}, M_{24}, J_1, J_2 have been omitted from the table due to having no $\mathrm{He}_3(p)$ subgroup with p coprime to $|H_3(P;\mathbb{Z})|$. The groups Ru, HN, Th, Fi_{23}, Fi'_{24}, and B have been omitted from the table because $H_3(G;\mathbb{Z})$ is not known.

The existence or non-existence of the $\mathrm{He}_3(p)$ subgroups is easily verified by consulting the ATLAS of finite simple groups [CCN+85]. The homology computations were obtained from the following sources: the low-dimensional homology of the Mathieu groups can be found in [DSE09] and the remaining computations are contained in [JFT19].

Self intersection number greater than one. As remarked in Section 3, when the self intersection number n satisfies $|n| > 1$ there is a surjection $\pi_1 Y \twoheadrightarrow \mathbb{Z}/n$ in which μ generates the image. Here we will show that for n divisible by a prime $p \geqslant 7$, we can take the simple group P to be isomorphic to $\mathrm{PSL}_2(p)$. The idea is that $\mathbb{Z}/n$ surjects $\mathbb{Z}/p$ and this is a subgroup of the simple group $\mathrm{PSL}_2(p)$. Thus, all that remains is to show that $H_3(\mathrm{PSL}_2(p);\mathbb{Z})$ has no elements of order p, since $H_3(\mathbb{Z}/p;\mathbb{Z}) \cong \mathbb{Z}/p$.

Let $P = \mathrm{PSL}_2(p)$. Note that the Sylow p-subgroup of P is isomorphic to $\mathbb{Z}/p$. The quadratic residues are an index two subgroup of $\mathbb{F}_p^\times$, so $N_P(\mathbb{Z}/p)$ has order $\frac{1}{2}p(p-1)$ and $C_P(\mathbb{Z}/p) = \mathbb{Z}/p$. By Swan's Theorem, the p-part of $H^k(P;\mathbb{Z})$ is the fixed points of $N_P(\mathbb{Z}/p)$-action on $H^k(\mathbb{Z}/p;\mathbb{Z})$. Now, the action on $H^{2k}(\mathbb{Z}/p;\mathbb{Z})$ is the kth power of the action on $H^2(\mathbb{Z}/p;\mathbb{Z})$. Hence, the degrees k where $H^k(P;\mathbb{Z})$ contains p-torsion are exactly the degrees $k = 2\ell$ with $\frac{1}{2}(p-1)|\ell$. In particular, for $p \geqslant 7$, by applying the Universal Coefficient Theorem, we see that the group $H_3(P;\mathbb{Z})$ has no elements of order p.

Finally, if n is divisible only by 2, 3, or 5, then one may resort to the sporadic simple groups in the previous section.

A classification? Whilst we have provided many examples of finite simple groups which can appear as complements of surfaces in simply-connected 4-manifolds we fall a long way short of a complete classification. First, note that the assumption P is a simple group in Lemma 2.2 is too strong and the proof goes through verbatim with the hypothesis that P is perfect and normally generated by $\varphi(\mu)$. In particular, many quasi-simple extensions of the previous examples will also show up as fundamental groups of complements. To this end we raise the following question.

Question 4.1. Let X be a simply connected 4-manifold and let $\Sigma \subset X$ be a smoothly embedded surface with $\Sigma \cdot \Sigma = n$. Which finite groups can appear as $\pi_1(X - \Sigma)$?

5. The non-primitive case

The proof of the main theorem produces surface complements $W = X - \Sigma$ with with $H_1(W) = 0$. In this case, Poincaré duality implies that the homology class of Σ must be primitive. For non-primitive classes we can use an explicit construction, not requiring any of our group-theoretic arguments, to get surface complements with non-abelian fundamental groups.

Suppose that the homology class $[\Sigma] \in H_2(X;\mathbb{Z})$ is dA where A is a primitive class in $H_2(X;\mathbb{Z})$. Then $n = \Sigma \cdot \Sigma = d^2 A \cdot A$, and the complement of Σ has first homology $\mathbb{Z}_d$ (In particular, n is not square-free, and hence not part of Kronheimer's original question.) Recall Zeeman's d-twist spinning construction [Zee65], which from a knot K in S^3 produces a fibred knot $\tau_d(K)$ in S^4. The fibre is $\Sigma_d(K)$, the d-fold branched cyclic cover of S^3 branched along K, minus a ball, and the monodromy is a generator of the covering transformations.

Write $n = d^2 m$, and assume without loss of generality (as in Remark 5) that $m > 0$. It is easy to find a surface Σ_0 embedded in a simply connected manifold X with simply connected complement, with $\Sigma_0 \cdot \Sigma_0 = m$. For instance, one could take an algebraic curve in $\mathbb{C}P^2$ of sufficiently high degree, and then blow up enough points to lower the self-intersection to m. Let X be the resulting blowup of $\mathbb{C}P^2$. Then d times the homology class of Σ_0 is represented by a smoothly embedded surface $\Sigma_1 \subset X$ with $\pi_1(X - \Sigma_1) \cong \mathbb{Z}_d$.

Now replace Σ_1 by $\Sigma = \Sigma_1 \# \tau_d(K)$ where K is any non-trivial knot in S^3. It is argued in [Kim06] that the fundamental group of $X - \Sigma$ is $\pi_1(S^3 - K)/\mu^d$, and that this contains the fundamental group of the d-fold cyclic cover of S^3 branched along K as an index d subgroup. By the solution of the Smith conjecture [MB84], this latter group is non-trivial. Since the abelianization of $\pi_1(S^3 - K)/\mu^d$ is $\mathbb{Z}_d$, it follows that it is non-abelian.

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