CONJUGACY SEPARABILITY IN POLYNOMIALLY GROWING FREE-BY-CYCLIC GROUPS

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Abstract. We show that free-by-cyclic groups with polynomially growing monodromies are conjugacy separable and that double cosets of cyclic subgroups are separable. As a corollary of our results, we show that the outer automorphism group of every polynomially growing free-by-cyclic group is residually finite.

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1. Introduction

A group G is *conjugacy separable* if the conjugacy class of every element is closed in the profinite topology on G. More explicitly, for any pair of non-conjugate elements $g, h \in G$, there exists a finite quotient of G such that the image of g is not conjugate to the image of h.

Any finitely presented conjugacy separable group has solvable conjugacy problem [Mos66]. Conjugacy separability is known to hold for a number of groups

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appearing in low-dimensional topology, for example: virtually free groups [Ste70, Dye79], polycyclic groups [Rem69, For76], Fuchsian groups [FR90], virtually surface groups [Mar07], limit groups [CZ07], and fundamental groups of compact orientable 3-manifolds [HWZ13]. Moreover, Minasyan–Zalenskii show that every virtually compact special hyperbolic group is conjugacy separable [MZ16].

On the other hand conjugacy separability is sensitive to passing to finite index subgroups and finite extensions. Indeed, there exists a non-conjugacy separable group G with a conjugacy separable index two subgroup [Gor86]. More surprisingly, Martino–Minasyan construct non-conjugacy separable subgroups of finite index in a conjugacy separable group, which are finitely presented and have solvable conjugacy problem [MM12]. See [Min17] for even more pathologies.

We say that a group G is *free-by-cyclic* if it admits a finite rank free normal subgroup $F \trianglelefteq G$ such that the quotient G/F is infinite cyclic. A lift of the generator of G/F to G acts by conjugation on F. Any two lifts induce automorphisms that differ by an inner automorphism. We call the corresponding outer class the *monodromy* of the splitting.

An outer automorphism $\Phi \in \mathrm{Out}(F)$ of a finite rank free group F is *polynomially growing* if, roughly, the minimal word lengths of conjugacy classes grow polynomially under the iterations of Φ (see Section 2.4 for the precise definition). We call a free-by-cyclic group *polynomially growing* if it has a polynomially growing monodromy.

Despite the ubiquity of free-by-cyclic groups in geometric group theory, the study of their profinite topology has received little attention and has almost entirely focussed on subgroup separability and rigidity [Kud24, HK25, BP25, AHLP25].

It was recently shown by G. Bartlett that every free-by-cyclic group with finite order monodromy is conjugacy separable [Bar25]. In this paper prove that free-by-cyclic groups with unipotent polynomially growing monodromies are conjugacy separable (see Theorem 7.2) and then use G. Bartlett's result to obtain the following generalization:

Theorem A. *Every polynomially growing free-by-cyclic group is conjugacy separable.*

In contrast to Theorem A, polynomially growing free-by-cyclic groups are almost never subgroup separable (LERF). Indeed, a free-by-cyclic group is LERF exactly when its monodromy has finite order [Kud24].

As an application of our results we obtain the following:

Corollary B. *Let G be a polynomially growing free-by-cyclic group. Then* $\mathrm{Out}(G)$ *is residually finite.*

The corollary follows from a theorem of Grossman, who shows that if G is a finitely generated conjugacy separable group such that every pointwise inner automorphism is inner then $\mathrm{Out}(G)$ is residually finite [Gro75]. The latter condition holds for all torsion-free acylindrically hyperbolic groups by [AMS16, Corollary 1.5]. Free-by-cyclic groups with non-periodic monodromies are acylindrically hyperbolic by the work of Genevois–Horbez [GH21, Corollary 1.5].

1.1. **Proof strategy and further results.** An outer automorphism $\Phi \in \mathrm{Out}(F)$ of a finite rank free group F is *unipotent* if it induces a unipotent element of $\mathrm{GL}(H_1(F;\mathbb{Z}))$. Every polynomially growing element has a unipotent power.

It is by now a well-known fact that free-by-cyclic groups with unipotent and polynomially growing monodromies admit acylindrical graphs-of-groups splittings over abelian subgroups [Mac02, BFH05, Hag19, AHK24, AM22, DT24, KV25]. Furthermore the vertex groups of these splittings are themselves free-by-cyclic groups with unipotent monodromies whose polynomial growth is of strictly lower degree, thus giving rise to a hierarchical decomposition. Our aim is to exploit such splittings and induction on the degree of polynomial growth to construct finite quotients that separate conjugacy classes.

In the base case of linearly growing monodromy, by [AM22, Proposition 5.2.2] the corresponding free-by-cyclic group G splits as a graph of groups with vertex groups of the form $F_v \times \mathbb{Z}$ where F_v is a finite rank free group, and $\mathbb{Z}^2$-edge groups. The action on the Bass–Serre tree is 4-acylindrical.

Our strategy is to construct a virtually free quotient of G for a given pair of non-conjugate elements so that the images remain non-conjugate. Since virtually free groups are conjugacy separable, we may pass to a further finite quotient where the images are non-conjugate.

The virtually free quotients arise through a *vertex filling* procedure, which works by replacing each vertex group in the splitting of G by an appropriate finite quotient, resulting in a graph of *finite* groups with the same underlying graph as that of the original splitting of G, and such that G maps onto the fundamental group of the new graph of groups.

Standard arguments reduce to the case of non-conjugate elements of G which both act loxodromically on the Bass–Serre tree of the splittings with the same translation length and same sequence of double cosets representatives in their *short-position representatives* (see Section 5.2 for the definition). In this case, we must work harder to construct the required virtually free fillings.

A key tool in constructing virtually free fillings in this case is the property of *strong command* for independent elements of vertex groups. Two infinite-order elements $g_1, g_2 \in G$ are *independent* if the conjugacy class of the cyclic subgroup $\langle g_1 \rangle$ intersects the conjugacy class of $\langle g_2 \rangle$ in exactly the trivial subgroup. The group G *strongly commands* independent elements g_1 and g_2 if there is a finite quotient of G such that the images of g_1 and g_2 have prescribed orders and their cyclic subgroups have trivial intersection. Bridson–Wilton showed that a (virtually) free group strongly commands any tuple of independent elements [BW15, Theorem 4.3], and thus the same holds true for groups of the form $\mathbb{F}_v \times \mathbb{Z}$. Being able to ensure that independent elements have images in finite groups whose orders have prescribed divisors and that generate trivially intersecting subgroups is crucial to the combinatorial arguments in the linear growth case. It is also interesting to note that while strong command is typically used to construct finite index subgroups of graphs of groups, which we do, we will we also use this property to construct vertex fillings.

In the case of superlinear growth, the corresponding free-by-cyclic group G splits as a graph of groups with infinite cyclic edge groups and vertex groups which are free-by-cyclic with unipotent and polynomially growing monodromy of strictly lower degree growth. The action on the Bass–Serre tree is 2-acylindrical. In this case, we may apply the Combination Theorem of Wilton–Zalesskii [WZ10] (see Theorem 7.1) and argue by induction.

One of the hypotheses in the theorem of Wilton–Zalesskii is that edge groups are double coset separable in the vertex groups. Thus, we show the following:

Theorem C. *Let G be a free-by-cyclic group with polynomially growing monodromy. Then, for any cyclic subgroups $H, K \leq G$ and $g \in G$, the double coset $HgK \subseteq G$ is separable.*

As before, we prove Theorem C by constructing virtually free quotients via vertex fillings. This time, we need good control over the orders and the pairwise intersections of cyclic subgroups in the image of the vertex groups. In the case that the vertex groups split as products $\mathbb{F}_v \times \mathbb{Z}$, we can use strong command of the fibre F_v as in the previous arguments. We are not able to prove command for all free-by-cyclic groups with unipotent and polynomially growing monodromies, however we show the following, which might be of independent interest:

Proposition D. *Let G be a free-by-cyclic group with unipotent and polynomially growing monodromy. Let $S \subseteq G$ be a collection of non-trivial elements and let $g, h \in G$ be such that $\langle g \rangle \cap \langle h \rangle = 1$ or $[g, h] \neq 1$. There exists $N = N(S)$ such that for every prime $p > N$ there is a p-periodic quotient of G such that the image of each element of S is non-trivial and the cyclic subgroups generated by g and h have trivial intersection.*

The key tool for proving Theorem D is the observation that any free-by-cyclic group with unipotent and polynomially growing monodromy is residually torsion-free nilpotent, and thus residually p-finite for every prime p. We note that it is known that free-by-cyclic groups are *virtually* residually p-finite for every prime p by [AF13, Corollary 4.32].

Theorem D differs from strong command in that we are no longer able to prescribe arbitrary divisors to the orders of the images of element in our independent set. This proposition is therefore not suitable for the linear growth case, but it is good enough for the inductive step. This fits into a typical pattern, starting with [Mac02], where the linear growth and the superlinear growth cases requires substantially different techniques, the latter case being less pathological.

1.2. **Structure of the paper.** In Section 2 we give the necessary background on group actions on trees, graphs of groups, profinite topologies on groups, automorphisms of free groups, and free-by-cyclic groups.

In Section 3 we prove that unipotent polynomially growing free-by-cyclic groups are residually torsion-free nilpotent (Theorem 3.12) and deduce Theorem D.

In Section 4 we construct virtually free vertex fillings for cyclic splittings of free-by-cyclic groups. We also prove general results about conjugacy distinguished elements and cyclic subgroups in graphs of groups that admit virtually free vertex fillings.

In Section 5 we specialise our study of vertex fillings to the case of unipotent linear monodromy. This section uses many results from [DT24]. We conclude this section by proving Theorem A in the special case of a free-by-cyclic group with unipotent and linearly growing monodromy (Theorem 5.17).

In Section 6 we prove Theorem C.

Finally, in Section 7 we prove Theorem A by combining Theorems 5.17 and C with work of Wilton–Zalesskii [WZ10] and Chagas–Zalesskii [CZ10].

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2. Preliminaries

We begin by establishing conventions that will be used throughout the paper. For $g, h \in G$, we write $\mathrm{ad}_g(h) = g^h = h^{-1}gh$.

2.1. **Actions on trees.** Let T be a simplicial tree and define a metric d_T on T such that each edge is isometrically identified with the unit interval and the distance between two points in T is the length of the shortest path between them. Let $\mathrm{Isom}(T)$ be the group of isometries of (T, d_T).

For any element $g \in \mathrm{Isom}(T)$, the *translation length* of g is

$$\ell_T(g) := \inf\{d_T(x, g\cdot x) \mid x \in X\}.$$

It is classical that if $\ell_T(g) = 0$ then g fixes a point in T, and otherwise there exists a unique line in T called the *axis* of g on which g acts as translation by $\ell_T(g)$. If $g \in \mathrm{Isom}(T)$ fixes a point then it is called *elliptic* and otherwise it is *hyperbolic.*

The action of $G \leqslant \mathrm{Isom}(T)$ on T is said to be κ*-acylindrical* if the pointwise stabiliser of any edge path of length $\kappa + 1$ is trivial. We say that the action of G on T is *acylindrical* if it is κ-acylindrical for some non-negative integer κ.

2.2. **Graphs of groups, Bass–Serre theory.** A *(combinatorial) graph* X consists of a tuple of sets $(V(X), E(X))$ where $V(X)$ is called the *vertex set*, and $E(X)$ the *edge set*, together with a pair of maps $\iota{:}\, E(X) \to V(X)$ and $\tau{:}\, E(X) \to V(X)$, and a fixed-point-free involution $^-: E(X) \to E(X)$, such that for every edge $e \in E(X)$ we have that $\iota(e) = \tau(\bar{e})$.

A *graph of groups* $\mathcal{G}$ is a triple $(X, \mathcal{G}_\bullet, \iota_\bullet)$ where X is a graph, $\mathcal{G}_\bullet$ encodes the assignment of a group $\mathcal{G}_v$ to every vertex $v \in V(X)$ and a group $\mathcal{G}_e$ to every edge $e \in E(X)$ so that $\mathcal{G}_e = \mathcal{G}_{\bar{e}}$, and $\iota_\bullet$ determines monomorphisms $\iota_e{:}\, \mathcal{G}_e \hookrightarrow \mathcal{G}_{\iota(e)}$ for all edges $e \in E(X)$. We also define the monomorphism $\tau_e{:}\, \mathcal{G}_e \to \mathcal{G}_{\tau(e)}$, to be $\tau_e = \iota_{\bar{e}}$ for each $e \in E(X)$. Throughout the paper we will assume that the underlying graph X is finite.

Let $\mathcal{G}$ be a graph of groups and denoting by $F_{E(X)}$ the free group with basis $E(X)$. The *Bass group* $\mathrm{Bass}(\mathcal{G})$ is defined to be

$$\mathrm{Bass}(\mathcal{G}) = \left(\left(\mathop{\ast}_{v\in V(X)} \mathcal{G}_v\right) \ast F_{E(X)}\right) \Big/ \langle\!\langle e\bar{e} = 1,\ \bar{e}\iota_e(g)e = \tau_e(g)\rangle\!\rangle\ .$$

For a vertex $v \in V(X)$, the *fundamental group of* $\mathcal{G}$ *based at* v, denoted by $\pi_1(\mathcal{G}, v)$, is the subgroup of $\mathrm{Bass}(\mathcal{G})$ given by the elements of the form

$$a_0 e_1 a_1 e_2 \cdots e_n a_n$$

where $e_i \in E(X)$, $a_i \in \mathcal{G}_{\tau(e_i)} = \mathcal{G}_{\iota(e_{i+1})}$ (when the indices occur) and the edge path $e_1 \cdots e_n$ is a closed loop based at v in the graph X.

For any vertex $v \in V(X)$, the fundamental group $\pi_1(\mathcal{G}, v)$ sits as a free factor inside $\mathrm{Bass}(\mathcal{G})$ and for any two vertices $v, w \in V(X)$, the subgroups $\pi_1(\mathcal{G}, w)$ and $\pi_1(\mathcal{G}, v)$ are conjugate in $\mathrm{Bass}(\mathcal{G})$. We call an element $g \in \mathrm{Bass}(\mathcal{G})$ a $\mathcal{G}$*-loop* if it is an element of $\pi_1(\mathcal{G}, v)$ for some $v \in V(X)$. We will sometimes omit the basepoint and simply write $\pi_1(\mathcal{G})$ when there is no risk of confusion.

For any two graphs of groups $\mathcal{G} = (X_\mathcal{G}, \mathcal{G}_\bullet, \iota_\bullet)$ and $\mathcal{H} = (X_\mathcal{H}, \mathcal{H}_\bullet, \kappa_\bullet)$, a *morphism* $f{:}\, \mathcal{G} \to \mathcal{H}$ consists of the tuple

$$(f_X, \{f_v\}_{v\in V(X)}, \{f_e\}_{e\in E(X)}, \{\gamma_e\}_{e\in E(X)})$$

where $f{:}\, X_\mathcal{G} \to X_\mathcal{H}$ is a morphism of graphs, each $f_v{:}\, \mathcal{G}_v \to \mathcal{H}_{f(v)}$ for $v \in V(X_\mathcal{G})$ and $f_e{:}\, \mathcal{G}_e \to \mathcal{H}_{f(e)}$ for $e \in E(X_\mathcal{G})$ is a homomorphism, and $f_e = f_{\bar{e}}$, and $\gamma_e \in \mathcal{G}_{f(e)}$ for every $e \in E(X_\mathcal{G})$. We also require that for every $e \in E(X_\mathcal{G})$,

$$f_{\iota(e)} \circ \iota_e = \mathrm{ad}_{\gamma_e} \circ \iota_{f(e)} \circ f_e.$$

A morphism of graphs of groups $f{:}\, \mathcal{G} \to \mathcal{H}$ induces a homomorphism of the corresponding fundamental groups $f_*{:}\, \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{H}, f(v))$ for any $v \in V(X_\mathcal{G})$ [Bas93, Proposition 2.4].

Let $X_{\mathcal{G}}$ be a graph and fix $v_0 \in V(X_{\mathcal{G}})$. The universal cover of a graph of groups $\mathcal{G} = (X_{\mathcal{G}}, \mathcal{G}_\bullet, \iota_\bullet)$ is the graph $T = T_{\mathcal{G}}$ with vertices

$$V(T) = \coprod_{v \in V(X_{\mathcal{G}})} \pi_1(\mathcal{G}, v_0)/\mathcal{G}_v,$$

edges

$$E(T) = \coprod_{e \in E(X_{\mathcal{G}})} \pi_1(\mathcal{G}, v_0)/\iota_e(\mathcal{G}_e),$$

and the adjacency map ι given by inclusions of cosets. The graph T comes equipped with an action of $\pi_1(\mathcal{G}, v_0)$ with cell stabilisers conjugates of the groups in $\mathcal{G}_\bullet$.

2.3. **Profinite topology.**

Definition 2.1. Let G be a discrete group and let $\{N_i\}_{i \in J}$ be the family of finite index normal subgroups of G. The *profinite completion* $\widehat{G}$ *of* G is the inverse limit of the inverse system $(G/N_i)_{i \in J}$,

$$\widehat{G} := \varprojlim_{i \in J} G/N_i.$$

There is a natural map $\iota\colon G \to \widehat{G}$ that sends each element g to the tuple $(gN_i)_{i \in J}$. The *profinite topology* is the coarsest topology on G so that the map $\iota\colon G \to \widehat{G}$ is continuous. Equivalently, it is the topology generated by a basis of open subsets consisting of cosets of finite index normal subgroups of G.

A subset $X \subseteq G$ is *separable* if it is closed in the profinite topology. Equivalently, for any element $g \in G \smallsetminus X$, there exists a finite quotient $\pi\colon G \to Q$ such that $\pi(g) \notin \pi(X)$.

A subgroup $H \leq G$ is *fully separable*, if every finite index subgroup $H' \leq_f H$ is separable in G. We say that G *induces the full profinite topology on* H if the closure of H in the profinite completion of G is isomorphic to $\widehat{H}$.

We will often use the following lemma:

Lemma 2.2 (Reid [Rei15, Lemma 4.6])**.** *Let G be a finitely generated group and $H \leq G$ a finitely generated subgroup. Then H is fully separable in G if and only if G induces the full profinite topology on H.*

A group G is *conjugacy separable* if the conjugacy class of any element is separable in G. We say that an element $g \in G$ is *conjugacy distinguished* if the conjugacy class of g is separable.

Theorem 2.3 (Stebe [Ste70], Dyer [Dye79])**.** *Let G be a finitely generated virtually free group. Then G is conjugacy separable.*

A group G is said to have the *unique roots property* if for any two elements $a, b \in G$ such that $a^n = b^n$ for some positive integer n, it follows that $a = b$. A subgroup $H \leq G$ of a torsion-free group is *root-closed* if for any $g \in G$ such that $g^n \in H$ for some positive integer n, it follows that $g \in H$.

Lemma 2.4 (Cotton-Barratt–Wilton [CBW12, Lemma 3.1])**.** *Let G be a finitely generated group with the unique roots property. If G contains a conjugacy separable finite index subgroup then G is conjugacy separable.*

Lemma 2.5. *Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be an acylindrical graph of groups such that the vertex groups have the unique roots property and the edge groups are root-closed. Then, $\pi_1(\mathcal{G})$ has the unique roots property.*

Proof. Let $G = \pi_1(\mathcal{G})$ and let T be the Bass–Serre tree corresponding to the splitting. Note that each vertex group has the unique roots property and thus is torsion free. Hence, G is torsion free.

Arguing as in [CBW12, Lemma 3.3], if $a^n = b^n$ for some positive integer n, then a and b are both hyperbolic or elliptic. If a and b are both hyperbolic then it must be the case that a and b have the same axis and $\ell_T(a) = \ell_T(b)$. Thus, the element ab^{-1} fixes an infinite line in T. Since the action of G on T is acylindrical, it follows that $ab^{-1} = e_G$ and $a = b$.

Suppose that both a and b are elliptic. If $\mathrm{Fix}(a) \cap \mathrm{Fix}(b) \neq \varnothing$, then $a, b \in \mathcal{G}_v$ for some vertex v and $a^n = b^n$ in $\mathcal{G}_v$. Hence, by the unique roots property of $\mathcal{G}_v$, it follows that $a = b$.

Suppose now that $\mathrm{Fix}(a) \cap \mathrm{Fix}(b) = \varnothing$. Let γ be the shortest path in T joining $\mathrm{Fix}(a)$ to $\mathrm{Fix}(b)$. Then $a^n = b^n$ must fix the path γ pointwise. Thus, a^n and b^n fix every edge in γ. Using root closure of edge groups, it follows that a and b fix every edge in γ. Hence, a and b fix a common vertex and we may argue as before. ☐

The group G is said to be *double coset separable*, if for any finitely generated subgroups $H, K \leq G$ and $g \in G$, the double coset $HgK \leq G$ is separable in G.

We will use the following lemma often throughout the text.

Lemma 2.6 (Niblo [Nib92, Proposition 2.2]). *Let G be a group and $H, K \leq G$ subgroups. Let $G' \leq_f G$ be a finite index subgroup and let $H' = H \cap G'$ and $K' = K \cap G'$. Then the double coset HK is separable in G if and only if $H'K'$ is separable in G'.*

Theorem 2.7 (Gitik–Rips [GR95]). *If G is a finitely generated free group, then G is double coset separable.*

Combining the previous two results we obtain that double coset separability also holds for virtually free groups.

Theorem 2.8 (Minasyan [Min23, Theorem 1.1]). *Let G be a residually finite group and let $H, K \leq G$ be subgroups. Suppose that for every finite index subgroup $H' \leq_f H$, we have that $H'K$ is separable in G. Then the intersection $\overline{H} \cap \overline{K}$ of the closures of H and K in the profinite completion of G is equal to $\overline{H \cap K}$.*

A finitely generated subgroup $H \leq G$ is *conjugacy distinguished*, if for any $g \in G$ which is not conjugate into H, there exists a finite quotient $\pi\colon G \to Q$ such that $\pi(g)$ is not conjugate into $\pi(H)$.

Theorem 2.9 (Ribes–Zalesskii [RZ16, Theorem A]). *Every finitely generated subgroup of a finitely generated virtually free group is conjugacy distinguished.*

Definition 2.10 (Wilton–Zalesskii [WZ10]). Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be a graph of groups. We say that the profinite topology on $\pi_1(\mathcal{G})$ is *efficient* if $\pi_1(\mathcal{G})$ is residually finite, every vertex and edge group is closed in the profinite topology on $\pi_1(\mathcal{G})$ and $\pi_1(\mathcal{G})$ induces the full profinite topology on each vertex and edge group.

Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be a graph of groups and suppose that the profinite topology on $G = \pi_1(\mathcal{G})$ is efficient. We write $\widehat{\mathcal{G}}$ to denote the graph of groups with underlying graph X, such that the vertex and edge groups are profinite completions of the corresponding groups in $\mathcal{G}$, and the edge inclusions $\widehat{\mathcal{G}}_e \hookrightarrow \widehat{\mathcal{G}}_{i(e)}$ are the natural maps induced by the edge inclusions in $\mathcal{G}$. Since the topology on G is efficient, the profinite completion $\widehat{G}$ of G is isomorphic to the profinite fundamental group of $\widehat{\mathcal{G}}$. Moreover, there is a simply-connected profinite graph, which we denote by $S(\widehat{G})$ and call the *profinite Bass–Serre tree*, which admits an action of $\widehat{G}$. The Bass–Serre tree corresponding to the splitting $G = \pi_1(\mathcal{G})$, which we denote here by $S(G)$, embeds as a dense subset of $S(\widehat{G})$. See [ZM88, Rib17] for further details about actions of profinite groups on profinite trees.

Definition 2.11. Let $\mathcal{G}$ be a graph of groups such that the profinite topology on $\pi_1(\mathcal{G})$ is efficient. Let $\widehat{\mathcal{G}}$ be the corresponding graph of profinite groups and $S(\widehat{G})$ the profinite Bass–Serre tree. We say that $\widehat{\mathcal{G}}$ is *profinitely κ-acylindrical* if every edge path of length $\kappa + 1$ in $S(\widehat{G})$ has trivial pointwise stabiliser for the action of $\widehat{G}$.

Lemma 2.12. *Let $\mathcal{G}$ be a 2-acylindrical graph of groups with efficient profinite topology. Suppose that for every vertex $v \in V(X)$ and incident edges e and f, the following conditions are satisfied:*

1. *the intersection $\mathcal{G}_e \cap \mathcal{G}_f$ is either trivial or $\mathcal{G}_e = \mathcal{G}_f$, and*
2. *the intersection of the closures of $\mathcal{G}_e$ and $\mathcal{G}_f$ in the profinite completion of $\mathcal{G}_v$ is given by $\bar{\mathcal{G}}_e \cap \bar{\mathcal{G}}_f = \widehat{\mathcal{G}_e \cap \mathcal{G}_f}$.*

Then, $\widehat{\mathcal{G}}$ is profinitely 2-acylindrical.

Proof. Our argument is modelled on the proof of [WZ10, Lemma 5.5].

Let ρ be a path of length 3 in $S(\widehat{G})$ that consists of the concatenation of the edges e_1, e_2 and e_3. We will show that the pointwise stabiliser of ρ in $\widehat{G}$ is trivial. After translating e_2 by an element of $\widehat{G}$, we may assume that e_2 is an element of $S(G)$. Let u be the initial point of e_2 and the final point of e_1. Since $e_2 \in S(G)$, we must have that u is also contained in $S(G)$. By assumption (1) the intersection of $\mathcal{G}_{e_1}$ and $\mathcal{G}_{e_2}$ is either trivial or $\mathcal{G}_{e_1} = \mathcal{G}_{e_2}$, so it follows from assumption (2) that $\bar{\mathcal{G}}_{e_1} \cap \bar{\mathcal{G}}_{e_2}$ is either trivial or is equal to $\widehat{\mathcal{G}}_{e_1} = \widehat{\mathcal{G}}_{e_2}$. Since $\mathcal{G}_u$ induces the full profinite topology on $\mathcal{G}_{e_1}$ and $\mathcal{G}_{e_2}$, we have that $\widehat{\mathcal{G}}_{e_1} \cap \widehat{\mathcal{G}}_{e_2}$ is either trivial, in the case that $\mathcal{G}_{e_1} \cap \mathcal{G}_{e_2} = 1$, or is equal to $\widehat{\mathcal{G}}_{e_1} = \widehat{\mathcal{G}}_{e_2}$, otherwise. Similarly, we have that $\widehat{\mathcal{G}}_{e_2} \cap \widehat{\mathcal{G}}_{e_3}$ is trivial if $\mathcal{G}_{e_2} \cap \mathcal{G}_{e_4} = 1$ or is equal to $\widehat{\mathcal{G}}_{e_2} = \widehat{\mathcal{G}}_{e_3}$.

If $\mathcal{G}_{e_1} \cap \mathcal{G}_{e_2}$ is non-trivial, then we must have that $\mathcal{G}_{e_2} \cap \mathcal{G}_{e_3} = 1$ by 2-acylindricity of $\mathcal{G}$ and by the first assumption. Then, $\widehat{\mathcal{G}}_{e_2} \cap \widehat{\mathcal{G}}_{e_3}$ is trivial by the argument above and thus $\widehat{\mathcal{G}}_{e_1} \cap \widehat{\mathcal{G}}_{e_2} \cap \widehat{\mathcal{G}}_{e_3} = 1$. On the other hand, if $\mathcal{G}_{e_1} \cap \mathcal{G}_{e_2}$ is trivial then $\widehat{\mathcal{G}}_{e_1} \cap \widehat{\mathcal{G}}_{e_2}$ is trivial and thus $\widehat{\mathcal{G}}_{e_1} \cap \widehat{\mathcal{G}}_{e_2} \cap \widehat{\mathcal{G}}_{e_3} = 1$. ☐

2.4. **Automorphisms of free groups.** Let F be a free group and X a free basis of F. An outer automorphism $\Phi \in \mathrm{Out}(F)$ acts on the set of conjugacy classes of elements in F. Given a conjugacy class $\bar{g}$ of an element $g \in F$, we write $|\bar{g}|_X$ to denote the word length of the shortest representative of the conjugacy class $\bar{g}$. We say that the conjugacy class $\bar{g}$ *grows polynomially of degree d under the iteration of Φ,* if there exist constants $A, B > 0$ such that for all $k \geq 1$

$$Ak^d - A \leq |\Phi^k(\bar{g})|_X \leq Bk^d.$$

For any two free generating sets S and S' of F, the word metrics with respect to S and S' are bi-Lipschitz equivalent. It follows that the growth of a conjugacy class under Φ does not depend on the specific choice of free basis for F.

We say the outer automorphism $\Phi \in \mathrm{Out}(F)$ *grows polynomially of degree d* if every conjugacy class in F grows polynomially of degree $\leq d$ and there exists an element $g \in F$ whose conjugacy class grows polynomially of degree exactly d.

An outer automorphism $\Phi \in \mathrm{Out}(F)$ is *unipotent* if Φ induces a unipotent element of $\mathrm{Out}(F_{\mathrm{ab}}) \cong \mathrm{GL}_n(\mathbb{Z})$. We will typically abbreviate *unipotent and polynomially growing* to *UPG*. An outer automorphism Φ is *neat* if for every $x \in F$ and every representative automorphism ϕ of Φ, if $\phi^k(x) = x$ for some $k \in \mathbb{Z} \smallsetminus 0$, then $\phi(x) = x$.

Lemma 2.13. *If $\Phi \in \mathrm{Out}(F)$ is UPG then it is neat.*

Proof. Apply the argument from the proof of [BFW23, Theorem 4.4] to the topmost splitting coming from an improved relative train track representative of a UPG element $\Phi \in \mathrm{Out}(F)$ [BFH00, Theorem 5.1.8]. □

2.5. **Free-by-cyclic groups.** A group G is *free-by-cyclic* if there exists an epimorphism $\chi\colon G \to \mathbb{Z}$ with $\ker \chi = F$, where F is a free group of finite rank. In this case, G splits as a semidirect product, $G \cong \mathbb{F} \rtimes_\phi \mathbb{Z}$ for $\mathbb{F} \cong F$ and $\phi \in \mathrm{Aut}(\mathbb{F})$. We will often abuse notation and write $G = F \rtimes_\phi \langle t\rangle$ where $t \in \chi^{-1}(1)$. Note that for any two elements of $\chi^{-1}(1)$, the corresponding automorphisms of F induced by the conjugation action represent the same element of $\mathrm{Out}(F)$. We call $F \trianglelefteq G$ the *fibre* and $\Phi := [\phi] \in \mathrm{Out}(F)$ the *monodromy* corresponding to the pair (G,χ). We will also often suppress χ from the notation.

Lemma 2.14. *Let $\phi \in \mathrm{Aut}(F)$ be a representative of a UPG outer automorphism Φ. Then $G = F \rtimes_\phi \langle t\rangle$ has the unique roots property.*

The proof is essentially [KV25, Lemma 3.3].

Proof. Let $n > 0$ and let $g, h \in G$ be such that $g^n = h^n$. Then there is some $k \in \mathbb{Z}$ such that $g = ut^k$ and $h = vt^k$ for $u, v \in F$. Let $\psi \in \mathrm{Aut}(F)$ be the automorphism induced by the conjugation action by g. Let $w = vu^{-1} \in F$. Then $h = wg$ and the equation $g^n = h^n$ evaluates to

$$g^n = g^n \psi^{n-1}(w)\dots\psi^2(w)\psi(w)w.$$

Hence, $\psi^{n-1}(w)\dots\psi^2(w)\psi(w)w = 1$ and thus $\psi^n(w) = w$. Note that ψ is in the outer automorphism class of Φ^k, and since Φ is UPG it follows that Φ^k is also UPG. Hence Φ^k is neat and so we must have that $\psi(w) = w$ and thus $w^n = 1$. Hence by the unique roots property of the free groups we have that $w = 1$ and so $g = h$. □

Lemma 2.15. *Let $\phi \in \mathrm{Aut}(F)$ be a representative of a UPG outer automorphism and $G = F \rtimes_\phi \langle t\rangle$. Let $H \leq G$ be a subgroup of the form $H = \langle vt\rangle$ or $H = K \oplus \langle vt\rangle$ for some $v \in F$, where $\mathbb{Z} \cong K \leq F$ is not generated by a proper power. Then H is root-closed.*

Proof. Suppose that there exists some $x \in G \smallsetminus 1$ such that $x^m \in H$ for some $m \in \mathbb{N}$.

We begin by considering the case where H is a cyclic subgroup. By replacing the automorphism ϕ with $\phi \circ \mathrm{ad}_v$, we may assume that $H = \langle t\rangle$. Let $x = ut^k$ for some $u \in F$ and $k \in \mathbb{Z}$. Since x is non-trivial and H is not a subgroup of F, it must be the case that $k \neq 0$ and

$$(ut^k)^m = t^{km}.$$

Then, by Theorem 2.14 it follows that $ut^k = t^k$ and thus $u = 1$. Hence $x \in H$.

Suppose now that $H = K \oplus \langle t\rangle$ where $\mathbb{Z} \cong K \leq F$ is not generated by a proper power. Let us first assume that $x \in F$. Then $x^m \in H \cap F = K$. Let $k \in K$ be a generator of K. Then x^m is a power of k. It follows that $x \in C_F(k) \cong \mathbb{Z}$ and thus k and x are powers of a common element $z \in F$. However, since k is not a proper power, it must be the case that $k = z^{\pm 1}$ and thus $x \in K \leq H$.

Suppose now that $x \notin F$. Then $x = ut^k$ for some $u \in F$ and $k \in \mathbb{Z} \smallsetminus 0$. Since $x^m \in H$, we have that

$$w := u\phi^k(u)\dots\phi^{(m-1)k}(u) \in K.$$

Then $\phi^k(w) = u^{-1}w\phi^{mk}(u)$, and also $\phi^k(w) = w$ since t centralises K. Hence

$$t^{-mk}ut^{mk} = \phi^{mk}(u) = w^{-1}uw.$$

Now, one checks that $t^{mk}w^{-1} = (t^ku^{-1})^m$ and thus

$$(t^ku^{-1})^{-m}u(t^ku^{-1})^m = u.$$

Hence, setting $\psi \in \mathrm{Aut}(F)$ to be the automorphism induced by the conjugation action of $t^k u^{-1}$ on F, we get that $\psi^m(u) = u$. Then, by neatness we must have that $\psi(u) = u$, and thus $\phi^k(u) = u$. Hence, $u^m \in K$ and by the argument above we must have that $u \in K$. Hence $ut^k \in H$. □

Note that if G is free-by-cyclic then $\mathrm{cd}_{\mathbb{Z}}(G) = 2$ and thus if $H \leq G$ is abelian then H is isomorphic to one of $1, \mathbb{Z}$, or $\mathbb{Z}^2$.

Lemma 2.16. *Let G be free-by-cyclic. Let $H, K \leq G$ be non-trivial abelian subgroups of G. Then there exists a finite index subgroup $G' \leq G$ such that $G' = F' \rtimes \langle s \rangle$ and for each $J \in \{H, K\}$, the following holds.*

1. *If $J \cong \mathbb{Z}$ then $J \cap G' = \langle vs \rangle$ for some $v \in F'$, or $J \cap G' = \langle u \rangle$ where $u \in F'$ is primitive.*
2. *If $J \cong \mathbb{Z}^2$ then $J \cap G' = \langle u, vs \rangle$ for some $v \in F'$, where $u \in F'$ is primitive.*

Moreover, we can pick the fibre F' to be a subgroup of any given fibre of F, and the monodromy to be a power of the corresponding monodromy in G.

Proof. Let $G = F \rtimes \langle t \rangle$ be a free-by-cyclic splitting and let $\chi\colon G \to \mathbb{Z}$ be the character corresponding to the splitting. For each $J \in \{H, K\}$, let $x_J \in F$ denote the (possibly trivial) generator of $J \cap F$. By [DT24, Lemma 6.5], there exists a finite index normal subgroup $F' \trianglelefteq_f F$ such that if $x_J \neq 1$ then $\langle x_J \rangle \cap F'$ is a free factor of F'.

Now for each $J \in \{H, K\}$, let $\mathcal{O}_J = |\mathbb{Z}/\chi(J)|$ and define

$$\mathcal{O} := \mathrm{lcm}\{\mathcal{O}_H, \mathcal{O}_K, [F : F']!\}.$$

Let $s = t^{\mathcal{O}}$. Then, $(F')^s = F'$ and we can set $G' := \langle F', t^s \rangle \cong F' \rtimes \langle s \rangle \leq_f G$. It follows that $J \cap F'$ is trivial or primitive, and if $\chi(J) \neq 0$, then there exists some $v \in F'$ such that $vs \in J$. □

We end this section by recording the graph-of-groups splittings for free-by-cyclic groups with polynomially growing monodromies that will be used throughout the paper.

Theorem 2.17 (see [DT24, §3.1])**.** *Let $\phi \in \mathrm{Aut}(F)$ be a representative of a unipotent and linearly growing outer automorphism and let $G = F \rtimes_\phi \langle t \rangle$ be the corresponding free-by-cyclic group. Then $G \cong \pi_1(\mathcal{G}, v_0)$ where $\mathcal{G}$ is a graph of groups with a bipartite underlying graph $(X, V_0(X), V_1(X))$ that satisfies the following properties.*

1. *For any vertex $b \in V_0(X)$, the group $\mathcal{G}_b$ is a maximal subgroup of G of the form $F_b \oplus \langle t_b \rangle$ where $F_b \leq F$ is a maximal cyclic subgroup and $t_b \in Ft$. We call the vertices in $V_0(X)$* black vertices.
2. *For any vertex $w \in V_1(X)$, the group $\mathcal{G}_w$ is maximal subgroup of the form $F_w \oplus \langle t_w \rangle$, where F_w is a finitely generated non-abelian subgroup of F and $t_w \in Ft$. We call the vertices in $V_0(X)$* white vertices. *The subgroup F_w is called the* local *fibre of $\mathcal{G}_w$ and t_w the* central element.
3. *Edge groups are isomorphic to maximal $\mathbb{Z}^2$ subgroups of G and map surjectively onto vertex groups in $V_0(X)$.*
4. *The action of G on the Bass–Serre tree corresponding to the splitting $G \cong \pi_1(\mathcal{G}, v_0)$ is 4-acylindrical.*

The following proposition (modulo the action being 2-acylindrical) is well known to experts and can be found in [Mac02], [BFH05, Theorem 4.22], [Hag19], and [AHK24, Proposition 2.5]. A proof that the action is 2-acylindrical can be found in [KV25, Lemma 5.2].

Proposition 2.18. *Let $\phi \in \mathrm{Aut}(F)$ be a representative of a unipotent and linearly growing outer automorphism and let $G = F \rtimes_\phi \langle t \rangle$ be the corresponding free-by-cyclic group. Then G admits a 2-acylindrical splitting $G \cong \pi_1(\mathcal{G})$. The vertex* *groups $\mathcal{G}_v$ are of the from $\mathcal{G}_v = F_v \rtimes_{\phi_v} \langle t_v \rangle$ for $F_v \leq F$ finitely generated, $t_v \in Ft$, and $\phi_v: F_v \to F_v$ an automorphism that represents a unipotent and polynomially growing outer automorphism of strictly lower degree growth. Moreover, each edge group is of the form $\mathcal{G}_e = \langle t_e \rangle$ where $t_e \in Ft$.*

Definition 2.19 (Standard splitting)**.** We will call the graph of groups splittings in Theorem 2.17 and Theorem 2.18 the *standard splitting* for (G, χ).

3. Residual nilpotency and periodic quotients

A group G is *residually (torsion-free) nilpotent* if for every non-trivial element $g \in G$ there exists a homomorphism $\alpha_g: G \to N$ such that N is (torsion-free) nilpotent and $\alpha_g(g)$ is non-trivial. In [BNB21, Proposition 7.8] it is shown that that all free-by-cyclic are virtually residually nilpotent (see also [AF13]). In this section we will show that free-by-cyclic groups with and polynomially growing monodromies are residually nilpotent.

For this section let F denote a finite rank free group and let ϕ be a polynomially growing automorphism in $\mathrm{Aut}(F)$. We denote the commutator of two elements by $[x, y] = x^{-1}y^{-1}xy$ and the commutator of two subgroups by

$$[A, B] = \langle [a, b] : a \in A, b \in B \rangle.$$

We set $\gamma_1 F = F$ and inductively define the terms of the *lower central series* $\gamma_{n+1} = [\gamma_n F, F]$. We say an element $g \in F$ has *weight* n, denoted $\mathrm{wt}(g) = n$ if and only if $g \in \gamma_n F$.

Lemma 3.1. *Let a, b, c be elements of a group. The following conclusions hold:*

(1) $[ab, c] = [b, [a, c]][a, c][b, c]$;
(2) $[a, b]^{-1} = [b, a]$;
(3) $[c, ab] = [c, b][c, a][[a, c], b]$;
(4) $\mathrm{wt}([a, b]) \geq \mathrm{wt}(a) + \mathrm{wt}(b)$;
(5) $\mathrm{wt}(ab) = \min(\mathrm{wt}(a), \mathrm{wt}(b))$.

Repeatedly applying this lemma, and using the fact that elements of weight n commute modulo $\gamma_{n+1} F$ we have.

Corollary 3.2. *Let $\mathrm{wt}(a_i) \geq n'$ and $\mathrm{wt}(b_i) \geq n''$ for $i = 0, \ldots, m$ then*

$$[a_0 a_1 \cdots a_m, b_0 b_1 \cdots b_m] = \prod_{i=1}^{m} \prod_{j=1}^{m} [a_i, bj]$$

modulo $\gamma_{n+1} F$, where $n = n' + n''$.

The following is a classical result of Magnus [Mag35] on free groups.

Theorem 3.3 (Magnus)**.** *If F is a free group then F is residually torsion-free nilpotent,*

$$\bigcap_{i=1}^{\infty} \gamma_i F = \{1\}.$$

In particular for any finite set $S \subset F$ there is some $c(S)$ such that the set S is mapped injectively via the canonical quotient $F/\gamma_{c(S)} F$.

Since $\gamma_n F$ is characteristic in F the automorphism ϕ descends to an automorphism $\bar{\phi}_n$ of $F/\gamma_n F$. We also have that $\gamma_n F \leq F \rtimes_\phi \mathbb{Z}$ is a normal subgroup and that

$$(F \rtimes_\phi \mathbb{Z})/\gamma_n F \cong (F/\gamma_n F) \rtimes_{\bar{\phi}_n} \mathbb{Z}$$

is polycyclic. Wolf's Theorem [Wol68] asserts that every polycyclic group is either virtually nilpotent or has exponential growth. We shall use the following which is actually a result of the proof of [DK18, Proposition 14.28].

Proposition 3.4 (see proof of [DK18, Proposition 14.28]). *The semidirect product* $(F/\gamma_n F)\rtimes_{\bar\phi_n}\langle t\rangle$ *is nilpotent if all the eigenvalues of the natural induced linear map*

$$\phi_n : \gamma_n F/\gamma_{n+1}F \to \gamma_n F/\gamma_{n+1}F$$

are precisely 1, i.e. each ϕ_n *is unipotent.*

Let $\phi\in \mathrm{Aut}(X)$ be unipotent, then we say that an ordered basis $X=(x_1,\dots,x_r)$ is *ϕ-ordered* if $\phi(x_i)=x_iW_i(x_{i+1},\dots,x_r)$ where $W_i(x_{i+1},\dots,x_r)$ denotes a word in $\{x_{i+1},\dots,x_r\}^{\pm1}$. We remark that F may not have a ϕ-ordered basis even though ϕ is unipotent, we address this issue in the proof of Theorem 3.12.

Definition 3.5. [Hal59, §11.1] Let $X=(x_1,\dots,x_r)$ be an ordered basis of F. The *basic commutators of F* form an ordered subset $(\mathcal{A},\le)\subset F$ consisting of all possible elements in F that satisfy the following properties:

(1) Either $c\in X$ or $c=[c',c'']$ where $c',c''\in\mathcal{A}$.
(2) The order $\le$ satisfies the following properties:
 (a) If $c_1,c_2\in\mathcal{A}$ and $\mathrm{wt}(c_1)>\mathrm{wt}(c_2)$ then $c_1>c_2$.
 (b) For elements in X we have $x_i\le x_j \Leftrightarrow i\le j$.
(3) If $c=[c',c'']\in\mathcal{A}$ then we must have
 (a) $c'>c''$, and
 (b) $c''\ge (c')''$, where $c'=[(c')',(c')'']$.
(4) If $\mathrm{wt}([c_1',c_1''])=\mathrm{wt}([c_2',c_2''])\ge 2$ then

$$[c_1',c_1'']\ge[c_2',c_2'']\Leftrightarrow\begin{cases}c_1''>c_2'', \text{ or}\\ c_1''=c_2'' \text{ and } c_1'\ge c_2''\end{cases}$$

The "anti-lexicographic ordering" Property (4) in Definition 3.5 is not standard, usually we are free to order the basic commutators any way we like within a weight class, but this specific ordering will be crucial to the results of this section. This terminology is also abusive since while the elements of X cannot be commutators, they are still *basic* commutators.

The *collection process* is a rewriting process that takes a given word $w=x_{i_1}\cdots x_{i_l}\in F$ and iteratively rewrites it as a product $w=c_1^{n_1}c_2^{n_2}\cdots$ with $c_i<c_{i+1}$ and $n_j\in\mathbb{Z}$, by iteratively taking the $\le$-minimal basic commutator that is "out of position" and migrating it to the left into position. Since $yx=xy[x,y]$, doing so inserts commutators, but if at each step we only move $\le$-minimal "out of position" commutators then all new commutators will be basic. If we work modulo γ_nF then this process will terminate since we can ignore high weight commutators.

Let $\mathcal{A}_n=\{c\in\mathcal{A}:\mathrm{wt}(c)=n\}$ and denote by $(\mathcal{A}_n,\le)$ the set $\mathcal{A}_n$ ordered by the basic commutator ordering. The following result, in particular, motivates the use of the term *basic*.

Theorem 3.6 (Basis Theorem [Hal59, Theorem 11.2.4]). *The set* $\mathcal{A}_n$ *of basic commutators of weight* n *maps bijectively to a basis of the free abelian group* $\gamma_nF/\gamma_{n+1}F$ *via the map*

$$c\mapsto c\gamma_{n+1}F.$$

Lemma 3.7 (see [Hal59, §11.1]). *Let* $v,u\in\mathcal{A}$ *and suppose* $[v,u]\in\mathcal{A}$. *Let* $v_0=v$ *and* $v_{i+1}=[v_i,u], i=0,1,2,3,\dots$. *Let* $w_1=[v,u]$ *and* $w_{t+1}=[w_t,v]$. *Then all* $v_i,w_t\in\mathcal{A}$ *and we have:*

$$\begin{aligned}vu&=uv[v,u]\\ vu^{-1}&=u^{-1}vv_2v_4\cdots v_3^{-1}v_1^{-1} &&=u^{-1}vW_{v,u^{-1}}[v,u]^{-1}\\ v^{-1}u&=uv^{-1}w_2w_4\cdots w_3^{-1}w_1^{-1} &&=uv^{-1}W_{v^{-1},u}[v,u]^{-1}\\ v^{-1}u^{-1}&=u^{-1}v_1v_3\cdots v_4^{-1}v_2^{-1}v^{-1} &&=u^{-1}[v,u]W_{v^{-1},u^{-1}}v^{-1},\end{aligned}$$

modulo $\gamma_n F$ *for all* $n > 0$. *Also* $\mathrm{wt}(W_{u^{\pm 1},v^{\pm 1}}) > \mathrm{wt}([v,u])$.

The factors $W_{v^{\pm 1},u^{\pm 1}}$ will be called W-factors. We will now extend the "antilexicographic" order $\leq$ on basic commutators to commutators of the form $[c', c'']$ where $c', c'' \in \mathcal{A}^{\pm 1}$ naturally as follows: firstly $c \leq c^{-1}$ and $c^{-1} \leq c$ and secondly if $\mathrm{wt}(c_1) = \mathrm{wt}(c_2)$ then $c_1 = [c_1', c_1''] \leq c_2 = [c_2', c_2'']$ if and only if either $c_1'' < c_2''$ or $c_1'' = c_2''$ and $c_1' \leq c_2'$.

The following three lemmas show how to rewrite "badly formed" commutators into products of $\leq$-larger basic commutators and their inverses.

Lemma 3.8. *Let* $c_0 = [a_0, b_0] \in \mathcal{A}$, *let* $y_i, y_j \in \mathcal{A}$ *and let* $c_{ij} = [y_i^{\epsilon_i}, y_j^{\epsilon} j]$, *where* $\epsilon_1, \epsilon_2 \in \{-1, 1\}$, *with* $c_0 < c_{ij}$ *in the extended ordering. Suppose* $\mathrm{wt}(c_0) = \mathrm{wt}(c_{ij}) = n$ *and suppose* $\mathrm{wt}(a_0) = \mathrm{wt}(b_0) = \mathrm{wt}(y_i) = \mathrm{wt}(y_j) = n/2$. *If* $c_{ij} \notin \mathcal{A}$ *then there is basic commutator* $b_{ij} \in \mathcal{A}$ *with* $c_0 < b_{ij}$ *such that*

$$c_{ij} = b_{ij}^{\epsilon'} \mod \gamma_{n+1} F,$$

for some $\epsilon' \in \{-1, 1\}$.

Proof. There are 4 cases to consider. In all cases, due to our hypotheses on weights we always have $y_j > y_i''$, where $y_i = [y_i', y_i'']$. In particuar $[y_i, y_j]$ is a basic commutator precisely when $y_i > y_j$. In all cases, even if $[y_i, y_j]$ is not a basic commutator, we still have $y_j \geq b_0$ and $y_i \geq a_0$. In particular, even if $y_j > y_i$, we still have $y_i \geq a_0 > b_0$.

Case 1: $\epsilon_i, \epsilon_j = 1$. In this case if $[y_i, y_j]$ is already a basic commutator there is nothing to show. If $[y_i, y_j]$ is not a basic commutator then $y_i < y_j$. Thus $[y_i, y_j] = [y_j, y_i]^{-1}$ is the inverse of basic commutator and as explained above $y_i > b_0$ so $[a_0, b_0] < [y_j, y_i]$.

Case 2 $\epsilon_i = 1, \epsilon_j = -1$. We have $[y_i, y_j^{-1}]$. We will use the collection process to express this as a product of basic commutators. Consider first the case where $y_i > y_j$ (so that $[y_i, y_j]$ is a basic commutator). We will start by migrating y_j symbols left and will be repeatedly using Lemma 3.7:

$$[y_i, y_j^{-1}] = y_i^{-1} y_j y_i y_j^{-1} = y_i^{-1} \cancel{y_j y_j^{-1}} y_i \cancel{W_{y_i, y_j^{-1}}} [y_i, y_j]^{-1} = [y_i, y_j]^{-1}.$$

Note that here we can cancel the W-factors since they will have weight $n + 1$ or more. Which, as seen in Case 1, is the inverse of a basic commutator that is greater than c_0. The next possibility is $b_0 < y_i < y_j$, this time we will start by migrating y_i and simply ignore W-factors and immediately cancel all commutators of weight more than n:

$$[y_i, y_j^{-1}] = y_i^{-1} y_j y_i y_j^{-1} = \cancel{y_i^{-1} y_i} y_j [y_j, y_i] y_j^{-1} = \cancel{y_j y_j^{-1}} [y_j, y_i] \cancel{[[y_j, y_i], y_j^{-1}]} = [y_j, y_i].$$

Again, in this case, $c_0 = [a_0, b_0] < [y_j, y_i]$.

Case 3 $\epsilon_i = 1, \epsilon_j = -1$. Calculations completely analogous to those in Case 2 will rewrite the commutator as a basic commutator or the inverse of a basic commutator that is greater than c_0.

Case 4 $\epsilon_i = \epsilon_j = -1$. We consider first the case where $y_i > y_j$. And proceed as before

$$\begin{aligned}[y_i^{-1}, y_j^{-1}] &= y_i y_j y_i^{-1} y_j^{-1} \\ &= y_i \cancel{y_j y_j^{-1}} [y_i, y_j] \cancel{W_{y_i^{-1}, y_j^{-1}}} y_i^{-1} = \cancel{y_i y_i^{-1}} \cancel{[[y_i, y_j], y_i^{-1}]} = [y_i, y_j].\end{aligned}$$

Which as we've seen before will be greater than c_0. The case $y_j > y_i > x_j$ is handled similarly. □

Lemma 3.9. *Let* $a, b, c \in \mathcal{A}$ *with* $a < b$, $c < a$ *and* $[c, a] > b$ *so that* $[[c, a], b]$ *is a basic commutator, say of weight* n. *Then we have*

(1) $[[c, a]^{-1}, b] = [[c, a], b]^{-1}$

(2) $[[c,a],b^{-1}] = [[c,a],b]^{-1}$
(3) $[[c,a]^{-1},b^{-1}] = [[c,a],b]$,

modulo $\gamma_{n+1}F$.

Proof. Consider the first equation. Noting the weight of the W-factors we have:

$$[[c,a]^{-1},b] = [c,a]b^{-1}[c,a]^{-1}b = [c,a]\cancel{b^{-1}b}[c,a]^{-1}W_{[c,a]^{-1},b}[[c,a],b]^{-1}$$
$$= [[c,a],b]^{-1}\left(\underbrace{W_{[c,a]^{-1},b}[W_{[c,a]^{-1},b},[[c,a],b]^{-1}]}_{\mathrm{wt}>n}\right)$$

The other equations follow similarly. □

Lemma 3.10. *Let* $a,b,c,[c,b]\in\mathcal{A}$ *with* $[c,b]>a$ *but* $b>a$ *so that* $[[c,b],a]$ *is not a basic commutator then if* $\epsilon_1,\epsilon_2\in\{-1,1\}$*,* $q=[[c,b]^{\epsilon_1},a^{\epsilon_2}]$ *can be rewritten as a product*

$$q=[[c,b]^{\epsilon_1},a^{\epsilon_2}]=b_1\cdots b_s \mod \gamma_{n+1}F$$

of (possibly repeated) basic commutators of weight n*, where* $\mathrm{wt}(q)=n$*, such that* $q<b_i, i=1,\dots,s$*.*

Proof. The hypotheses imply that $c>b>a$ and that $[[c,a],b]$ is a basic commutator. For this proof, we will be using the following terminology: we will say that basic commutators $q_1<q_2$ are *commutable* if $[q_2,q_1]\in\mathcal{A}$. If $q_1<q_2$ are not commutable then we will say q_1 *explodes* q_2 and if $q_2=[q_2',q_2'']$ we will say that q_2',q_2'' *are the debris* of the explosion. Explosions occur in the collection process when we have a subword $[q_2',q_2'']q_1$ with $q_1<[q_2',q_2'']$ and $q_2''>q_1$. Then the basic commutator $[q_2',q_2'']$ must be replaced by the product $q_2'^{-1}q_2''^{-1}q_2'q_2''$ of basic commutators and their inverses. The following is immediate from definitions, but central to the argument of the proof.

Fact: *If* q_1 *explodes* q_2 *then* $q_1<q_2',q_2''$ *where* q_2',q_2'' *is the debris of the explosion.*

Let us first consider the case where $\epsilon_1=\epsilon_2=1$ and where $[c,a],[b,a]\in\mathcal{A}$. We turn the non-basic commutator into a product of basic commutators by expanding it and applying the collection process

$$[[c,b],a]=[c,b]^{-1}a^{-1}[c,b]a$$

We see here that $[c,b]$ and a are not commutable, which means we must explode $[c,b]$:

$$[c,b]^{-1}a^{-1}[c,b]a=[c,b]^{-1}a^{-1}c^{-1}b^{-1}cba.$$

The smallest basic commutator is a so we migrate it to the left until it cancels with its inverse. We underline it to aid in keeping track of the process

$$\begin{aligned}
&[c,b]^{-1}a^{-1}c^{-1}b^{-1}cb\underline{a}\\
&= [c,b]^{-1}a^{-1}c^{-1}b^{-1}\underline{a}c[c,a]b[b,a]\\
&= [c,b]^{-1}a^{-1}c^{-1}\underline{a}b^{-1}W_{b^{-1},a}[b,a]^{-1}c[c,a]b[b,a]\\
&= [c,b]^{-1}\cancel{a^{-1}\underline{a}}c^{-1}W_{c^{-1},a}[c,a]^{-1}b^{-1}W_{b^{-1},a}[b,a]^{-1}c[c,a]b[b,a]\\
&= [c,b]^{-1}c^{-1}W_{c^{-1},a}[c,a]^{-1}b^{-1}W_{b^{-1},a}[b,a]^{-1}c[c,a]b[b,a].
\end{aligned}$$

Where from Lemma 3.7 we have

$$\begin{aligned}
W_{b^{-1},a} &= [[b,a],b]\cdot[[[[b,a],b],b],b]\cdots[[[b,a],b],b]^{-1}\\
W_{c^{-1},a} &= [[c,a],c]\left(\cancel{\cdots[[[c,a],c],c]^{-1}}\right).
\end{aligned}$$

where we cancel off the terms that clearly have weight greater than n. At this point in the collection process we have cancelled out all a symbols and we have a product of basic commutators that are all strictly greater than a.

Consider now the general case

$$[c,b]^{\epsilon_1}a^{\epsilon_2}[c,b]^{-\epsilon_1}a^{-\epsilon_2},$$

where we no longer assume that a is commutable with b or c. In all cases, the collection process will migrate the rightmost $a^{\pm1}$ to the left until it cancels with the other $a^{\mp1}$ symbol. Throughout the collection process basic commutators are created when a is commutable with its leftmost neighbour. Note that all created W-factors will have weight greater than a, or exploded, but the debris will remain strictly greater than a. It follows that once the a symbols cancel out, we will be left with a product of basic commutators that are all strictly greater than a.

Now the collection process will continue to rewrite the product. In doing so many basic commutators will be created, exploded, or cancelled out. Because of our fact about explosion debris, all basic commutators that will occur for the remainder of the collection process will remain strictly greater than a.

In the end no basic commutators of weight less than n will remain, furthermore in the final product of basic commutators, every basic comutator factor will be of the form $b_i = [b'_i, b''_i]$ where b'_i, b''_i are commutators that are strictly greater than a. In particular $b''_i > a$ so $b_i > [[c,b],a]$ as required. □

Lemma 3.11. *Let $\phi \in \mathrm{Aut}(F)$ be unipotent and let X be a ϕ-ordered basis of F that in turn induces the order $\leq$ on $\mathcal{A}$. If c is a basic commutator with* $\mathrm{wt}(c) = n$ *then*

$$\phi(c) = cc_1^{n_1}\cdots c_m^{n_m} \mod \gamma_{n+1}F$$

where $c < c_1 < \cdots c_m$ are basic commutators with the same weight as c, $n_i \in \mathbb{Z}$. In particular the induced the matrix representation of the induced linear map

$$\phi_n : \gamma_n F/\gamma_{n+1}F \to \gamma_n F/\gamma_{n+1}F$$

with respect to the ordered basis $(\mathcal{A}_n, \leq)$ is lower unitriangular.

Proof. We proceed by induction on $n = \mathrm{wt}(c)$. The base case is $n = 1$ where ϕ_1 is the induced automorphism of the abelianization of F. $C_1 = \{x_1, \dots, x_r\}$ and $x_i < x_j \Leftrightarrow i < j$. By definition of ϕ-ordered we immediately get

$$\phi(x_i) = x_i x_{i+1}^{n_{i+1}} \cdots x_r^{n_r} \mod \gamma_2 F$$

for some $n_{i+1}, \dots, n_r \in \mathbb{Z}$. The lower unitriangularity of the matrix representation follows immediately.

Suppose now that the result was true for all weights up to n. Let $c = [c', c'']$ be such that $\mathrm{wt}(c) = n+1$. By induction hypothesis and Corollary 3.2 we get

$$\begin{aligned}\phi(c) &= [\phi(c'), \phi(c'')] \\ &= [a_0 a_1^{n_1}\cdots a_p^{n_p} R, b_0 b_1^{m_1}\cdots b_q^{m_q} S] = [a_0, b_0]\left(\prod_{(i,j)\neq(0,0)} \left[a_i^{|n_i|/n_i}, b_j^{|m_j|/m_j}\right]^{n_i m_j}\right) T\end{aligned}$$

where $a_0 = c', b_0 = c''$, $\mathrm{wt}(a_i) = \mathrm{wt}(a_0)$ and $\mathrm{wt}(b_j) = \mathrm{wt}(b_0)$ for all i, j, and where $\mathrm{wt}(a_0) < \mathrm{wt}(R)$ and $\mathrm{wt}(b_0) < \mathrm{wt}(S)$. By induction hypothesis, the sequences of basic commutators $a_0, a_1, a_2, \dots$ and $b_0, b_1, b_2, \dots$ are strictly increasing with respect to $\leq$ and $\mathrm{wt}(T) > n+1$. Now for $(i,j) \neq (0,0)$ each of the commutators

$$\left[a_i^{|n_i|/n_i}, b_j^{|m_j|/m_j}\right]$$

is greater than $c = [c', c''] = [a_0, b_0]$ with respect to the extended ordering $\leq$ for $i, j > 0$, but some may not be a basic commutator due to a combination of the the signs $|n_i|/n_i, |m_j|/m_j$ possibly being negative or $[a_i, b_j]$ not being basic.

If $[a_i, b_j]$ is not basic because $b_j > a_i$, then $\mathrm{wt}(a_i) = \mathrm{wt}(b_j)$ and Theorem 3.8 lets us rewrite it as $[b_j, a_i]^{\pm 1}$ which is a basic commutator mod $\gamma_{n+2}F$.

Otherwise we have $\mathrm{wt}(a_i) > \mathrm{wt}(b_j)$ and Theorem 3.9 or Theorem 3.10 allows us to rewrite the commutator as a product of $\le$-strictly greater basic commutators. Finally noting that all commutators of weight $n+1$ commute modulo $\gamma_{n+2}F$ the first part of the result follows.

By Theorem 3.6 $(\mathcal{A}_n, \le)$ gives a basis of $\gamma_{n+1}F/\gamma_{n+2}F$ and ϕ_{n+1} is easily seen be lower unitriangular. The result now follows by induction. □

Theorem 3.12. *If $\phi \in \mathrm{Aut}(F)$ is unipotent then*

$$\left(F \rtimes_\phi \mathbb{Z}\right)/\gamma_n F$$

is torsion-free nilpotent.

Proof. If ϕ is unipotent then by [BFH00, Theorem 5.1.5] there is a connected directed graph Γ such that $E(\Gamma) = \{e_1, \dots, e_r\}$ and a homotopy equivalence $\Phi : \Gamma \to \Gamma$ that, for $i > 1$, maps e_i to a concatenation $e_i\mu_i$ where μ_i is a (possibly empty) loop that is itself a concatenation of edges (possibly traversed with or against orientation) that lie in $\{e_1, \dots, e_{i-1}\}$. We can pick any vertex $v \in \Gamma$ identify $F = \pi_1(\Gamma, v)$ and Φ will be a representative for the class $[\phi]$ in $\mathrm{Out}(F)$. Since $\Phi(v) = v$ we may assume without loss of generality that, under the π_1-functor, we have $\Phi_\sharp = \phi$.

Consider the quotient map $q : \Gamma \to \Gamma_\bullet$ obtained by identifying all the vertices of Γ to obtain a bouquet of circles with the single vertex $\bullet$. The map q is π_1-injective and maps F to a free factor of $F * F_r = \pi_1(\Gamma_\bullet, \bullet)$. Now, as a bouquet of circles, we can also view $\pi_1(\Gamma_\bullet, \bullet) = F(E(\Gamma))$, the free group on the (abstract) set $E(\Gamma)$. Let $X = E(\Gamma)$ and reverse its order so that $e_i < e_j$ if and only if $i > j$.

Now Φ naturally descends to a homotopy equivalence $\Phi_\bullet$ of $\Gamma_\bullet$ and since it preserves the unique basepoint it induces the automorphism $\phi_\bullet \in \mathrm{Aut}(F(X))$ given by

$$e_i \mapsto \mu_i,$$

where μ_i can be interpreted as a string in $X^{\pm 1}$. It follows that $(X, \le)$ is a $\phi_\bullet$-ordered basis for $\pi_1(\Gamma_\bullet) = F(X)$.

Now the basic commutators of weight n map to an ordered basis of the free abelian group

$$\gamma_n F(X)/\gamma_{n+1}F(X).$$

By Lemma 3.11 and Proposition 3.4, for all $N > 0$ the quotient

$$\left(F(X) \rtimes_{\phi_\bullet} \mathbb{Z}\right)/\gamma_N F(X) \cong \left(F(X)/\gamma_N F(X)\right) \rtimes_{\overline{\phi}_{\bullet_N}} \mathbb{Z}$$

is torsion-free nilpotent.

We note that although F is a free factor of $F(X)$ this particular free factorization is not obtained from a partition of X, besides none these free factorizations will be $\phi_\bullet$-invariant

By our construction we do have that the image of F, which we identify with F is $\phi_\bullet$ invariant and that $[\phi_\bullet|_F] \in \mathrm{Out}(F)$ as an outer automorphism is represented by Φ. Thus, without loss of generality we may assume that $\phi_\bullet|F = \phi$ and in fact that we have an embedding

$$\begin{array}{rcl} F \rtimes_\phi \langle s\rangle & \hookrightarrow & F(X) \rtimes_{\phi_\bullet} \langle t\rangle \\ fs^n & \mapsto & ft^n, \end{array}$$

where we identify $f \in F$ with its image in $F(X)$. We further note that since F is a free factor of $F(X)$ we have

$$\gamma_n F(X) \cap F = \gamma_n F$$

for all terms of the lower central series. It therefore follows that we have a natural embedding

$$\left(F \rtimes_\phi \mathbb{Z}\right)/\gamma_N F \hookrightarrow \left(F(X) \rtimes_{\phi_\bullet} \mathbb{Z}\right)/\gamma_N F(X).$$

The result now follows since torsion-free nilpotency is inherited by subgroups. □

Theorem 3.13 ([KM79, Theorem 17.2.5])**.** *Let G be a finitely generated torsion-free nilpotent group, then there exists an integer $n = n(G)$ such that G embeds in $\mathrm{UT}_n(\mathbb{Z})$, the group of $n \times n$ upper unitriangular integral matrices.*

The following proposition is a simple exercise.

Proposition 3.14. *For any prime $p > n$ the group $\mathrm{UT}_n(\mathbb{Z}/p\mathbb{Z})$ is a group with exponent p, i.e. every nontrivial element has order p.*

Corollary 3.15. *Let $\phi \in \mathrm{Aut}(F)$ be unipotent. For any finite set $S \subset F \rtimes_\phi \mathbb{Z}$ there exists some $N(S) \in \mathbb{Z}_{\geq 0}$ such that for any prime $p > N(S)$ there is a finite p-periodic quotient Q_p^S of $F \rtimes_\phi \mathbb{Z}$ in which S is mapped injectively.*

Furthermore if $g, h \in F \rtimes_\phi \mathbb{Z}$ and the commutator $[g, h]$ does not vanish in Q_p^S then the images of the cyclic groups $\langle g\rangle, \langle h\rangle$ will have order precisely p and will have trivial intersection.

Proof. Let $S = \{f_1 t^{n_1}, \dots f_m t^{n_m}\}$. By Theorem 3.3 there is a sufficiently large N so that if f_i is non-trivial then the image of f_i survives in $F/\gamma_N F$. Each element of S will therefore be mapped non-trivially to

$$G_N \coloneqq \left(F \rtimes_\phi \mathbb{Z}\right)/\gamma_N F \cong \left(F/\gamma_N F\right) \rtimes_{\overline{\phi}_N} \mathbb{Z},$$

which by Theorem 3.12 is torsion-free nilpotent. By Theorem 3.13 N embeds into $\mathrm{UT}_d(\mathbb{Z})$ for some $d = d(N)$. Looking at the matrix images of the elements of S in $\mathrm{UT}_d(\mathbb{Z})$, we see that if we pick a prime p greater than N_1, the maximal absolute value of the coefficients of the matrices occuring in the image of S, then S will be mapped injectively via

$$F \rtimes_\phi \mathbb{Z} \twoheadrightarrow G_N \hookrightarrow \mathrm{UT}_d(\mathbb{Z}) \twoheadrightarrow \mathrm{UT}_d(\mathbb{Z}/p\mathbb{Z}).$$

If p is chosen to be be greater than $\max(N_1, d) = N(S)$ then the image $Q_p^S$of $F \rtimes_\phi \mathbb{Z}$ will be p-periodic. The first part of the proof follows.

Suppose now that $[g, h]$ did not vanish in the quotient to Q_p^S. Then neither g nor h vanished so, by p-periodicity, their images both generate subgroups isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Suppose towards a contradiction the images $\langle g\rangle, \langle h\rangle$ had non-trivial intersection. Then by they structure of $\mathbb{Z}/p\mathbb{Z}$, the images must coincide and the images of both g and h will generate this intersection. This means that the image of g will be a power of the image of h, forcing their images to commute, contradicting the assumption that $[g, h]$ had non-trivial image. It follows that the images of $\langle g\rangle, \langle h\rangle$ must have trivial intersection and the proof is complete. □

We will also need the following:

Lemma 3.16. *Let $G = F \rtimes_\phi \langle t\rangle$ where $\phi \in \mathrm{Aut}(F)$ is unipotent and* $\mathrm{rank}(F) > 1$. *Let $g, h \in G \smallsetminus 1$ be two commuting elements such that $\langle g\rangle \cap \langle h\rangle = 1$. Let $S \subseteq G$ be a finite subset of elements that contains g and h. Then there exists some $N(S) \in \mathbb{Z}_{\geq 0}$ such that for every prime $p > N(S)$ there is a finite p-periodic quotient Q_p^S such that each element of S has non-trivial image and the images of $\langle g\rangle$ and $\langle h\rangle$ have trivial intersection.*

Proof. We begin with the following claim:

Claim 3.17. *The elements g, h are contained in a subgroup $H = F_H \oplus \langle s\rangle \leq G$, where $F_H \leq F$ is not abelian and $s \in F \cdot t$.*

Proof. We argue by induction on the degree of growth of ϕ. If $\deg(\phi) = 0$ then $\phi = \mathrm{id}$. Hence we can take H to be the entire group G. Now suppose that $\deg(\phi) > 0$. Let $G \cong \pi_1(\mathcal{G})$ be the standard splitting as in Theorem 2.19. Recall that the standard splittings are acylindrical.

Since g and h are elements of infinite order that commute, by acylindricity it must be the case that g and h are elliptic and $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \varnothing$. Pick a vertex v in the intersection. Then, $g, h \in \mathcal{G}_v$ and $\mathcal{G}_v = F_v \rtimes \langle t_v \rangle$ for some $F_v \leq F$ and $t_v \in F \cdot t$. The conjugation action of s on F_v induces a unipotent and polynomially growing automorphism of degree $d < \deg(\phi)$. If $\mathrm{rank}(F_v) > 1$ then we may conclude the required result by induction.

Hence, suppose that $\mathrm{rank}(F_v) = 1$ and assume first that $\deg(\phi) = 1$. It must be the case that $\mathcal{G}_v$ is a black vertex of the standard splitting (as defined in Theorem 2.17), and thus for any incident edge e, the map $\iota_e \colon \mathcal{G}_e \to \mathcal{G}_v$ is onto. Hence, $\mathcal{G}_v$ can be identified with a subgroup of $\mathcal{G}_w$ where $w = \tau(e)$. Then, $\mathcal{G}_w$ is a white vertex and thus is of the form $\mathcal{G}_w = F_w \oplus \langle t_w \rangle$ where $F_v \leq F$ is finitely generated of rank greater than 1. Again, the result follows.

Suppose now that $\mathrm{rank}(F_v) = 1$ and $\deg(\phi) > 1$. Suppose that there exists a non-loop edge e incident at v and let w be the other endpoint of e. We may write $\mathcal{G}_v = \langle x_v \rangle \oplus \langle t_v \rangle$ for $x_v \in F$, $t_v \in F \cdot t$, and $\mathcal{G}_w = F_w \rtimes_{\phi_w} \langle t_w \rangle$ for $F_w \leq F$, $t_w \in F \cdot t$, and let

$$K := \langle \mathcal{G}_v, \mathcal{G}_w \rangle \cong \mathcal{G}_v *_{x_v^l t_v = t_w} \mathcal{G}_w,$$

for some $l \in \mathbb{Z}$. Note that ϕ_w is unipotent and polynomially growing and $\deg(\phi_w) < \deg(\phi)$. Thus K admits a presentation

$$K \cong \left\langle F_w, x_v, t_w \middle| a^{t_w} = \phi_w(a) \forall a \in F_w, x_v^{t_w} = x_v \right\rangle \cong F' \rtimes_{\tilde{\phi}} \langle s \rangle$$

where $F' \leq F$ is finitely generated, $s \in F \cdot t$ and $\tilde{\phi}$ is a unipotent automorphism of degree $d < \deg(\phi)$. Again, the result follows by induction.

Suppose now that there is some loop edge e at v. Then, the stable letter a is an element of the fibre F and we have that

$$K := \langle \mathcal{G}_v, a \rangle \cong \mathcal{G}_v *_{{x_v}^l t_v \sim {x_v}^k t_v, a}$$

for some $l, k \in \mathbb{Z}$. One checks that

$$K \cong \langle x_v, a, s \mid x_v^s = x_v, a^s = a x_v^{k-l} \rangle,$$

where $s = x_v^l t_v \in F \cdot t$. Thus, either $K = F' \oplus \langle s \rangle$ where $F' = \langle x_v, a \rangle \leq F$, or $K = F' \rtimes_{\tilde{\phi}} \langle s \rangle$ where $\tilde{\phi}$ is unipotent and linearly growing. In the former case, the result follows immediately. For the latter case, since $\tilde{\phi}$ grows linearly and $\mathrm{rank}(F') > 1$, we may argue by induction.

The final case to consider is $\mathrm{rank}(F_v) = 0$. Then, $\mathcal{G}_v$ is infinite cyclic and $g, h \in \mathcal{G}_v$, contradicting the assumption that $\langle g \rangle \cap \langle h \rangle = 1$. □

Let $H = F_H \oplus \langle s \rangle$ be as in Theorem 3.17. Since g and h commute, it must be the case that $g = u^a s^b$ and $h = u^c s^d$ for some $u \in F_H$ not a proper power and integers $a, b, c, d \in \mathbb{Z}$ such that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

Since $\mathrm{rank}(F_H) > 1$, there exists some $v \in F_H$ which does not commute with u.

Let $S' = S \cup \{u, v, s, [u, v]\}$ and let $N(S') > 0$ be the integer so that for each prime $p > N(S')$, there is a p-periodic quotient $\pi \colon G \to Q_p^{S'}$ so that the image of each element of S' is non-trivial, which exists by Theorem 3.15.

Fix $p > N(S')$ and suppose, for contradiction, that $\langle \pi(u) \rangle \cap \langle \pi(s) \rangle \neq 1$. Since $Q_p^{S'}$ is p-periodic, we must have that $\langle \pi(u) \rangle = \langle \pi(s) \rangle$ and thus $\pi(u)$ commutes with $\pi(v)$. However, we constructed $Q_p^{S'}$ so that $\pi([u, v]) \neq 1$.

It follows that $\langle\pi(u),\pi(s)\rangle = \langle\pi(u)\rangle \oplus \langle\pi(s)\rangle \cong \mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$. Now if we choose $M(S) = \max\{N(S'),\det(A)+1\}$, then for any prime $p > M(S)$ it follows that

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \not\equiv 0 \mod p.$$

Hence, $\pi(u^a s^b) \notin \langle\pi(u^c s^d)\rangle$. It follows that $\langle\pi(g)\rangle \cap \langle\pi(h)\rangle = 1$. □

Combining Theorem 3.15 and Theorem 3.16 we obtain Theorem D from the introduction.

Theorem D. *Let G be a free-by-cyclic group with unipotent and polynomially growing monodromy. Let $S \subseteq G$ be a collection of non-trivial elements and let $g, h \in G$ be such that $\langle g\rangle \cap \langle h\rangle = 1$ or $[g,h] \neq 1$. There exists $N = N(S)$ such that for every prime $p > N$ there is a p-periodic quotient of G such that the image of each element of S is non-trivial and the cyclic subgroups generated by g and h have trivial intersection.*

4. Virtually free vertex fillings

The aim of this section is to introduce and construct virtually free vertex fillings, in particular for free-by-cyclic groups with unipotent polynomial monodromy.

4.1. Constructing fillings.

Definition 4.1. Let $\mathcal{G} = (X_{\mathcal{G}}, \mathcal{G}_\bullet, \iota_\bullet)$ and $\mathcal{H} = (X_{\mathcal{H}}, \mathcal{H}_\bullet, \kappa_\bullet)$ be graphs of groups. A morphism $f\colon \mathcal{G} \to \mathcal{H}$ is a *vertex filling* if the corresponding graph morphism $f\colon X_{\mathcal{G}} \to X_{\mathcal{H}}$ is an isomorphism, for each vertex $v \in V(X)$ the homomorphism $f_v\colon \mathcal{G}_v \twoheadrightarrow \mathcal{H}_{f(v)}$ is onto, and $\gamma_e = 1$ for each $e \in E(X_{\mathcal{G}})$. A vertex filling induces a quotient of the corresponding fundamental groups

$$\Theta_f\colon \pi_1(\mathcal{G},*) \twoheadrightarrow \pi_1(\mathcal{H}, f(*)).$$

Let $f\colon \mathcal{G} \to \mathcal{H}$ be a vertex filling. Let $T_{\mathcal{G}}$ and $T_{\mathcal{H}}$ be the Bass–Serre trees corresponding to $\mathcal{G}$ and $\mathcal{H}$, respectively, and let $N = \ker\Phi$. Then $T_{\mathcal{H}} = N\backslash T_{\mathcal{G}}$ and the quotient

$$\pi\colon T_{\mathcal{G}} \to T_{\mathcal{H}} = N\backslash T_{\mathcal{G}}$$

is a Φ-equivariant morphism, i.e. for every $g \in \pi_1(\mathcal{G},*)$ and $x \in T_{\mathcal{G}}$, $\pi(g\cdot x) = \Phi(g)\cdot\pi(x)$.

Definition 4.2. Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be a graph of groups. For every $v \in V(X)$, let $N_v \leq_f \mathcal{G}_v$ be a finite index subgroup. A *virtually free vertex filling associated to the collection* $\{N_v\}_{v\in V(X)}$ consists of a collections of finite index normal subgroups $N'_v \trianglelefteq_f \mathcal{G}_v$ such that $N'_v \leq N_v$, and a vertex filling $f\colon \mathcal{G} \to \mathcal{H}$ such that for every $v \in V(X)$, the vertex group $\mathcal{H}_{f(v)} = \mathcal{G}_v/N'_v$, and $f_v\colon \mathcal{G}_v \to \mathcal{H}_{f(v)}$ is the natural quotient.

If $S \subseteq V(X)$ is a subset of vertices, and $N_v \leq_f \mathcal{G}_v$ a collection of finite index subgroups for all $v \in S$, then a virtually free vertex filling corresponding to $\{N_v\}_{v\in S}$ is the virtually free vertex filling corresponding to $\{N_v\}_{v\in S} \cup \{\mathcal{G}_v\}_{v\in V(X)\smallsetminus S}$.

We say that $\mathcal{G}$ admits *arbitrarily deep virtually free vertex fillings* if for every collection $\{N_v\}_{v\in V(X)}$ where $N_v \leq_f \mathcal{G}_v$ is a finite index subgroup, there exists a virtually free vertex filling associated to it.

Lemma 4.3. *Let $f\colon \mathcal{G} \to \mathcal{H}$ be a vertex filling with a corresponding morphism of trees $\pi\colon T_{\mathcal{G}} \to T_{\mathcal{H}}$. Fix a basepoint $* \in V(X)$ and let $G = \pi_1(\mathcal{G},*)$ and $H = \pi_1(\mathcal{H}, f(*))$. Let ρ and ρ' be a vertex or an edge in $T_{\mathcal{G}}$. Then ρ and ρ' are in the same G-orbit if and only if $\pi(\rho)$ and $\pi(\rho')$ are in the same H-orbit.*

Proof. By the definition of a vertex filling we have a commutative square

$$\begin{array}{ccc} T_{\mathcal{G}} & \xrightarrow{\pi} & T_{\mathcal{H}} \\ \downarrow & & \downarrow \\ X_{\mathcal{G}} & \xrightarrow{f} & X_{\mathcal{H}}, \end{array}$$

where f is a graph isomorphism and the vertical arrows are the covering maps. The claim follows immediately from commutativity since ρ and ρ' are in the same orbit if and only if they have the same image in $X_{\mathcal{G}}$ and similarly for the images of $\pi(\rho)$ and $\pi(\rho')$ in $X_{\mathcal{H}}$. □

Lemma 4.4. *Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be a graph of groups that admits arbitrarily deep virtually free vertex fillings. Suppose that all the vertex groups are residually finite and all the edge groups are separable in the vertex groups. Then for every vertex $v \in V(X)$, every collection of (possibly repeated) images $H_{e_1}, \dots, H_{e_s} \leq \mathcal{G}_v$ of edge group monomorphism into $\mathcal{G}_v$, every finite set of elements $g_1, \dots, g_s \in \mathcal{G}_v$ that respectively do not lie in the subgroups $H_{e_1}, \dots, H_{e_s} \leq \mathcal{G}_v$, and any finite set of elements $h_1, \dots, h_m \in \mathcal{G}_v$, there exists a vertex filling of $\mathcal{G}$ such that the image of each vertex group $\bar{\mathcal{G}}_v$ is finite, and for $i = 1, \dots, s$, the image of g_i in $\bar{\mathcal{G}}_v$ does not lie in the image of H_{e_i} and none of the elements h_j are mapped to the identity.*

Proof. Using the hypotheses that the edge groups are separable in the vertex groups and that the vertex groups are residually finite, for each vertex $v \in V(X)$ we can find a finite index normal subgroup $N_v \trianglelefteq_f \mathcal{G}_v$ such that the image of each g_i is not mapped into the image of the edge group H_{e_i} and each h_j is mapped to a non-trivial element in the quotient $\mathcal{G}_v/N_v$. Then, we may pass to a virtually free vertex filling associated to the collection $\{N_v\}_{v \in V(X)}$ to obtain the required vertex filling. □

If $\mathcal{G}$ satisfies the conclusion of Theorem 4.4 then we say that $\mathcal{G}$ admits *arbitrarily deep edge separating virtually free fillings.*

Lemma 4.5. *Let $G = H \rtimes_\phi \langle t \rangle$ where H is finitely generated and let $\chi\colon G \to \mathbb{Z}$ be the map associated to the splitting. Let $N \leq_f G$ be a subgroup of finite index and fix a positive integer k. Then, there exists a positive integer L such that for every $i \in k\mathbb{Z}_{>0}$, there is a finite index normal subgroup $N_i \trianglelefteq_f G$ such that $N_i \leq N$ and every element $g \in G$ with $\chi(g) = k$ has order $iL/\gcd(iL, k)$ in the quotient G/N_i*

Proof. Fix a positive integer k and a subgroup $N \leq_f G$ of finite index. After possibly passing to a further subgroup of finite index, which we also call N by an abuse of notation, we may assume that $\chi(N) = n\mathbb{Z}$ for some positive integer n and $t^n \in N$. Then, $N = (N \cap H) \rtimes \langle t^n \rangle$. Let $K \coloneqq C_H(N \cap H) \trianglelefteq_f H$ be the characteristic core of the subgroup $N \cap H$ in H. Let $M = [H : K]!$ and $L = Mn$. Pick $i \in k\mathbb{Z}_{>0}$ and define $N_i \coloneqq \langle K, t^{iL} \rangle \cong K \rtimes \langle t^{iL} \rangle$. Note that N_i is a finite index normal subgroup of G and $N_i \leq N$.

Now to prove the claim about orders of elements in the quotient. For every $x \in H$ and positive integers $k, m > 0$, let

$$\Phi_{k,m}(x) \coloneqq x\phi^k(x)\phi^{2k}(x)\cdots\phi^{(m-1)k}(x).$$

Let $g \in G$ be such that $\chi(g) = k > 0$. Then, $g = xt^k$ for some $x \in H$. Let $G_i = G/N_i$, and denote the natural quotient by

$$\begin{aligned} \bar{\cdot} : G &\to G_i \\ z &\mapsto \bar{z}. \end{aligned}$$

Note that $G_i \cong H/K \rtimes \mathbb{Z}/(iL)\mathbb{Z}$. Let $l = iL/\gcd(iL, k)$. Let $m > 0$ and write $m = ql + r$ for some $0 \leq r < l$ and $q \in \mathbb{Z}$. Then,

$$\bar{g}^m = \overline{\Phi_{k,l}(x)}^q \cdot \overline{\Phi_{k,r}(x)} \cdot \bar{t}^r.$$

It follows that the order of $\bar{g}$ is given by

$$\mathrm{ord}_{G_i}(\bar{g}) = \mathrm{lcm}\{\mathrm{ord}_{G_i}(\overline{\Phi_{k,l}(x)}), l\}.$$

However, the order of $\overline{\Phi_{k,l}(x)}$ in G_i is the order of $\overline{\Phi_{k,l}(x)}$ in H/K, and thus $\mathrm{ord}_{G_i}(\overline{\Phi_{k,l}(x)})$ divides $[H : K]$. By definition, $[H : K]$ divides L, and since i is a multiple of k, L divides l. It follows that $\mathrm{ord}_{G_i}(\overline{\Phi_{k,l}(x)})$ divides l and thus $\mathrm{ord}_{G_i}(\bar{g}) = l$. □

Corollary 4.6. *Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be a graph of groups with infinite cyclic edge groups. Suppose that there exists an epimorphism $\chi\colon \pi_1(\mathcal{G}) \to \mathbb{Z}$ with finitely generated kernel such that $\chi(\mathcal{G}_e) = \mathbb{Z}$ for every edge $e \in E(X)$. Let $N_v \leq_f \mathcal{G}_v$ be a subgroup of finite index for each $v \in V(X)$. Then there exists a positive integer L such that for every positive integer i, there is a virtually free filling $\mathcal{G} \to \overline{\mathcal{G}}$ associated to $\{N_v\}_{v \in V(X)}$ such that the order of each edge group is iL.*

Proof. For each vertex $v \in V(X)$, let $\chi_v\colon \mathcal{G}_v \to \mathbb{Z}$ be the restriction of χ to $\mathcal{G}_v$. Since χ is surjective on edge groups, each restriction χ_v is surjective. Let $L_v > 0$ be the constant from Theorem 4.5 for the finite index subgroup $N_v \leq \mathcal{G}_v$ and $k = 1$. Let $L = \mathrm{lcm}\{L_v\}_{v \in V(X)}$. Fix integer $i > 0$. Then, for every $v \in V(X)$, let $N_v' \trianglelefteq_f \mathcal{G}_v$ be the finite index normal subgroup contained in N_v such that for every $g \in \mathcal{G}_v$ with $\chi_v(g) = 1$, we have that the order of g in the quotient $\mathcal{G}_v/N_v'$ is iL, which exists by Theorem 4.5. In particular, the generators of the images of the incident edge groups have order iL. Now we may construct a vertex filling of $\mathcal{G}$ by replacing each vertex group $\mathcal{G}_v$ with $\mathcal{G}_v/N_v'$. □

We now specialise to the setting of splittings of free-by-cyclic groups with unipotent and superlinearly growing monodromy.

Proposition 4.7. *Let G be a free-by-cyclic group with unipotent superlinearly growing monodromy and let $G \cong \pi_1(\mathcal{G})$ be the standard splitting. Then the following properties are satisfied.*

1. *(1) The profinite topology on $\pi_1(\mathcal{G})$ is efficient.*
2. *(2) The edge groups are separable in the vertex groups.*
3. *(3) The edge groups are root-closed in $\pi_1(\mathcal{G})$.*
4. *(4) $\mathcal{G}$ admits arbitrarily deep virtually free vertex fillings.*

Proof. The group G is residually finite by [Bau71] and every vertex and edge group is fully separable by [HK25, Proposition 2.9]. By Theorem 2.2 it follows that G induces the full profinite topology on every edge and vertex group and thus the profinite topology on $\pi_1(\mathcal{G})$ is efficient, proving (1).

All infinite cyclic subgroups are separable in free-by-cyclic groups by [HK25, Proposition 2.9] and thus the edge groups are separable in the vertex groups, proving (2).

The edge groups are root-closed since they are maximal cyclic subgroups of G. The latter follows from the fact that the epimorphism $\chi\colon G \to \mathbb{Z}$ is surjective on the edge groups. This proves (3).

Finally, $\mathcal{G}$ admits arbitrarily deep virtually free vertex fillings by Theorem 4.6, proving (4). □

4.2. **Conjugacy separability.** In this subsection we will utilise virtually free vertex fillings to show that certain pairs of elements can be conjugacy distinguished in finite quotients.

Lemma 4.8. *Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be a graph of groups such that the edge groups are separable in the vertex groups. Suppose that $\mathcal{G}$ admits arbitrarily deep virtually free vertex fillings. Let $g_1, g_2 \in \pi_1(\mathcal{G}, v)$. Then there exists a virtually free vertex* *filling $\mathcal{G} \to \overline{\mathcal{G}}$ inducing a morphism of the associated Bass–Serre trees $T \to \overline{T}$ such that $\ell_T(g_1^k) = \ell_{\overline{T}}(\bar{g}_1^k)$ and $\ell_T(g_2^k) = \ell_{\overline{T}}(\bar{g}_2^k)$ for all $k \in \mathbb{Z}$.*

Proof. Note that the image of an elliptic element is elliptic in any vertex filling. Thus we may assume that g_1 and g_2 are loxodromic.

Let $G = \pi_1(\mathcal{G}, v)$. Let α_i be the axis of g_i in T for $i = 1, 2$. Let $D_i \subseteq \alpha_i$ be a minimal closed connected fundamental domain for the action of G on α_i and note that D_i is isometric to the closed interval $[1, n_i]$ for some n_i. Let $\widetilde{D}_i \subset \alpha_i$ be obtained from the subinterval D_i by taking the union with an edge on either side of D_i. Then, for any pair of edges f_1 and f_2 on α_i with $i(f_1) = i(f_2)$, there exist edges e_1 and e_2 on $\widetilde{D}_i$ and some $g \in G$ such that $f_1 = g \cdot e_1$ and $f_2 = g \cdot e_2$.

Fix a vertex v in D_1 and let e_1 and e_2 be the distinct edges on α_1 with $i(e_1) = v = i(e_2)$. If e_1 and e_2 are in the same G-orbit, let $x_v \in \mathcal{G}_v$ be such that $x_v \cdot e_1 = e_2$. Since $\mathcal{G}_{e_1} \leq \mathcal{G}_v$ is separable, we may pick a finite index normal subgroup $N_v \trianglelefteq_f \mathcal{G}_v$ such that $x_v \notin N_v \cdot \mathcal{G}_{e_1}$. If e_1 and e_2 are not in the same G-orbit, take $N_v = \mathcal{G}_v$. Do the same for the vertices in the interior of D_2, obtaining a collection of finite index normal subgroups $\{M_v\}_{v \in V(D_2)}$. Finally, extend each collection $\{N_v\}_{v \in V(D_1)}$ and $\{M_v\}_{v \in V(D_2)}$ to equivariant families $\{N_v\}_{v \in V(T)}$ and $\{M_v\}_{v \in V(T)}$, in the obvious way. That is, for any $w \in V(T)$ with $g \cdot v = w$ for some $v \in V(D_1)$, we set $N_w = g \cdot N_v \cdot g^{-1}$. If $w \in V(T)$ is not in the G-orbit of any vertex in D_1, then set $N_w = \mathcal{G}_w$. Carry out the analogous procedure for the collection $\{M_v\}_{v \in V(T)}$.

Finally, set $L_v \coloneqq N_v \cap M_v$ for each $v \in V(T)$. Then, $\{L_v\}_{v \in V(T)}$ is an equivariant collection of finite index normal subgroups $L_v \trianglelefteq_f \mathcal{G}_v$. Construct a vertex filling $\mathcal{G} \to \overline{\mathcal{G}}$ corresponding to the system $\{L_v\}_{v \in V(X)}$ and let $T \to \overline{T}$ be the associated morphism of Bass–Serre trees.

We claim that α_1 and α_2 are mapped injectively onto their image via the quotient $T \to \overline{T} = \langle L_v \rangle_{v \in V(T)} \backslash T$. Since $T \to \overline{T}$ maps edges to edges, and since $\overline{T}$ is a tree, it must be the case that if two edges of α_i are identified under the map $T \to \overline{T}$, then there exist two edges f_1 and f_2 in α_i with $i(f_1) = i(f_2)$ that are identified. By equivariance, we may assume such edges to be contained in $\widetilde{D}_i$.. However, our construction of the subgroups $L_v \leq N_v \cap M_v$ precludes this.

Now let $n_i = \ell_T(g_i) \in \mathbb{N}$ and let $\bar{\alpha}_i$ be the image of the axis α_i under the vertex filling morphism $T \to \overline{T}$. Then $\bar{g}_i$ acts as translation by n_i on the line $\bar{\alpha}_i$. Hence $\bar{g}_i$ is loxodromic and $\ell_{\overline{T}}(\bar{g}_i) = n_i$. It follows that for all $k \in \mathbb{Z}$,

$$\ell_{\overline{T}}(\bar{g}_i^k) = |k| \cdot \ell_{\overline{T}}(\bar{g}_i) = |k| \cdot n_i = \ell_T(g_i^k). \qquad \square$$

A direct consequence of Theorem 4.8 is the following:

Lemma 4.9. *Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be a graph of groups such that the edge groups are separable in the vertex groups. Suppose that $\mathcal{G}$ admits arbitrarily deep virtually free vertex fillings. Let $G = \pi_1(\mathcal{G}, v)$ and let $g_1, g_2 \in G$ be two elements with distinct translation lengths. Then there exists a finite quotient of G such that the images of g_1 and g_2 are not conjugate.*

Lemma 4.10. *Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be an acylindrical graph of groups that satisfies the following.*

(1) Vertex groups are conjugacy separable.
(2) Edge groups are conjugacy distinguished in vertex groups.
(3) $\mathcal{G}$ admits arbitrarily deep virtually free vertex fillings.

Let $G = \pi_1(\mathcal{G}, v)$. Then for any pair $g_1, g_2 \in G$ of non-conjugate elements which are elliptic, there exists a virtually free quotient $G \twoheadrightarrow \overline{G}$ such that the image of g_1 is not conjugate to the image of g_2. Hence, there exists a finite quotient of G such that the image of g_1 is not conjugate to the image of g_2.

Proof. Let T be the Bass–Serre tree corresponding to the splitting $G = \pi_1(\mathcal{G}, v)$.

Using the fact that edge groups are conjugacy distinguished in the vertex groups, and that $\mathrm{Fix}(g_1)$ and $\mathrm{Fix}(g_2)$ have bounded diameter by acylindricity, we may construct a virtually free vertex filling $\pi: T \to \overline{T}$ such that $\pi(\mathrm{Fix}(g_1)) = \mathrm{Fix}(\bar{g}_1)$ and $\pi(\mathrm{Fix}(g_2)) = \mathrm{Fix}(\bar{g}_2)$. Note also that for any vertex fillings, two edges e, e' incident at the same vertex are identified by π if only if they are contained in the same G-orbit Theorem 4.3.

Suppose first that $\mathrm{Fix}(g_1)$ consists of a single vertex v. Then, if $\mathrm{Fix}(g_2)$ contains an edge, it follows that $\mathrm{Fix}(\bar{g}_1)$ and $\mathrm{Fix}(\bar{g}_2)$ are not isomorphic as graphs and thus $\bar{g}_1$ and $\bar{g}_2$ are not conjugate.

Hence, suppose that $\mathrm{Fix}(g_2)$ consists of a single vertex w. If v and w are not in the same G-orbit in T, then $\bar{v}$ and $\bar{w}$ are not in the same $\bar{G}$-orbit and thus $\bar{g}_1$ and $\bar{g}_2$ are not conjugate in $\bar{G}$. Hence, suppose that v and w are in the same orbit. Then, there exists $x \in G$ such that $g_2' := g_2^x$ fixes v. Now g_2' is not conjugate to g_1 and $g_1, g_2' \in \mathcal{G}_v$. Hence, by conjugacy separability of $\mathcal{G}_v$, there is a finite quotient $q: \mathcal{G}_v \to Q_v$ such that $q(g_1)$ and $q(g_2')$ are not conjugate in Q_v. Hence there is a virtually free vertex filling $\mathcal{G} \to \bar{\mathcal{G}}$ such that $\mathrm{Fix}(\bar{g}_1) = \mathrm{Fix}(\bar{g}_2') = \{\bar{v}\}$ and $\bar{g}_1$ and $\bar{g}_2'$ are not conjugate in $\bar{\mathcal{G}}_v = \mathrm{stab}_{\bar{G}}(\bar{v})$. But if there exists some $\bar{y} \in \bar{G}$ such that $\bar{g}_2'^{\bar{y}} = \bar{g}_1$, then $\bar{y} \cdot \mathrm{Fix}(\bar{g}_1) = \mathrm{Fix}(\bar{g}_2')$. Hence, $\bar{y} \in \mathrm{stab}_{\bar{G}}(\bar{v})$. It follows that $\bar{g}_1$ and $\bar{g}_2'$ are not conjugate in $\bar{G}$. Thus $\bar{g}_1$ and $\bar{g}_2$ are not conjugate.

Assume now that $\mathrm{Fix}(g_1)$ and $\mathrm{Fix}(g_2)$ both contain an edge. If no pair of edges $e_1 \in \mathrm{Fix}(g_1)$ and $e_2 \in \mathrm{Fix}(g_2)$ are in the same G-orbit, then their images in a virtually free filling are not in the same $\bar{G}$-orbit and thus $\bar{g}_1$ and $\bar{g}_2$ are not conjugate. Hence, there exists a pair e_1 and e_2 that are in the same G-orbit. Replacing g_2 by a conjugate, we may assume that there exist some edge e that is fixed by g_1 and g_2. If $\mathrm{Fix}(g_1) = \mathrm{Fix}(g_2)$ consists of a single edge, then as before, we may use conjugacy separability of the vertex groups to pass to a virtually free quotient where $\bar{g}_1$ and $\bar{g}_2$ are not conjugate.

Suppose finally that $\mathrm{Fix}(g_1)$ contains at least two edges. Since $\mathcal{G}$ is acylindrical, each $\mathrm{Fix}(g_i)$ is a subtree of T of finite diameter. In particular, it contains finitely many non-leaf vertices. Let u be an endpoint of an edge $e \in \mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)$ that is not a leaf of $\mathrm{Fix}(g_1)$.

Consider the set X of all elements $x \in G$ such that $x \cdot \mathrm{Fix}(g_1) = \mathrm{Fix}(g_2)$. Since G acts by isometries, each $x \in X$ maps u to a non-leaf vertex of $\mathrm{Fix}(g_2)$. Let $\{v_1, \dots, v_k\}$ be the non-leaf vertices of $\mathrm{Fix}(g_2)$. Let $X_i \subset X$ be the set of all $x \in X$ such that $x \cdot u = v_i$. Then the X_i partition X into finitely many subsets, and for any two $x, y \in X_i$, we must have that $y^{-1}x \in \mathcal{G}_u$. Hence, X consists of finitely many $\mathcal{G}_u$-orbits. For each i, pick some $x_i \in X_i$. Then $x_i^{-1} g_2 x_i \in \mathcal{G}_u$ for every i. Since $\mathcal{G}_u$ is conjugacy separable, we may pass to a further virtually free filling such that $x_i^{-1} g_2 x_i$ and g_1 are not conjugate in $\bar{\mathcal{G}}_u$ for every i. Then, by the same argument as above, we conclude that $\bar{g}_1$ and $\bar{g}_2$ are not conjugate.

The final claim follows from the fact that finitely generated virtually free groups are conjugacy separable Theorem 2.9. □

4.3. **Conjugacy distinguished cyclic subgroups.** We mimic the proof of Theorem 4.10 to construct virtually free fillings that separate the conjugacy class of an element g_2 from the cyclic subgroup $\langle g_1 \rangle$ in the case that g_1 and g_2 are both elliptic.

For the remainder of this article we will use the notation $\mathcal{G} \to \overline{\mathcal{G}}$ to denote a vertex filling, $T \to \overline{T}$ the corresponding morphism of the associated Bass–Serre trees, and $G \to \overline{G}, g \mapsto \bar{g}$ the quotient of the correpsoninng fundamental groups.

Lemma 4.11. *Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be an acylindrical graph of groups that satisfies the following.*

(1) Vertex groups are conjugacy separable.
(2) Vertex groups are $\mathbb{Z}$-conjugacy distinguished.
(3) Edge groups are conjugacy distinguished in the vertex groups.
(4) Edge groups are root-closed in $\pi_1(\mathcal{G})$.
(5) $\mathcal{G}$ admits arbitrarily deep virtually free vertex fillings.

Let $G = \pi_1(\mathcal{G}, v)$. Then, for any elements $g_1, g_2 \in G$ such that at least one of g_1 or g_2 is elliptic and g_2 is not conjugate into $\langle g_1 \rangle$, there exists a virtually free quotient of G such that the image of g_2 is not conjugate into the image of $\langle g_1 \rangle$. Hence, there exists a finite quotient of G such that the image of g_2 is not conjugate into the image of $\langle g_1 \rangle$.

Proof. Let $g_1, g_2 \in G$ be as in the statement of the lemma. Let T be the Bass–Serre tree corresponding to $\mathcal{G}$.

Suppose first that g_1 is loxodromic and g_2 is elliptic. By Theorem 4.8, there exists a virtually free vertex filling $T \to \overline{T}$ such that $\bar{g}_1$ is loxodromic. Moreover, g_2^k remains elliptic in any virtually free filling. Hence g_1 is not conjugate into $\langle g_2 \rangle$ in the image. The same argument holds if g_1 is elliptic and g_2 is loxodromic.

Suppose now that both g_1 and g_2 are elliptic. We begin by noting that for any $k, l \in \mathbb{Z} \smallsetminus 0$,

$$\mathrm{Fix}(g_1^k) \subseteq \mathrm{Fix}(g_1^{kl}),$$

and since the action of G on T is acylindrical, the diameter of $\mathrm{Fix}(g_1^k)$ is uniformly bounded over all $k \in \mathbb{Z} \smallsetminus 0$. Using the fact that edge groups are conjugacy distinguished in the vertex groups, and that $\mathrm{Fix}(g_1)$ and $\mathrm{Fix}(g_2)$ have bounded diameter by acylindricity, we may pass to a virtually free vertex filling $\pi: T \to \overline{T}$ such that $\pi(\mathrm{Fix}(g_1^k)) = \mathrm{Fix}(\bar{g}_1^k)$ for all $k \in \mathbb{Z} \smallsetminus 0$, and $\pi(\mathrm{Fix}(g_2)) = \mathrm{Fix}(\bar{g}_2)$.

Suppose now that $\mathrm{Fix}(g_2)$ consists of a single vertex v. If there is no power g_1^k of g_1 such that $\mathrm{Fix}(g_1^k)$ consists of a single vertex, then $\mathrm{Fix}(\bar{g}_1^k)$ is not isomorphic to $\mathrm{Fix}(\bar{g}_2)$ for any $k \in \mathbb{N}$ and thus $\bar{g}_2$ is not conjugate into $\langle \bar{g}_1 \rangle$.

If there is a power g_1^k of g_1 such that $\mathrm{Fix}(g_1^k)$ consists of a single vertex, then we must have that $\mathrm{Fix}(g_1)$ consists of a single vertex, since $\mathrm{Fix}(g_1) \subseteq \mathrm{Fix}(g_1^k)$ for all $k \in \mathbb{Z}$. Let $\mathrm{Fix}(g_1) = \{w\}$. If v and w are not in the same G-orbit in T, then $\bar{v}$ and $\bar{w}$ are not in the same $\bar{G}$-orbit and thus $\bar{g}_2$ and $\bar{g}_1^k$ are not conjugate in $\bar{G}$ for all $k \in \mathbb{Z}$.

Hence, suppose that v and w are in the same orbit. Then, there exists $x \in G$ such that $g_2' = g_2^x$ fixes v. Now g_2' is not conjugate to g_1^k for all $k \in \mathbb{N}$, and $g_1^k, g_2' \in \mathcal{G}_v$. Then, since $\langle g_1 \rangle$ is conjugacy distinguished in $\mathcal{G}_v$, there is a finite quotient $q: \mathcal{G}_v \to Q_v$ such that $q(g_1^k)$ and $q(g_2')$ are not conjugate in Q_v, for all $k \in \mathbb{Z}$. Hence there is a virtually free vertex filling $\mathcal{G} \to \bar{\mathcal{G}}$ such that $\mathrm{Fix}(\bar{g}_1^k) = \mathrm{Fix}(\bar{g}_2') = \{\bar{v}\}$ and $\bar{g}_1^k$ and $\bar{g}_2'$ are not conjugate in $\bar{\mathcal{G}}_v = \mathrm{stab}_{\bar{G}}(\bar{v})$. But if there exists some $\bar{y} \in \bar{G}$ such that $\bar{g}_2'^{\bar{y}} = \bar{g}_1^k$, then $\bar{y} \cdot \mathrm{Fix}(\bar{g}_1^k) = \mathrm{Fix}(\bar{g}_2')$. Hence, $\bar{y} \in \mathrm{stab}_{\bar{G}}(\bar{v})$. It follows that $\bar{g}_1^k$ and $\bar{g}_2'$ are not conjugate in $\bar{G}$. Thus $\bar{g}_1^k$ and $\bar{g}_2$ are not conjugate.

Assume now that $\mathrm{Fix}(g_2)$ contains an edge and there exists $k \in \mathbb{Z} \smallsetminus 0$ such that $\mathrm{Fix}(g_1^k)$ also contains an edge. If no pair of edges $e_1 \in \mathrm{Fix}(g_2)$ and $e_2 \in \mathrm{Fix}(g_1^k)$, for any $k \in \mathbb{Z} \smallsetminus 0$, are in the same G-orbit, then their images in a virtually free filling are not in the same $\bar{G}$-orbit and thus $\bar{g}_1^k$ and $\bar{g}_2$ are not conjugate. Hence, there exists a pair e_1 and e_2 that are in the same G-orbit. Replacing g_2 by a conjugate, we may assume that there exist some edge e that is fixed by g_2 and g_1^k for some $k \in \mathbb{Z} \smallsetminus 0$.

Suppose that $\mathrm{Fix}(g_2)$ only contains the edge e. If $\mathrm{Fix}(g_1)$ also consists of a single edge, then for every $k \in \mathbb{Z} \smallsetminus 0$ such that $\mathrm{Fix}(g_2^k)$ consists of an edge, we must have that $\mathrm{Fix}(g_1^k) = \mathrm{Fix}(g_2) = \{e\}$. Let u be an endpoint of e. Then u is fixed by g_2 and g_1^k for every $k \in \mathbb{Z}$. Hence, we may use the conjugacy distinguished property of the vertex group $\mathcal{G}_u$ to pass to a virtually free quotient such that $\bar{g}_2$ is not conjugate into $\langle \bar{g}_1 \rangle$.

Finally, suppose that $\mathrm{Fix}(g_2)$ contains at least two edges. Since $\mathcal{G}$ is acylindrical, each $\mathrm{Fix}(g_2)$ is a subtree of T of finite diameter. In particular, it contains finitely many non-leaf vertices. Let u be an endpoint of an edge $e \in \mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)$ that is not a leaf of $\mathrm{Fix}(g_1)$.

We will now use the root-closed assumption. Since G is root-closed, $\mathrm{Fix}(g_1^k) = \mathrm{Fix}(g_1)$ for all $k \in \mathbb{Z} \setminus 0$. Now we argue as in the last part of Theorem 4.10. Consider the set X of all elements $x \in G$ such that $x \cdot \mathrm{Fix}(g_1) = \mathrm{Fix}(g_2)$. Then each $x \in X$ maps u to a non-leaf vertex of $\mathrm{Fix}(g_2)$. Let $\{v_1, \dots, v_k\}$ be the non-leaf vertices of $\mathrm{Fix}(g_2)$. Let X_i be the subset of $x \in X$ such that $x \cdot u = v_i$. Then the X_i partition X into finitely many subsets, and for any two $x, y \in X_i$, we must have that $y^{-1}x \in \bar{\mathcal{G}}_u$. Hence, X consists of finitely many $\bar{\mathcal{G}}_u$ orbits. For each i, pick some $x_i \in X_i$. Now $x_i^{-1} g_2 x_i, g_1^k \in \mathcal{G}_u$ for every i and $k \in \mathbb{Z} \setminus 0$. Since $\mathcal{G}_u$ is conjugacy distinguished, we may pass to a further virtually free filling such that the image of $x_i^{-1} g_2 x_i$ is not conjugate into the image of $\langle g_1 \rangle$.

The final claim follows from the fact that cyclic subgroups are conjugacy distinguished in virtually free groups [CZ15]. □

Lemma 4.12. *Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be an acylindrical graph of groups such that the edge groups are separable in the vertex groups. Suppose that $\mathcal{G}$ admits arbitrarily deep virtually free vertex fillings. Suppose also that hyperbolic elements of $G = \pi_1(\mathcal{G}, v)$ are conjugacy distinguished.*

Then, for any hyperbolic elements g_1 and g_1 such that g_2 is not conjugate into $\langle g_1 \rangle$, there exists a virtually free quotient of G such that the image of g_2 is not conjugate into the image of $\langle g_1 \rangle$. Hence, there exists a finite quotient of G such that the image of g_2 is not conjugate into the image of $\langle g_1 \rangle$.

Proof. By Theorem 4.8, we may pass to a virtually free vertex filling $G \to \overline{G} = G/N_1$ that preserves the translation lengths of g_1 and g_2. Hence, there exists at most one positive integer $k > 0$ such that

$$\ell_T(g_2) = \ell_{\overline{T}}(\bar{g}_2) = \ell_{\overline{T}}(\bar{g}_1^k) = \ell_T(g_1^k).$$

If no such integer exists, then $\bar{g}_2$ is not conjugate into $\langle \bar{g}_1 \rangle$ and we are done.

Suppose that there exists some $k > 0$ such that $\ell_{\overline{T}}(g_2) = \ell_{\overline{T}}(g_1^k)$. Since hyperbolic elements in G are conjugacy distinguished, there exists a finite quotient $\pi \colon \pi_1(\mathcal{G}) \to Q$ such that $\pi(g_2)$ is not conjugate to $\pi(g_1^{\pm k})$. Let N_2 be the kernel of π.

Then, the quotient $G \to G/N_1 \cap N_2$ corresponds to a virtually free filling such that the image of g_2 is not conjugate into the image $\langle g_1 \rangle$. The final claim follows from the fact that cyclic subgroups are conjugacy disntinguished in virtually free groups.

□

5. Piecewise trivial suspensions

The goal of this section is to prove that free-by-cyclic groups with unipotent linear monodromy are conjugacy separable. This is done in Theorem 5.17.

5.1. **Some background results.** Recall that a free-by-cyclic group with unipotent and linearly growing monodromy admits a standard splitting as in Theorem 2.17. Following the terminology established in [DT24], we will call such a splitting a *piecewise trivial suspension with free local fibres*, sometimes omitting reference to the local fibres if it is clear from the context.

A piecewise trivial suspension $\mathcal{K}$ is said to be *clean* if whenever w is a white vertex of the underlying graph we have $\mathcal{K}_w \cong F_w \oplus \langle t_w \rangle$ and if $H \leq \mathcal{K}_w$ is the image of any edge group in $\mathcal{K}_w$, then $H = \langle c_H, t_w \rangle$ where c_H is a primitive element of F_w. We say that a finite index subgroup $K \leq \pi_1(\mathcal{G}, v)$ is a *clean cover* if the induced splitting $\mathcal{K}$ of K is clean and furthermore if there is a power k such that every black vertex group in $\mathcal{K}$ is conjugate to a *k-congruence subgroup*, i.e. the subgroup $k\mathbb{Z} \oplus k\mathbb{Z} \leq \mathbb{Z}^2$ of a black vertex group of $\mathcal{G}$.

Proposition 5.1. *Let $\mathcal{G}$ be a piecewise trivial suspension with free local fibres. Then the following properties are satisfied.*

1. *The profinite topology on $\pi_1(\mathcal{G}, v)$ is efficient; in particular, the edge groups are separable in the vertex groups.*
2. *The vertex groups are conjugacy separable.*
3. *Cyclic subgroups are conjugacy distinguished in the vertex groups.*
4. *The edge groups are conjugacy distinguished in the vertex groups.*
5. *The splitting is 4-acylindrical.*

Moreover, $\pi_1(\mathcal{G}, v)$ contains a finite index subgroup that admits a splitting as a clean piecewise trivial suspension with free fibres.

Proof. By definition of piecewise trivial suspension the splitting is 4-acylindrical, the vertex groups are of the form $F_v \times \mathbb{Z}$ where F_v is a finitely generated free group and the edge groups are maximal $\mathbb{Z}^2$-subgroups. The group G is residually finite by [Bau71], and by [HK25, Proposition 2.9] every edge and vertex group is fully separable. Hence the profinite topology on $\pi_1(\mathcal{G})$ is efficient.

Since finitely generated free groups are conjugacy separable, it follows that the vertex groups are conjugacy separable. Since finitely generated subgroups are conjugacy distinguished in finitely generated free groups, it follows that edge groups and cyclic subgroups are conjugacy distinguished in the vertex groups. Similarly, edge groups are separable in the vertex groups.

By [DT24, Proposition 6.14], there is a finite index subgroup $G' \leq G$ that admits a splitting as a clean piecewise trivial suspension. □

Clean piecewise trivial suspensions admit virtually free vertex fillings of a particularly nice form.

Proposition 5.2 (Virtually free vertex fillings ([DT24, Proposition 6.17]))**.** *Let $\mathcal{K}$ be a clean piecewise trivial suspension. Suppose for every white vertex w we are given a finite index subgroup $N_w \leq F_w$. Then there are finite index characteristic subgroups $D_w \leq N_w \leq F_W$ such that there exists a positive integer $N > 0$ and a graph of groups $\bar{\mathcal{K}}$ and with the same underlying graph such that for every white vertex we have $\bar{\mathcal{K}}_w \cong F_w/D_w \oplus (\mathbb{Z}/N\mathbb{Z})$, every black vertex is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$ and there is a natural surjection $\pi_1(\mathcal{K}, v) \twoheadrightarrow \pi_1(\bar{\mathcal{K}}, v)$ induced by a morphism of graphs of groups, where each vertex group morphism $\mathcal{K}_v \twoheadrightarrow \bar{\mathcal{K}}_v$ is the quotient by $\langle D_w, t_w^N \rangle$.*

Remark 5.3. By strong omnipotence of the free groups D_w, there exists a positive integer $N' > 0$ such that for every $i \in \mathbb{N}$, there is a virtually free vertex filling $\bar{\mathcal{K}}$ of $\mathcal{K}$ with images of white vertex groups of the form $\bar{\mathcal{K}}_w \cong F_w/D_w \oplus (\mathbb{Z}/iN\mathbb{Z})$ and black vertex groups $(\mathbb{Z}/iN\mathbb{Z})^2$.

5.2. **Short positions and vertex fillings.** Let $\mathcal{G} = (X, \mathcal{G}_\bullet, \iota_\bullet)$ be a piecewise trivial suspension. A *turn* is a pair of edges $(e, f) \in E(X)^{\pm 1} \times E(X)^{\pm 1}$ such that $\tau(e) = \iota(f)$. Every turn defines a system of double coset representatives in $\mathcal{G}_{\tau(e)}$ for the double cosets $I_e \backslash \mathcal{G}_{\tau(e)} / O_f$ where I_e is the image of $\tau_e \colon \mathcal{G}_e \to \mathcal{G}_{\tau(e)}$ and O_f is the image of $\iota_f \colon \mathcal{G}_f \to \mathcal{G}_{\iota(f)}$. For a piecewise trivial suspension we can always assume that the double coset representatives are the trivial element in the black vertex groups and elements of the local fibre F_v in the white vertex groups.

Recall that an element $g \in \mathrm{Bass}(\mathcal{G})$ is a *$\mathcal{G}$-loop* if it is an element of $\pi_1(\mathcal{G}, v)$ for some $v \in V(X)$. We say that a $\mathcal{G}$-loop based at a white vertex is in *normal* *form* when it is explicitly written out as

$$\ell_0^{l_0}\tilde{a}_0 e_1 e_2 \ell_2^{l_2}\tilde{a}_2 e_3 \cdots e_{n-1} e_n \ell_n^{l_n}\tilde{a}_n r_n^{m_n} t^k$$

where the $\tilde{a}_i$ are double coset representatives with $\tilde{a}_0$ and $\tilde{a}_n$ being representatives for the turn (e_n, e_1); for each $i \geq 1$, $\ell_i \in I_{e_i}$, $\ell_0 \in I_{e_n} \cap F_{\tau(e_n)}$ and $r_n \in O_{e_1} \cap F_{\iota(e_1)}$; and t is the generator of the center of $\mathcal{G}_{\tau(e_n)}$.

We say a $\mathcal{G}$-loop g is in *short position* within its $\mathcal{G}$-conjugacy class if it has translation length 2 and its normal form looks like

$$e_1 e_2 \tilde{a}_2 r_2^d t^k,$$

or if if has translation length 4 or more and its normal form looks like

$$e_1 e_2 \tilde{a}_2 e_3 e_4 \ell_{e_4}^d \tilde{a}_4 \cdots t^k$$

where $d \in \mathbb{Z}$ is minimal (w.r.t. some fixed well-order of $\mathbb{Z}$) among all conjugates of g. Note that by conjugating by the right element of the edge group corresponding to e_1 it is possible to fix the initial prefix $e_1 e_2 \tilde{a}_2$ while changing the exponent d.

We say that g is in *almost short position* if it has a prefix of the form $e_1 e_2$. Note that all conjugates in almost short position are conjugates of elements in short position by elements that lie in the edge group $\iota_{e_1}(\mathcal{G}_{e_1})$. The following proposition follows by 4-acylindricity of the splitting.

Proposition 5.4 ([DT24, Proposition 5.6])**.** *An element of $\pi_1(\mathcal{G}, v)$ with translation length $2m$ has at most m $\mathrm{Bass}(\mathcal{G})$-conjugates in short position.*

Now given a $\mathcal{G}$-loop g, let $DCR(g) = (\tilde{a}_0, \ldots, \tilde{a}_n)$ denote the sequence of double coset representatives that occur in the normal form for g if g is in short position, and if g' is a conjugate of g by an element of an edge group that is in almost short position then $DCR(g') = DCR(g)$. As a consequence of double coset separability of free groups we have:

Lemma 5.5. *If for any $\mathrm{Bass}(\mathcal{G})$ conjugates k_1, k_2 of g_1, g_2 in short position with the same underlying edge part we have $DCR(k_1) \neq DCR(k_2)$ then there is a virtually free quotient of G where g_1, g_2 have non-conjugate image.*

Proof. By hypothesis, we can find sufficiently deep finite index subgroups of the vertex groups so that the kth coset representative $\tilde{b}_{2k}$ occurring in $DCR(k_2)$ is separated from the double coset $I_{e_{2k}}\tilde{a}_{2k}O_{e_{2k+1}}$. Let $\bar{\mathcal{G}}$ be a sufficiently deep virtually free vertex filling. Then all almost short conjugates of the image of g_1 will be different from the image of all almost short conjugates of the image of g_2 in $\mathrm{Bass}(\bar{\mathcal{G}})$. Since any element has almost short conjugates, it follows that g_1 and g_2 have non-conjugate images. □

5.3. **Separating elements with the same double coset sequence.** Suppose now that we are given two non-conjugate elements in short position with normal forms

$$g = e_1 e_2 a_2 e_3 e_4 c_4^{n_4} a_4 e_5 \cdots e_l c_l^{n_l} a_l \bar{c}_1^{n_1} t_1^r \quad (1)$$

$$h = e_1 e_2 a_2 e_3 e_4 c_4^{m_4} a_4 e_5 \cdots e_l c_l^{m_l} a_l \bar{c}_1^{m_1} t_1^r \quad (2)$$

where we allow $e_i = e_j^{\pm 1}$ even if $i \neq j$ and where the c_j are elements that generate the intersection of the fibre and the image of the edge group. We have the following migration relations:

$$\bar{c}_i e_i e_{i+1} = e_i e_{i+1} c_{i+1} \quad (3)$$

$$t_i e_i e_{i+1} = e_i e_{i+1} c_{i+1}^{\epsilon_{i+1}} t_{i+1}, \quad (4)$$

where t_i is the generator of the center of either $\mathcal{G}_{\tau(e_i)}$ or $\mathcal{G}_{\iota(e_i)}$; which ever happens to be a non-abelian vertex group. The exponents ϵ_j that occur in (4) are called *twisting numbers* since these are the exponents that occur in the Dehn multitwists.

Let us now look at the effect of conjugating g from (1) by t_1. The sequence of equalities is obtained by migrating the t_i symbols to the right.

$$\begin{aligned} t_1 g t_1^{-1} &= t_1 e_1 e_2 a_2 e_3 e_4 c_4^{n_4} a_4 e_5 \cdots e_l c_l^{n_l} a_l \bar{c}_1^{n_1} t_1^{r-1} \\ &= e_1 e_2 c_2^{\epsilon_2} t_2 a_2 e_3 e_4 c_4^{n_4} a_4 e_5 \cdots e_l c_l^{n_l} a_l \bar{c}_1^{n_1} t_1^{r-1} \\ &= e_1 e_2 c_2^{\epsilon_2} a_2 e_3 e_4 c_4^{n_4+\epsilon_4} t_4 a_4 e_5 \cdots e_l c_l^{n_l} a_l \bar{c}_1^{n_1} t_1^{r-1} \\ &= \cdots \end{aligned}$$

$$t_1 g t_1^{-1} = e_1 e_2 c_2^{\epsilon_2} a_2 e_3 e_4 c_4^{n_4+\epsilon_4} a_4 e_5 \cdots e_l c_l^{n_l+\epsilon_l} a_l \bar{c}_1^{n_1} t_1^{r} \tag{5}$$

From this we see that the subgroup that preserves the prefix $e_1 e_2 a_2 e_3 e_4$ of g under conjugation is $\langle t \bar{c}_1^{-\epsilon_2} \rangle$. We also see that

$$(t_1 \bar{c}_1^{-\epsilon_2}) g (t_1 \bar{c}_1^{-\epsilon_2})^{-1} = e_1 e_2 a_2 e_3 e_4 c_4^{n_4+\epsilon_4} \dots e_l c_l^{n_l+\epsilon_l} a_l \bar{c}_1^{n_1+\epsilon_2} t_1^{r}. \tag{6}$$

Up to this point we did not specify the ordering on $\mathbb{Z}$ we wanted for short position. For our purposes we will now require our well-ordering on $\mathbb{Z}$ to be any well-order that extends

$$0 < 1 < \cdots < |\epsilon_4|.$$

Lemma 5.6. *If g, h as given in (1), (2), respectively, are in short position then we have*

$$0 \le n_4, m_4 < |\epsilon_4|.$$

Proof. Suppose towards a contradiction that g is in short position but either $n_4 < 0$ or $n_4 > \epsilon_4$. This means that by the division algorithm there exists $q \in \mathbb{Z}$ such that

$$n_4 = q|\epsilon_4| + r$$

with $0 \le r < |\epsilon_4|$. By equation 6, by repeatedly conjugating g by $(t_1 \bar{c}_1^{\epsilon_2})^{\pm 1}$, we can add integer multiples of ϵ_4 to he exponent in the c_4 position while preserving the $e_1 e_2 a_2 e_3 e_4$-prefix. By the division algorithm we can therefore make this exponent the remainder of division, contradicting that g was in short position. The same argument works for h □

We can now make sense of and then justify the following definition.

Definition 5.7. If (g, h) is the pair of elements given by (1) and (2) then we define the *persistent vector* $\vec{\epsilon}$ and the *difference vector* $\vec{\delta}$ to be

$$\vec{\epsilon} = \begin{bmatrix} \epsilon_4 \\ \epsilon_6 \\ \vdots \\ \epsilon_l \\ \epsilon_2 \end{bmatrix} \text{ and } \vec{\delta} = \begin{bmatrix} m_4 - n_4 \\ m_6 - n_6 \\ \vdots \\ m_l - n_l \\ m_1 - n_1 \end{bmatrix}.$$

We also define the *persistent equation* to be

$$\vec{\delta} = \lambda \vec{\epsilon}.$$

The reason why these vectors are important is that firstly, if the persistent equation does in fact have a solution $\lambda_0 \in \mathbb{Z}$ then by (6) we would have $(t_1 \bar{c}_1^{-\epsilon_2})^{\lambda_0} g (t_1 \bar{c}_1^{-\epsilon_2})^{-\lambda_0} = h$. That said, we also have the following.

On the one hand, 4-acylindricity implies that all the ϵ_i are nonzero. Lemma 5.6 implies $|m_4 - n_4| < |\epsilon_4|$, so if $|m_4 - n_4| \neq 0$ then there is no integer solution to the equation and if $|m_4 - n_4| = 0$ then the only solution is $\lambda = 0$, which implies $\vec{\delta} = 0$ and therefore that $g = h$.

We now wish to pass to a virtually free vertex filling $\bar{\mathcal{G}}$ in which the images of g, h remain non-conjugate. To simplify notation we will continue to denote the images of g,h in $\mathrm{Bass}(\bar{\mathcal{G}})$ as (1) and (2), but assuming that the non-$E(X)$ symbols denote elements in the finite, filled, vertex groups. Every abelian vertex group in $\bar{\mathcal{G}}$ is isomorphic to $\mathbb{Z}_N \times \mathbb{Z}_N$ so we'll call N the *abelian exponent* of $\bar{\mathcal{G}}$. The image of a double coset $\langle c_n\rangle a_n \langle \bar{c}_{n+1}\rangle$ is said to be *permeable* if there are $p,q \in \mathbb{Z}$, at least one of which is not zero, such that

$$c_n^p a_n =_{\bar{\mathcal{G}}} a_n (\bar{c}_{n-1})^q.$$

Otherwise the image of the double coset is said to be *impermeable*. It is easy to see that the image of double coset is impermeable if and only it has cardinality N^2.

Suppose all the double cosets from $DCR(g)$ have impermeable images in a virtually free vertex filling. We will now describe potential conjugators that bring the image of g to the image of h. We note that (6) still holds in $\bar{\mathcal{G}}$ only here the exponents can be taken $\mod N$. In particular, if there is a solution λ to the persistent equation

$$\vec{\delta} = \lambda \vec{\epsilon} \mod N$$

then the images of g and h will be conjugate. The persistent equation comes from the determining which conjugators preserve the prefix $e_1 e_2 a_2 e_3 e_4$ of g or h for that matter. It is worth noting that in $\mathcal{G}$, the group of such elements coincides with the stabilizer of a segment of length 4 in the dual Bass–Serre tree. Equation (5) also holds with exponents modulo N and we immediately get that conjugation by t_1^{N/ϵ_2} also preserves the prefix $e_1 e_2 a_2 e_3 e_4$. Looking at the effect of this conjugation by repeatedly applying (6) motivates the following.

Definition 5.8. If (g,h) is the pair of elements given by (1) and (2) then for a given N the we have the *content vector* and *variable vector*

$$\vec{\kappa} = \begin{bmatrix} \epsilon_4 \\ \epsilon_6 \\ \vdots \\ \epsilon_l \\ 0 \end{bmatrix} \text{ and } \vec{\nu}_N = \frac{N}{\epsilon_2} \cdot \vec{\kappa}$$

respectively. We further define the *modulo N conjugacy equation* to be

$$\vec{\delta} = \lambda_1 \vec{\epsilon} + \lambda_2 \vec{\nu}_N \mod N. \tag{7}$$

Obviously, a solution (λ_1, λ_2) to the modulo N conjugacy equation would imply that the images of g and h are conjugate in $\bar{\mathcal{G}}$. The following shows that we can arrange for this not to happen.

Proposition 5.9. *If (g,h) is the pair of elements given by (1) and (2) then there is some M such that, if $M|N$, the modulo N conjugacy equation (7) has no solution.*

Proof. The equation

$$\vec{\delta} = \lambda_1 \vec{\epsilon} + \lambda_2 \vec{\kappa} \tag{8}$$

has a solution if and only if $\vec{\delta}$ lies in the subgroup $\langle \vec{\epsilon}, \vec{\kappa}\rangle$. The hypothesis that g,h are non conjugate implies that $\delta \neq \vec{0}$, so $(\lambda_1, \lambda_2) = (0,0)$ is not a solution. Lemma 5.6 implies that there are also no integer solutions with $\lambda_2 = 0$.

Recall that all the twisting numbers ϵ_i are non-zero. It follows that the persistent and content vectors, $\vec{\epsilon}$ and $\vec{\kappa}$, are linearly independent. So if there is a solution to (8), then this solution must be unique. This means that we can take L to be divisible by ϵ_2 and so large that

$$\vec{\delta} = \lambda_1 \vec{\epsilon} + \lambda_2 \frac{L}{\epsilon_2} \vec{\kappa} \tag{9}$$

does not have a solution in $\mathbb{Z}^2$. It follows that we can always find some sufficiently large L that is divisible by ϵ_2 such that

$$\vec{\delta} \notin \left\langle \vec{\epsilon}, \frac{L}{\epsilon_2}\vec{\kappa} \right\rangle.$$

Since $\mathbb{Z}^{l/2}$ is subgroup separable then we can find some M that is divisible by L such that the image of $\vec{\delta}$ does not lie in the image of $\langle \vec{\epsilon}, \frac{L}{\epsilon_2}\vec{\kappa}\rangle$ in $(\mathbb{Z}/M\mathbb{Z})^{l/2}$ and therefore neither in any $(\mathbb{Z}/N\mathbb{Z})^{l/2}$ where $M|N \Rightarrow N = qL$ for some $q \in \mathbb{Z}$.

Suppose finally towards a contradiction that the modulo N conjugacy equation (7) had a solution, then we have

$$\begin{aligned} \vec{\delta} &= \lambda_1\vec{\epsilon} + \lambda_2\vec{\nu}_N \mod N \\ \Rightarrow \quad \vec{\delta} &= \lambda_1\vec{\epsilon} + \lambda_2\frac{N}{\epsilon_2}\vec{\kappa} \mod N \\ \Rightarrow \quad \vec{\delta} &= \lambda_1\vec{\epsilon} + (\lambda_2 q)\frac{L}{\epsilon_2}\vec{\kappa} \mod N \end{aligned}$$

which implies that the image of $\vec{\delta}$ lies in the image of $\langle \vec{\epsilon}, \frac{L}{\epsilon_2}\vec{\kappa}\rangle$ in $(\mathbb{Z}/N\mathbb{Z})^{l/2}$, which is a contradiction and the result follows. □

Proposition 5.10. *Let g, h be as in (1),(2) and suppose they are non-conjugate in $\mathcal{G}$. Let $g_1, \ldots, g_c$ be all the conjugates of g that are in short position and where $DCR(g_i) = DCR(g), i = 1, \ldots, c$. If we can find a vertex filling $\bar{\mathcal{G}}$ such that all double coset images are impermeable, distinct double cosets have distinct images in their respective vertex groups, and where the abelian exponent N is such that the modulo N conjugacy equation*

$$\vec{\delta}_i = \lambda_1\vec{\epsilon} + \lambda_2\vec{\nu}_N \mod N$$

has no solution, where δ_i is the difference vector for (g_i, h), then the images of g, h are not conjugate in $\bar{\mathcal{G}}$.

Proof. Consider the image of g as given in (1) in $\bar{\mathcal{G}}$. By impermeability distinct vectors of exponents $[n_4, \ldots, n_l, n_1]^\top \mod N$ will give distinct elements. Suppose towards a contradiction that the images of g and h were conjugate in $\bar{\mathcal{G}}$. Let k be the conjugator that gives $kgk^{-1} =_{\bar{\mathcal{G}}} h$. If k is hyperbolic then, it can be decomposed as $k = ep$ where p cyclically permutes syllables and e is elliptic. It follows that there will be some elliptic k such that $(k')g_i(k')^{-1} =_{\bar{\mathcal{G}}} h$ for some $1 \le i \le c$.

Equation (5) implies that k' must lie in the image of the subgroup $\langle t\bar{c}_1^{-\epsilon_2}, t^{N/\epsilon_2}\rangle$ and in particular that (7), the modulo N conjugacy equation, must hold for δ_i on the left hand side, which is impossible by hypothesis. The result follows. □

5.3.1. *Designer vertex fillings.* Let g_1, g_2 be two hyperbolic elements in G. Our goal is to find a finite quotient of G where the images $\bar{g}_1$ and $\bar{g}_2$ are not conjugate. We will do this by finding a virtually free quotient of a finite index subgroup of G in which powers of images of so-called elevations of g_1 and g_2 are non-conjugate. The finite index subgroup and virtually free quotients we will construct will work specifically for the pair g_1, g_2.

We write $H \leqslant_{\text{fi}} G$ to indicate that H is a finite index subgroup of G. If $H \leqslant_{\text{fi}} G$ then there is some power $e(H)$ such that $g^{e(H)} \in H$ for all $g \in G$. The following result is a consequence of the proof of [CBW12, Lemma 3.1], we reproduce the relevant part here for completeness.

Lemma 5.11. *Let g_1, g_2 be elements of a group G. Suppose there exists $\gamma \in \widehat{G}$ such that $g_1^\gamma = g_2$, i.e. g_1, g_2 are conjugate in the profinite completion of G. Let*

$H \leqslant_{\mathrm{fi}} G$. *Then there exists some* $d \in G$ *and some* $\gamma' \in \hat{H}$ *such that* $(g_1^{e(H)})^{\gamma'} = (g_2^{e(H)})^d$.

Proof. Since $\widehat{G} = \widehat{H}G$ we may write $\gamma = \gamma' d^{-1}$ with $\gamma' \in \widehat{H}$ and $d \in G$. We have $g_2^{e(H)} = \left(g_1^{e(H)}\right)^{\gamma} = \left(g_1^{e(H)}\right)^{\gamma' d^{-1}}$, that is $\left(g_1^{e(H)}\right)^{d} = \left(g_2^{e(H)}\right)^{\gamma'}$. □

Our goal will therefore be, given nonconjugate elements $g_1, g_2 \in G$, to construct some $H \leqslant_{\mathrm{fi}} G$ such that for any G-conjugate of $g_2^{e(H)}$ lying in H there will be some finite quotient of H such that its image will be nonconjugate to the image of $g_1^{e(H)}$. Lemma 5.11 will then imply that g_1 and g_2 have nonconjugates image in some finite quotient of G.

An important tool to achieve our goal is Shepherd's strong commanding property, which is a generalization of Wise's omnipotence.

Definition 5.12 (Commanding group elements [She23, Definition 1.1] (c.f. [Wis00, Definition 3.2]))**.** A group G *commands* a set of elements $\{g_1, \dots, g_n\} \subset G$ if there exists an integer $N > 0$ such that for any integers $r_1, \dots, r_n > 0$ there exists a homomorphism to a finite group $G \to \bar{G}, g \mapsto \bar{g}$ such that the order of $\bar{g}_i$ is Nr_i. If this can always be done with $\langle \bar{g}_i \rangle \cap \langle \bar{g}_j \rangle = \{1\}$ for all $i \neq j$ then we say that G *strongly commands* $\{g_1, \dots, g_n\}$.

Typically, command (or omnipotence) has been used to construct finite degree covering spaces of graphs of spaces - we will be using command in this way and we will also use command in a somewhat novel way to construct vertex fillings. The following lemma indicates how the notion of *strong command* will be useful to construct impermeable double cosets.

Lemma 5.13. *Let* $H = \langle h \rangle$ *and* $K = \langle k \rangle$ *be non-trivial cyclic subgroups of a group* F. *If* $H \cap gKg^{-1} = \{1\}$ *then the double coset* HgK *is impermeable.*

Proof. We will prove the contrapositive, to this end suppose HgK is not impermeable. Then there are distinct pairs of integers $(n_1, m_1) \neq (n_2, m_2)$ such that

$$h^{n_1} g k^{m_1} = h^{n_2} g k^{m_2}.$$

This immediately implies

$$h^{n_1 - n_2} = g k^{m_2 - m_1} g^{-1}$$

and since either $n_1 = n_2$ or $m_2 - m_1$ is non-zero, both must be non-zero so $H \cap gKg^{-1} \neq \{1\}$ and the result follows. □

Let us now first focus on constructing a finite index subgroup that will enable us to do the necessary vertex fillings. This finite index subgroup will be constructed from a finite degree covering space.

Let X be a topological space and let $\ell : S^1 \looparrowright X$ be a loop (i.e. an continuous immersion from a circle to the space). Let $\rho : Y \twoheadrightarrow X$ be a finite degree covering space of X. An *elevation* $\hat{\ell} : S^1 \looparrowright Y$ of ℓ to Y is an immersion such that the following diagram commutes:

$$\begin{array}{ccc} S^1 & \xhookrightarrow{\ \hat{\ell}\ } & Y \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \rho} \\ S^1 & \xhookrightarrow{\ \ell\ } & X \end{array}$$

for some covering map c. The degree of c is the *degree of the elevation.* On the group theoretic level, if an orientation is given to S^1 then the loop ℓ represents a conjugacy class $[g]$ in $G = \pi_1(X, x_0)$ where x_0 is any basepoint. The covering space Y corresponds to a conjugacy class of a finite index subgroup $H \leqslant_{\mathrm{fi}} G$, if

Y is a regular cover, i.e. the group of deck transformations of $\rho : Y \twoheadrightarrow X$ acts transitively on point preimages, then H will in fact be a normal subgroup.

A fixed loop ℓ will admit only finitely many elevations to a finite cover Y. If Y is regular, then if non-empty, the set of elevations $\hat{\ell}_1, \dots, \hat{\ell}_m$ correspond to the set of H-conjugacy classes $[(g^{d_1})^n]_H, \dots, [(g^{d_m})^n]_H$ of generators of the intersection of conjugates of $\langle g \rangle$ and H. The power n is the common degree of the elevations. We call the H-conjugacy classes $[(g^{d_i})^n]_H$ the *algebraic elevations* of g to H. A covering space argument gives $n \cdot m = [G : H]$.

From the graph of groups $\mathcal{G}$ we have a corresponding graph of spaces $\mathcal{X}_{\mathcal{G}}$ constructed from the canonical splitting $\mathcal{G}$ of a piecewise trivial suspension as follows: the vertex space associated to a white vertex w is of the form $X_w \times S^1$ where X_w is a bouquet of circles with vertex x_w and $F_w \cong \pi_1(X_w, x_w)$, the vertex space associated to a black vertex b is a 2-torus $S^1 \times S^1$, edge spaces are also 2-tori. Any non-elliptic $\mathcal{G}$-loop in short position will give rise to an immersed loop $\ell : S^1 \to \mathcal{X}_{\mathcal{G}}$.

Recall that to every $\mathcal{G}$-loop we can associate a sequence of turns. We call a turn of the form (e, e^{-1}) a *sharp turn*. We will need to eliminate sharp turns to enable impermeability. Sharp turns occur precisely when an immersed loop ℓ enters and then exits a vertex space via the same edge space. A finite covering space of $\mathcal{X}_{\mathcal{G}}$ is also naturally a graph of spaces. Our goal is to construct a covering space that ensures that elevations of ℓ no longer have sharp turns. We will achieve this using subgroup separability of free groups.

A finite cover of $\mathcal{X}_{\mathcal{G}}$ corresponds to a (conjugacy class) of a finite index subgroup of G. In particular, if G is a piecewise trivial suspension, then so are all its finite index subgroups. A sharp turn will occur along the loop ℓ if in the $\mathcal{G}$-loop representing its π_1-image we have a subword

$$eae^{-1}.$$

Note that for any white vertex w, $\mathcal{G}_w = F_w \oplus \langle t_w \rangle$ and t_w is contained in the image of every incident edge group. Thus, we may always assume that a_i is contained in the fibre.

Let $H \le G$ be a finite index subgroup. Suppose that for every white vertex $w \in V(X)$, the subgroups in the induced graph-of-groups $\mathcal{G}^H$ intersect F_w as a normal subgroup $H_w \trianglelefteq_{\text{fi}} F_w$. Suppose also that for any sharp turn eae^{-1} with $\tau(e) = w$, we have that the intersections of subgroups I_e and aI_ea^{-1} with H_w are non-conjugate in H_w. Then, the elevations of ℓ will always enter and exit a vertex space from different edge spaces.

Lemma 5.14. *For every F_w, there is a normal finite index subgroup $H_w \trianglelefteq_{\text{fi}} F_w$ such that for any sharp turn eae^{-1} that occurs in any conjugate of g_1 or g_2 in short form with $a \in F_w$, conjugates of $I_e \cap H_w$ and $aI_ea^{-1} \cap H_w$ have trivial intersection in H_w.*

Proof. Let eae^{-1} with $a \in F_w$ be a sharp turn. By hypothesis, the sharp turn occurs in a reduced word so $a \notin I_e \cap F_w = \langle c \rangle$. Also note that by hypothesis $\langle a \rangle$ is a maximal cyclic subgroup of F_w. Suppose that for some $H \trianglelefteq_{\text{fi}} F_w$ there exists $h \in H$ such that

$$\langle c \rangle \cap h \langle aca^{-1} \rangle h^{-1} \neq \{1\}.$$

Then this implies that $[ha, c] = 1$, which in a free group, by maximality of the cyclic group $\langle c \rangle$ implies that $ha \in \langle c \rangle$. In particular the image of a in F_w/H lies in the image of $\langle c \rangle$.

Now F_w is subgroup separable so there exists $H_{a,c} \trianglelefteq_{\text{fi}} F_w$ that separates a from $\langle c \rangle$. In particular, the subgroups $\langle c \rangle \cap H_{a,w}$ and $\langle aca^{-1} \rangle \cap H_{a,w}$ are not conjugate in $H_{a,w}$.

Since any subgroup that is deeper than $H_{a,c}$ will also separate a from $\langle c\rangle$, we may take such a normal finite index subgroup for every sharp turn that occurs in F_w and then take their intersection to obtain H_w with the desired properties. Note that there are finitely many sharp turns to consider since the short form representatives of g_1 and g_2 have finite length. □

Lemma 5.15. *Let $G = \pi_1(\mathcal{G}, v)$ be a piecewise trivial suspension and let $g_1, g_2 \in G$. There exists a finite index subgroup $H \leqslant_{\text{fi}} G$ with an induced graph of groups $\mathcal{G}^H$ such that the following hold:*

(1) Elevations of the conjugates in short form of g_1 and g_2 to H have no sharp turns with respect to $\mathcal{G}^H$.
(2) There is some exponent k_H such that every black vertex group of $\mathcal{G}^H$ is conjugate to some subgroup of k_H powers $k_H \cdot \mathcal{G}_b$ where $b \in V(X)$ a black vertex.
(3) Every white vertex group $\mathcal{G}^H_{\hat{w}}$ is conjugate to a subgroup $\langle H_w, t_w^{k_H}\rangle \leq F_w \oplus \langle t_w\rangle = \mathcal{G}_w$ and the image of every edge group in $\mathcal{G}^H_{\hat{w}}$ is conjugate to a subgroup of the form $\langle c^{k_H}, t_w^{k_H}\rangle \cong \mathbb{Z}^2$ where c is a fibrewise generator of an edge group.

Proof. By Theorem 5.14, for every white vertex $w \in V(X)$, there is a finite index normal subgroup $J_w \trianglelefteq_{\text{fi}} F_w$ such that for every sharp turn eae^{-1} occurring in any conjugate of g_1 or g_2 in short position, we have that the $I_e \cap J_w$ and $aI_ea^{-1} \cap J_w$ have no conjugates in J_w with non-trivial intersection.

Fix a white vertex $w \in V(X)$. For each incident edge e at w, let h_e be the generator of $\mathcal{G}_e \cap J_w$ and let $\mathcal{H}_w$ be the set of all the elements h_e. The hypotheses imply that $\mathcal{H}_w$ is an independent set and thus J_w commands the elements in $\mathcal{H}_w$ by [Wis00, Theorem 3.5]. Each h_e is an algebraic elevation of a fibrewise generator $c_{e_i} \in F_w$ of the image of an incident edge group.

Now, using command, we may find some positive integer D and for each white vertex $w \in V(X)$ a finite index subgroups $H_w \trianglelefteq_f F_w$, such that all algebraic elevations of the fibrewise generators of edge groups in F_w have a common degree D. We note that for every fibrewise generator $c_{e_i} \in F_w$ the sum of the algebraic degrees of all its algebraic elevations is the index $[F_w : H_w]$.

We can now construct finite index subgroups

$$\langle H_w, t_w^D\rangle \leq \mathcal{G}_w \cong F_w \oplus \langle t_w\rangle$$

of the white vertex groups and take the D-congruence subgroups $D\cdot\mathbb{Z}^2$ of the black vertex groups. By following the construction of [DT24, Proposition 6.14] it is possible to construct a finite degree covering graph of spaces, and therefore a finite index subgroup H with the desired properties with $k_H = D$. □

We call the finite index subgroup $H \leq \pi_1(\mathcal{G})$ constructed in Lemma 5.14 *unsharpened relative to* g_1, g_2. We now construct the vertex filling of $H = \pi_1(\mathcal{G}^H)$ that witnesses the non-conjugacy of g_1 and g_2.

Proposition 5.16. *Let g_1, g_2 be non-conjugate elements of $G = \pi_1(\mathcal{G})$ with $DCR(g_1) = DCR(g_2)$. Then it is possible to find a finite index subgroup $H = \pi_1(\mathcal{G}^H)$ that admits a virtually free vertex filling $\mathcal{G}^H \to \overline{\mathcal{G}^H}$ in which all algebraic elevations of g_1 and g_2 have non-conjugate images.*

Proof. We first apply Lemma 5.14 in order to find the finite index subgroup $H = \pi_1(\mathcal{G}^H)$ that is unsharpened relative to g_1, g_2. Let P be the product of the positive integers M given in Theorem 5.9 coming from all pairs of non-conjugate algebraic elevations of g_1, g_2 in H.

Consider now a white vertex group $\mathcal{G}_w^H \cong H_w \oplus \langle t^{k_H} \rangle \leq \mathcal{G}_{[w]}$, where $\mathcal{G}_{[w]}$ is the vertex group of $\mathcal{G}$ that naturally contains $\mathcal{G}_w^H$. The elements

$$E_w = \{c_e \in H_w : e \text{ is incident to } w\}$$

are independent. Let N_w be the integer given in Definition 5.12 given for H_w and the set E_w, and let

$$N_H = M \cdot \prod_w N_w$$

where w ranges over the white vertices of $\mathcal{G}^H$. For each turn $\tau = e'ae^{-1}$ that occurs in a normal form of an algebraic elevation of g_1, g_2 we want to find a finite quotient of $H_w \to \overline{H_w}$ such that the images of $\langle c_{e'} \rangle$ and $\langle ac_e a^{-1} \rangle$ have trivial intersection. By [BW15, Theorem 4.3], H_w strongly commands E_w and replacing some c_e with a conjugate $ac_e a^{-1}$ we can ensure by strong command that there is a finite quotient H_w^τ where the images of all the elements c_e have order N_H and where the images of $\langle c_{e'} \rangle$ and $\langle ac_e a^{-1} \rangle$ have trivial intersection. Let K_w^τ be the kernel of this homomorphism and let $K_w = \bigcap_\tau K_w^\tau$ where τ ranges over the turns that occur in w. Then the normal subgroup $\mathcal{K}_w = K_w \oplus \langle t^{k_H N_H} \rangle \trianglelefteq_{\text{fi}} H_w \oplus \langle t^{k_H} \rangle$ is such that for every turn $\tau = e'ae^{-1}$, the element a is in the fibre, the corresponding double coset images are impermeable and the image of every edge group is a N_H-congruence quotient of the image of the edge group in $\mathcal{G}_w^H$. It follows that that the $\mathcal{K}_w$ give a system of vertex fillings and the result follows from Proposition 5.10. □

5.4. **Conjugacy separability in the unipotent linear case.**

Theorem 5.17. *Let G be a free-by-cyclic group with unipotent and linearly growing monodromy. Then G is conjugacy separable and every cyclic subgroup of G is conjugacy distinguished..*

Proof. We begin by showing that G is conjugacy separable. Since G has the unique roots property by Theorem 2.14, bu Theorem 2.4 it suffices to show that G contains a conjugacy separable subgroup of finite index.

Up to replacing G by a finite index subgroup we may assume that G admits a splitting $G = \pi_1(\mathcal{G})$ where $\mathcal{G}$ is a clean piecewise trivial suspension satisfying the properties in Theorem 5.1. Let $g_1, g_2 \in G$ be non-conjugate elements.

If both g_1 and g_2 are elliptic, then by Theorem 4.10 there exists a finite quotient of G such that the image of g_1 is not conjugate to the image of g_2. Moreover, if one of the g_i is loxodromic, or if both g_1 and g_2 are loxodromic with different translation lengths, then by Theorem 4.9 there exists a finite quotient where the images are not conjugate.

If no conjugates of g_1 and g_2 in short position have the same sets of double coset representatives then their conjugacy classes can be separated in a finite quotient by Theorem 5.5. Otherwise, we combine Theorem 5.10 with Theorem 5.16, noting that we may pass to a finite index subgroup by Theorem 5.11.

We also claim that cyclic subgroups are conjugacy distinguished in G. We will again use the piecewise trivial splitting of G, noting that the edge groups are root-closed by Theorem 2.15. Since we have already established that G is conjugacy separable, we can apply Theorem 4.11 and Theorem 4.12 to conclude the claim. □

6. Double-$\mathbb{Z}$-coset separability

The aim of this section is to prove that double cosets of cyclic subgroups are separable in free-by-cyclic groups with polynomially growing monodromy (see Theorem C). Note that for any element $g \in G$, the double coset $HgK \subset G$ is separable if and only if $HgKg^{-1} \subset G$ is separable. Hence we only need to consider separability of double cosets of the form $HK' \subset G$ where H and K' are cyclic.

Lemma 6.1. *Let $\mathcal{G}$ be a κ-acylindrical graph of groups and let h, k be two hyperbolic elements of $\pi_1(\mathcal{G}, v_0)$ with distinct respective axes α_h, α_k in the dual Bass–Serre tree T. Let $H = \langle h \rangle$ and $K = \langle k \rangle$. Let v be a vertex in α_k and let $T_{HK} \subset T$ be the convex hull of $HK \cdot v$. Then the vertices of T_{HK} have valence at most 4. Furthermore, for any vertex $w \in T_{HK}$ there are at most finitely many elements $x \in HK$ such that $x \cdot v = w$.*

Proof. Since h, k have distinct axes, acylindricity forces the intersection of the axes to be contained in a finite interval. The tree T_{HK} must therefore have one of the following configurations shown in Figure 1, depending on whether the axes α_h, α_k are disjoint, intersect in a point, or intersect in an arc. The advertised bound on vertex degrees follows immediately. Noting that h and

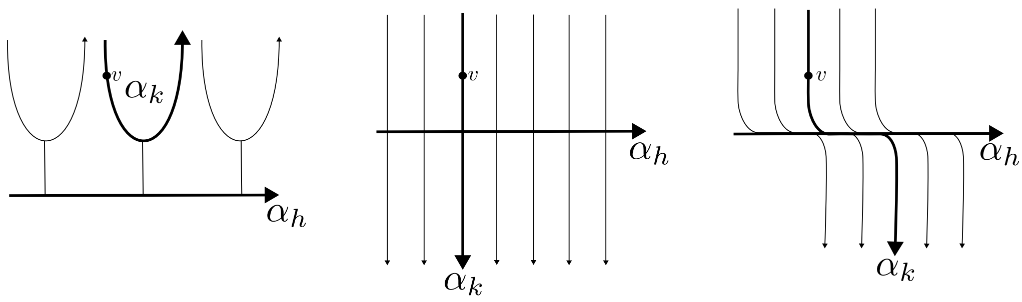

Figure 1. The three configurations for T_{HK}. The axes α_h, α_k are drawn thicker.

k translate points along the axes α_h and α_k, respectively, we immediately see that in the first two configurations the set $HK \cdot v = \{h^m k^n \cdot v : m, n \in \mathbb{Z}\}$ is in bijective correspondence with HK. In the third case, where the axes have an intersection that contains a non-trivial arc I, the only way for it to be possible that $h^{m_1} k^{n_1} \cdot v = h^{m_2} k^{n_2} \cdot v$, is if both $k^{n_1} v, k^{n_2} v \in I$ and the different powers of h make these two points coincide. Since there are only finitely many powers k^n of k such that $k^n \cdot v \in I$ the finiteness result follows. □

Lemma 6.2. *Let $\mathcal{G}$ be a κ-acylindrical graph of groups that admits arbitrarily deep edge-group-separating virtually free vertex fillings. Assume that $G = \pi_1(\mathcal{G}, v_0)$ does not contain elements of order 2. Let $H, K \leq G$ be cyclic non-elliptic subgroups. Then the double coset HK is separable in G.*

Proof. Let $g \in G \smallsetminus HK$. We begin by constructing a virtually free quotient of G such that the image of g is not mapped to the image of HK.

We will split into two cases depending on whether the axes of the cyclic subgroups H and K are distinct or coincide.

Case 1: *H and K have distinct axes.*

Let $T_{HK} \subset T$ be the tree given in the statement of Lemma 6.1 and let us fix a vertex v contained in α_k, the axis of k. We split into two subcases depending on the behaviour of $g \cdot v$.

Subcase i: $g \cdot v \in T_{HK}$.

By Theorem 6.1, the set $g\mathcal{G}_v \cap HK$ is at most finite. Let $\mathcal{A}$ be the set of all elements of the form $y^{-1}g$ for $y \in g\mathcal{G}_v \cap HK$. Note that all elements of $\mathcal{A}$ are non-trivial since $g \notin HK$.

Since $\mathcal{G}_v$ is residually finite, there exists a finite index normal subgroup $N_v \trianglelefteq_f \mathcal{G}_v$ such that each element of $\mathcal{A} \subseteq \mathcal{G}_v$ is mapped to a non-trivial element via the quotient $\mathcal{G}_v \to \mathcal{G}_v / N_v$. Let $\mathcal{G} \to \overline{\mathcal{G}}$ be a virtually free vertex filling associated to N_v.

Now, suppose for contradiction that $\bar{g} \in \overline{HK}$. Then $\bar{g} \in \bar{g}\overline{\mathcal{G}}_v \cap \overline{HK}$. Hence, there exists some $y \in g\mathcal{G}_v \cap HK$ such that $\bar{y}^{-1}\bar{g} = 1$. This contradicts the fact that the elements in $\mathcal{A}$ have non-trivial image in the quotient.

Subcase ii: $g \cdot v \notin T_{HK}$.

By construction, there is a finite connected subgraph $\mathcal{D} \subset T_{HK}$ such that

$$T_{HK} \subset \bigcup_{x \in HK} x \cdot \mathcal{D}.$$

In particular, if w is any vertex in $\mathcal{D}$ and e is an edge of $\mathcal{D}$ that is adjacent to w, then if $w' \in HK \cdot w$ is another vertex in T_{HK}, it follows that there are finitely many elements $x \in HK$ such that $x \cdot w = w'$ and $x \cdot e$ is an edge of T_{HK} adjacent to w'.

Then, by the edge group separating hypothesis, there exists a vertex filling $\mathcal{G} \to \overline{\mathcal{G}}$ that doesn't fold any pair of edges of T_{HK} incident to any vertex $w \in \mathcal{D}$. It follows that the tree T_{HK} is mapped injectively via the folding map $T \to \overline{T}$.

Let ρ be the path of minimal length joining $g \cdot v$ to T_{HK}. Note that by Lemma 6.1 the maximal possible valency of any vertex in the tree $T_{HK} \cup \rho$ is 5. Thus if we take a vertex filling deep enough so that T_{HK} is mapped injectively, we can then take an ever deeper filling so that none of the finitely many edges of $T_{HK} \cup \rho$ adjacent to vertices in ρ get folded together. It now follows that

$$\bar{g} \cdot \bar{v} \notin \overline{T_{HK}} \supset \overline{HK}\bar{v}$$

which means that $\bar{g} \notin \overline{HK}$ as required. ◆

We have exhausted all possibilities for $g \cdot v$ completing the proof of Case 1. ∎

Case 2: *The axes of H and K coincide.*

Since the action of G on T is acylindrical, it must be the case that H and K are contained in a maximal cyclic subgroup. Here we are using acylindricity and that there are no elements of order 2 (otherwise they could be contained in D_∞). Thus, we see that HK coincides with a cyclic subgroup C. The result in this case follows by the same arguments as those given above, the only difference is that T_{HK} is the minimal C-invariant tree T_C.

Finally, since $g \in G \smallsetminus HK$ was arbitrary, and since finitely generated virtually free groups are double coset separable, we conclude that HK is separable in G. □

Lemma 6.3. *Let $\mathcal{G}$ be a κ-acylindrical graph of groups that admits arbitrarily deep edge separating virtually free vertex fillings and let $G = \pi_1(\mathcal{G}, v_0)$. Let $H, K \leq G$ be cyclic subgroups and suppose that H is generated by a loxodromic element and K is generated by an elliptic element. Then the double cosets HK and KH are separable in G.*

Proof. Let $H = \langle h \rangle$ and let α_h denote the axis of h. Let $v \in \mathrm{Fix}(K)$ be the element of $\mathrm{Fix}(K)$ closest to α_h. Let T_{HK} be the convex hull of the set $HK \cdot v$ in the Bass–Serre tree T associated to $\mathcal{G}$. If v is contained in α_h then $T_{HK} = \alpha_h$. Otherwise, T_{HK} is as in Fig. 2. In particular, T_{HK} is locally finite and for any vertex $w \in T_{HK}$, there is at most one $m \in \mathbb{Z}$ such that $h^m k \cdot v = w$ for some $k \in K$. Now the argument is the same as in Theorem 6.2. To show that KH is separable, we note that the map $G \to G, g \mapsto g^{-1}$ is continuous with respect to the profinite topology on G. Hence, if $HK \subseteq G$ is closed then so is $K^{-1}H^{-1} = KH$. □

Lemma 6.4. *Let $\mathcal{G}$ be a κ-acylindrical graph of groups such that double cosets of cyclic subgroups are separable in the vertex groups. Let $G = \pi_1(\mathcal{G}, v_0)$. Suppose that all the vertex groups are fully separable in G. Let $H, K \leq G$ be cyclic subgroups generated by elliptic elements such that* $\mathrm{Fix}(H) \cap \mathrm{Fix}(K) \neq \varnothing$. *Then HK is separable in G.*

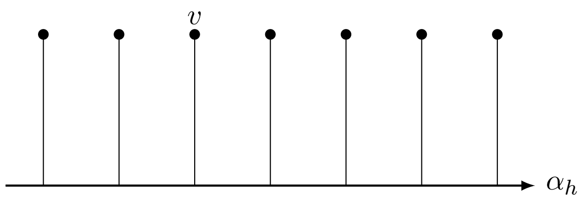

FIGURE 2. The configuration for T_{HK} when H is hyperbolic and K elliptic.

Proof. Pick a vertex $v \in \mathrm{Fix}(H) \cap \mathrm{Fix}(K)$. Hence, $H, K \leq \mathcal{G}_v$. Then, since double cosets of cyclic subgroups are separable in $\mathcal{G}_v$, it follows that HK is closed in the profinite topology on $\mathcal{G}_v$. By assumption, G induces the full profinite topology on $\mathcal{G}_v$ and thus $HK \subseteq G$ is separable. □

Lemma 6.5. *Let $\mathcal{G}$ be a clean piecewise trivial suspension with free local fibres and let $G = \pi_1(\mathcal{G}, v_0)$. Let $H, K \leq G$ be cyclic subgroups generated by elliptic elements. Suppose that $\mathrm{Fix}(H) \cap \mathrm{Fix}(K) = \varnothing$. Then HK is separable in G.*

We postpone the proof of Theorem 6.5 until the end of the section.

Proposition 6.6. *Let G be free-by-cyclic with unipotent and linearly growing monodromy. Then double cosets of cyclic subgroups are separable.*

Proof. By Theorem 2.6, it suffices to show that G admits a finite index subgroup with separable double cosets of cyclic subgroups. Hence, assume that G admits a splitting $G \cong \pi_1(\mathcal{G})$ where $\mathcal{G}$ is a clean piecewise trivial suspension with free local fibres. Then, all vertex groups are residually finite and all edge groups are separable in the vertex groups. Hence, $\mathcal{G}$ admits edge separating virtually free vertex fillings by Theorem 5.2 and Theorem 4.4.

If at least one of the subgroups is generated by a hyperbolic element, then the result follows by Theorem 6.2 and Theorem 6.3. Otherwise, the result holds by Theorem 6.4 and Theorem 6.5. □

We are now ready to prove the main result of this section.

Theorem C. *Let G be a free-by-cyclic group with polynomially growing monodromy. Then double cosets of cyclic subgroups are separable in G.*

Proof. By Theorem 2.6, it suffices to show that G admits a finite index subgroup with separable double cosets of cyclic subgroups. Hence, we may assume without loss of generality that G has unipotent and polynomially growing monodromy.

We will argue by induction on the degree d of growth. If $d = 0$ then $G \cong F \times \mathbb{Z}$ where F is a free group of finite rank. Double cosets of finitely generated subgroups of free groups are separable by Theorem 2.7. It follows that double cosets of cyclic subgroups are separable in G.

The case $d = 1$ is proved in Theorem 6.6.

Now suppose that $d \geq 2$. Let $G = \pi_1(\mathcal{G})$ be the standard splitting and let T be the associated Bass–Serre tree. Let H and K be finitely generated cyclic subgroups of G.

By Theorem 2.16, we may find a finite index subgroup $G' = F \rtimes_\phi \langle t \rangle$ where $\phi \in \mathrm{Aut}(F)$ is unipotent and polynomially growing of degree d, and such that $H' := H \cap G' = \langle ut \rangle$ or $H' = \langle v \rangle$, and $K' := K \cap G' = \langle u't \rangle$ or $K' = \langle v' \rangle$ for some $u, u', v, v' \in F$ and v and v' not proper powers. By Theorem 2.6, it suffices to show that the double coset $H'K' \subset G'$ is separable. Hence, without loss of generality, we may assume that H and K are of this form.

If at least one of H or K is generated by a hyperbolic element, then HK is separable by Theorem 6.2 and Theorem 6.3. If both H and K are elliptic and

$\mathrm{Fix}(H) \cap \mathrm{Fix}(K) \neq \varnothing$ then the result follows by Theorem 6.4. Hence, let us assume that H and K are elliptic and $\mathrm{Fix}(H) \cap \mathrm{Fix}(K) = \varnothing$. Let $[a,b] \subset T$ be the segment that realises the distance between $\mathrm{Fix}(H)$ and $\mathrm{Fix}(K)$ where a is fixed by K and b is fixed by H. We define

$$T_{HK} = \bigcup_{h \in H} h \cdot [a,b]$$

and let $g \in G \setminus HK$.

Note that by the root-closed property of the edge groups, if there is an edge on $[a,b]$ fixed by a proper power of the generator of H, then it is fixed by all of H. Hence, no edge on $[a,b]$ is fixed by a non-trivial element of H. Let e be the edge on $[a,b]$ adjacent to b.

Case 1: $g \cdot a \in HK \cdot a$.

Let $h \in H$ be such that $g \cdot a = h \cdot a$, note that such h exists because K fixes a by assumption. Since $g \notin HK$, we have that $h^{-1}g \in \mathcal{G}_a \setminus K$.

Claim 6.7. *For a sufficiently high prime p, there exists a p-periodic quotient $\mathcal{G}_a \to \overline{\mathcal{G}}_a$ such that $\overline{h^{-1}g} \notin \overline{K}$ and the image of every incident edge group has order p.*

Proof. We will show that $\langle k \rangle \cap \langle h^{-1}g \rangle = 1$ or $[k, h^{-1}g] \neq 1$.

Suppose first that k and $h^{-1}g$ are elements of the fibre F and $\langle k \rangle \cap \langle h^{-1}g \rangle \neq 1$. Then k and $h^{-1}g$ must be powers of a common element. Since $k \in F$ is not a proper power (by our initial assumptions on H and K), it follows that $h^{-1}g \in K$, a contradiction. If exactly one of $\{k, h^{-1}g\}$ is contained in the fibre, then it is clear that $\langle k \rangle \cap \langle h^{-1}g \rangle = 1$. Finally, if both are not contained in the fibre then in particular $k \in F \cdot t$ (by our initial assumptions on H and K), and thus $C_G(k) = (C_G(k) \cap F) \oplus \langle k \rangle$. Hence, if k and $h^{-1}g$ commute and $\langle k \rangle \cap \langle h^{-1}g \rangle \neq 1$ then it must be the case that $h^{-1}g \in \langle k \rangle$, another contradiction.

Now, by Theorem D there exists a positive integer $L_a > 0$ such that for any prime $p > L_a$, there is a p-periodic quotient $\mathcal{G}_a \to \overline{\mathcal{G}_a}$ such that the images of the generators of incident edge groups and the elements $h^{-1}g$ and k are non-trivial and such that $\overline{\langle h^{-1}g \rangle} \cap \overline{\langle k \rangle} = 1$. In particular, $\overline{h^{-1}g} \notin \overline{K}$.

□

Claim 6.8. *For a sufficiently high prime p, there exists a p-periodic quotient $\mathcal{G}_b \to \overline{\mathcal{G}}_b$ such that the image of H and $\mathcal{G}_e$ have trivial intersection and each incident edge group has order p.*

Proof. This is essentially the same argument as Theorem 6.7.

□

Claim 6.9. *There is a virtually free filling $\mathcal{G} \to \overline{\mathcal{G}}$ such that the images of $\mathcal{G}_e$ and H have trivial intersection, and the image of $h^{-1}g$ is not contained in the image of K.*

Proof. For each vertex $v \in V(X) \setminus \{a,b\}$, we use Theorem 3.15 to construct a finite quotient such that each incident edge group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for a sufficiently high prime p. By Theorem 6.7 and Theorem 6.8, there is a sufficiently high prime p such that we may find a p-periodic quotient of $\mathcal{G}_a$ and $\mathcal{G}_b$ such that all incident edge groups are isomorphic to $\mathbb{Z}/p\mathbb{Z}$, the image of $\mathcal{G}_e$ and H have trivial intersection, and the image of $h^{-1}g$ is not contained in the image of K. Then, we may assemble all such quotients to construct a virtually free vertex filling $\mathcal{G} \to \overline{\mathcal{G}}$. □

We now complete the proof of Case 1. To this end, consider the filling $\mathcal{G} \to \overline{\mathcal{G}}$ constructed in Theorem 6.9 and suppose for contradiction that $\bar{g} \in \overline{HK}$. Then $\bar{g} = \bar{h}'\bar{k}$ for some $h' \in H$ and $k \in K$. Thus $\bar{h}'^{-1}\bar{g} \in \overline{\mathcal{G}}_a$. Hence $(\bar{h}'^{-1}\bar{g})(\bar{g}^{-1}\bar{h}) \in \overline{\mathcal{G}}_a$. It follows that $\bar{h}'^{-1}\bar{h} \in \overline{\mathcal{G}}_a \cap \overline{\mathcal{G}}_b \leq \overline{\mathcal{G}}_e$. Hence, $\bar{h}'^{-1}\bar{h} \in \overline{\mathcal{G}_e} \cap \overline{H}$. However, by Theorem 6.9 we have constructed a virtually free vertex filling such that the images of $\mathcal{G}_e$ and H have trivial intersection. Hence, we must have that $\bar{h}'^{-1}\bar{h} = 1$. Thus, $\bar{h}^{-1}\bar{g} \in \overline{K}$. Again by Theorem 6.9, in the constructed virtually free vertex filling the image of $h^{-1}g$ is not contained in the image of K and thus we have arrived at a contradiction. It follows that $\bar{g} \notin \overline{HK}$ as required. ∎

Case 2: $g \cdot a \notin HK \cdot a$.

Let e' be the edge on $[g \cdot a, b]$ adjacent to b.

Subcase i: *There is some $h \in H$ such that $h \cdot e = e'$.*

By assumption, we have that $h^{-1}g \cdot a \neq a$ and e is an edge in the intersection $h^{-1}[g \cdot a, b] \cap [a, b]$. Let η be the geodesic in T that joins a to $h^{-1}g \cdot a$. Note that the path η does not contain b. Using Theorem 3.15 there exists some $L > 0$ such that for every prime $p > L$ there is a virtually free vertex filling of T such that η is mapped injectively to $\overline{T}$ and the order of every edge group is p. Moreover, we may find some further $L' > L$ such that for any prime $p > L'$, there is a finite quotient Q_b of $\mathcal{G}_{b'}$ so that the images of H and $\mathcal{G}_e$ have trivial intersection and every incident edge has order p. As before, we may replace the image of $\mathcal{G}_b$ in the original virtually free vertex filling with the quotient Q_b, to obtain a virtually free vertex filling $\mathcal{G} \to \overline{\mathcal{G}}$ so that the images of H and $\mathcal{G}_e$ have trivial intersection and the segment η is mapped injectively via the folding map $T \to \overline{T}$.

Suppose now that $\bar{g} = \bar{h}' \cdot \bar{k}$ for $h' \in H$ and $k \in K$. Then $\bar{h}' \cdot \bar{e} = \bar{h} \cdot \bar{e}$ and thus $\bar{h}'^{-1}\bar{h} \in \overline{\mathcal{G}_e} \cap \overline{H}$. But by construction of the virtually free vertex filling, we have that $\overline{H} \cap \overline{\mathcal{G}_e} = 1$ and thus $\bar{h}' = \bar{h}$. It follows that $\bar{h}^{-1}\bar{g} \cdot \bar{a} = \bar{a}$. However, the fact that the geodesic η is mapped injectively via the vertex filling $T \to \overline{T}$ implies that $\bar{h}^{-1}\bar{g} \cdot \bar{a} \neq \bar{a}$, which gives a contradiction. Hence in this case also we have that $\bar{g} \notin \overline{HK}$. ◆

Subcase ii: *There is no $h \in H$ such that $h \cdot e = e'$.*

Let $\mathcal{G} \to \overline{\mathcal{G}}$ be a virtually free vertex filling such that no edges in $[a, b]$ are folded in the image and the same is true for $[g \cdot a, b]$. If e and e' are not in the same G-orbit then it cannot be the case that $\bar{e}$ and $\bar{e}'$ are in the same $\overline{G}$-orbit by Theorem 4.3. Then, we must have that $\bar{g} \notin \overline{HK}$.

Let us then assume then that there exists some $x \in G$ such that $x \cdot e = e'$. It must then be the case that $x \notin H\mathcal{G}_e$. By induction, using double coset separability of abelian subgroups in the vertex groups, we may construct a deeper virtually free vertex filling such that the image of x is not in the image of the double coset $H\mathcal{G}_e$. Hence, again we have that $\bar{g} \cdot \bar{a}$ will not be in the image of $\overline{HK} \cdot \bar{a}$. ◆

The two subcases together complete the proof of Case 2 ∎

Cases 1 and 2 exhaust all possibilities and the result follows. □

We end the section with a sketch of the proof of Theorem 6.5. The argument follows the same outline as the proof of Theorem C and thus we give a detailed account of only one of the cases, leaving the others as an exercise.

Proof of Theorem 6.5. As before, after possibly passing to a further finite index subgroup, we may assume that $G = F \rtimes \langle t \rangle$ and $H = \langle ut \rangle$ or $H = \langle v \rangle$, and $K = \langle u't \rangle$ or $K = \langle v' \rangle$ for some $u, u', v, v' \in F$ and v and v' not proper powers.

Let $[a, b] \subset T$ be the segment realising the distance between $\mathrm{Fix}(K)$ and $\mathrm{Fix}(H)$, with $a \in \mathrm{Fix}(K)$ and $b \in \mathrm{Fix}(H)$. Note that since the edge groups map surjectively onto the black vertex groups, it must be the case that both a and b are white vertices and thus are of the form $\mathcal{G}_a = F_a \oplus \langle t_a \rangle$ and $\mathcal{G}_b = F_b \oplus \langle t_b \rangle$ where F_a and F_b are finite rank non-abelian subgroup of the fibre F.

Let $T_{HK} = \bigcup_{h \in H} h \cdot [a, b]$. Let e be the edge in $[a, b]$ adjacent to b. Let $g \in G \smallsetminus HK$.

We begin by assuming that $g \cdot a \in HK \cdot a$. Thus, there is some $h \in H$ such that $h^{-1}g \in \mathcal{G}_a$ and thus $h^{-1}g \in \mathcal{G}_a \smallsetminus K$. Now since $\mathcal{G}_a$ is cyclic subgroup separable, there is a finite index normal subgroup $N_a \trianglelefteq \mathcal{G}_a$ such that the image of $h^{-1}g$ is separated from the image of K in of $\mathcal{G}_a/N_a$. Let $t_a \in \mathcal{G}_a$ be the central element. By Theorem 5.2 and Theorem 5.3, we may find some positive integer $N > 0$ such that for every $i \in \mathbb{N}$, there is a virtually free vertex filling of $\mathcal{G}$ associated to $\{N_a\}$ such that the image of each edge group is of the form $(\mathbb{Z}/iN\mathbb{Z})^2$ and quotients of white vertices are of the form $F_w \oplus \langle t_w \rangle \twoheadrightarrow F_w/D_w \oplus \langle \bar{t}_w \rangle$ where $\bar{t}_w$ has order iN.

The image of $\mathcal{G}_e$ in $\mathcal{G}_b$ is $\langle c_e, t_b \rangle$ for some $c_e \in F \cap \mathcal{G}_b =: F_b$ and $h = ut_b^k$ for some $k \in \mathbb{Z}$ and $u \in F_b$. Note that no conjugate of a power of h fixes the edge e. Hence, u and c_e are independent in F_b. Now for every other incident edge e' at b, let $c_{e'}$ be a generator of the cyclic subgroup $\mathcal{G}_{e'} \cap F$. By strong omnipotence of free groups, there exists some positive integer $L > 0$ such that for any positive integer $j > 0$, there is a finite index normal subgroup $(F_b')_j \trianglelefteq_f F_b$ such that the images of the subgroups generated by u and c_e have trivial intersection and the order of the image of $c_{e'}$ for every edge e' incident at b is jL. In particular, the quotient $\mathcal{G}_b \to \mathcal{G}_b/\langle (F_b')_{LN}, t_b^{LN} \rangle$ is such that the image of each incident edge group is of the form

$$\langle \bar{c}_e \rangle \oplus \langle \bar{t}_b \rangle \cong \mathbb{Z}/LN\mathbb{Z} \times \mathbb{Z}/LN\mathbb{Z},$$

and the intersection of the image of the cyclic subgroup generated by h with the image of the incident edge group $\mathcal{G}_e$ is trivial.

Now, construct a vertex filling $\mathcal{G} \to \overline{\mathcal{G}}$ as described two paragraphs above, setting $i = L$. Replace the image of the vertex group corresponding to the vertex b in the filling by the quotient constructed in the previous paragraph,

$$\overline{\mathcal{G}}_b = \mathcal{G}_b/\langle (F_b')_{LN}, t_b^{LN} \rangle.$$

As a result, we obtain a filling of $\mathcal{G}$ such that $\langle \bar{h} \rangle \cap \overline{\mathcal{G}_e} = 1$ and $\overline{h^{-1}g} \notin \overline{K}$. Now we may apply the same argument as in the proof of Section 6 to conclude that $\bar{g} \notin \overline{HK}$.

The other case, that is when $g \cdot a \notin HK \cdot a$, follows the same outline as Section 6, again using the strong omnipotence of free groups to construct finite quotients of the white vertex groups with the required intersection properties of subgroups. □

7. Conjugacy separability

7.1. **The unipotent case.** In order to prove that all free-by-cyclic groups with unipotent monodromies are conjugacy separable, we will induct on the degree of growth and use the following combination theorem of Wilton–Zalesskii:

Theorem 7.1 (Wilton–Zalesskii [WZ10])**.** *Let $\mathcal{G}$ be a graph of groups with conjugacy separable vertex groups and suppose that the profinite topology on $G = \pi_1(\mathcal{G})$ is efficient. Suppose that the following conditions are satisfied for all vertices $v \in V(X)$ and edges $e, f \in E(X)$ that are adjacent to v.*

1. *The edge group $\mathcal{G}_e$ is conjugacy distinguished in $\mathcal{G}_v$.*
2. *For any $g \in \mathcal{G}_v$, the double coset $\mathcal{G}_e g \mathcal{G}_f$ is separable in $\mathcal{G}_v$.*
3. *The intersection $\overline{\mathcal{G}_e} \cap \overline{\mathcal{G}_f}$ of the closures of $\mathcal{G}_e$ and $\mathcal{G}_f$ in the profinite completion of $\mathcal{G}_v$, is equal to the profinite completion $\widehat{\mathcal{G}_e \cap \mathcal{G}_v}$.*
4. *The graph of groups $\mathcal{G}$ is profinitely 2-acylindrical.*

Then G is conjugacy separable.

We are now ready to prove the main result for unipotent and polynomially growing automorphisms.

Theorem 7.2. *Let G be a free-by-cyclic group with unipotent and polynomially growing monodromy. Then G is conjugacy separable and every cyclic subgroup is conjugacy distinguished.*

Proof. We will argue by induction on the degree d of growth. If $d = 0$ then $G \cong F \times \mathbb{Z}$ where F is a finitely generated free group. It is not hard to see that G is conjugacy separable and that every infinite cyclic subgroup is conjugacy distinguished, since those properties hold for the fibre F.

When $d = 1$, both results follow by Theorem 5.17.

Suppose now that $d \geq 2$. Let $G \cong \pi_1(\mathcal{G}, v)$ be the standard splitting (as in Theorem 2.18). We will show that all the hypotheses of Theorem 7.1 are satisfied and thus G is conjugacy separable.

We begin by noting that the vertex groups are of the form $F_w \rtimes_\phi \mathbb{Z}$ where $\phi \in \mathrm{Aut}(F_w)$ is a representative of a unipotent and polynomially growing outer automorphism with polynomial growth of degree $d_w \leq d-1$, and thus each vertex group is conjugacy separable by induction. Moreover, the edge groups are cyclic and thus conjugacy distinguished in the vertex groups by induction.

The profinite topology on $G \cong \pi_1(\mathcal{G}, v)$ is efficient by Theorem 4.7.

The vertex groups are double-$\mathbb{Z}$-coset separable by Theorem C, and thus for any vertex v and incident edge groups $\mathcal{G}_e$ and $\mathcal{G}_f$, and for any finite index subgroup $\mathcal{G}'_f \leq_f \mathcal{G}_f$, the double coset $\mathcal{G}_e\mathcal{G}'_f \subseteq G$ is separable. It follows by Theorem 2.8 that $\overline{\mathcal{G}}_e \cap \overline{\mathcal{G}}_f = \overline{\mathcal{G}_e \cap \mathcal{G}_f} = \widehat{\mathcal{G}_e \cap \mathcal{G}_f}$.

Finally, the group G admits an epimorphism $\phi\colon G \to \mathbb{Z}$ that is surjective on edge groups. Hence, for any two edge groups $\mathcal{G}_e$ and $\mathcal{G}_f$, we must have that $\mathcal{G}_e \cap \mathcal{G}_f$ is either trivial or $\mathcal{G}_e = \mathcal{G}_f$. It follows by Theorem 2.12 that $\widehat{\mathcal{G}}$ is profinitely 2-acylindrical.

Hence, the conditions of Theorem 7.1 are satisfied and thus G is conjugacy separable.

To show that cyclic subgroups are conjugacy distinguished, we apply Theorem 4.11 and Theorem 4.12, noting that the hypotheses are satisfied by Theorem 4.7. □

7.2. **The bootstrap.** We will show that all polynomially growing free-by-cyclic groups are conjugacy separable using the following:

Theorem 7.3 (Chagas–Zalesskii [CZ10, Theorem 2.4])**.** *Let G be a finitely generated torsion free group that admits a conjugacy separable normal subgroup of finite index. Suppose also that for every $g \in G \smallsetminus 1$, the centraliser $C_G(g)$ is conjugacy separable and $\overline{C_G(g)} = \widehat{C_G(g)}$. Then G is conjugacy separable.*

Lemma 7.4. *Let G be free-by-cyclic. Then for every $g \in G \smallsetminus 1$, the centraliser $C_G(g)$ is either infinite cyclic or isomorphic to a free-by-cyclic subgroup with finite order monodromy.*

Proof. Let $G = F \rtimes_\phi \langle t \rangle$ and let $\chi\colon G \to \mathbb{Z}$ be the map that sends $F \mapsto 0$ and $t \mapsto 1$. Let $g \in G \smallsetminus 1$.

Suppose first that $g \in F$. Then,

$$C_G(g) \cap F = C_F(g) \cong \mathbb{Z}.$$

Hence, either $\chi(C_G(g)) = 0$, in which case

$$C_G(g) = C_G(g) \cap F \cong \mathbb{Z},$$

or χ is non-trivial on $C_G(g)$, and

$$C_G(g) \cong \mathbb{Z} \rtimes \mathbb{Z}.$$

Suppose now that $\chi(g) \neq 0$. Then,

$$C_G(g) = (C_G(g) \cap F) \rtimes \langle s \rangle,$$

for some $s \in G \smallsetminus F$. We must have that $g = ys^l$ for some $y \in C_G(g) \cap F$ and $l \in \mathbb{Z} \smallsetminus 0$. Then $C_G(g)$ admits a finite index subgroup generated by

$$\langle C_G(g) \cap F, ys^l \rangle = (C_G(g) \cap F) \oplus \langle g \rangle.$$

Let $\psi \in \mathrm{Aut}(F)$ be the automorphism induced by conjugation action by g. Then, $C_G(g) \cap F = \mathrm{Fix}_F(\psi)$. By the work of Gersten, $\mathrm{Fix}_F(\psi)$ is finitely generated [Ger87]. It follows that $\langle C_G(g) \cap F, ys^l \rangle \cong F' \times \mathbb{Z}$ where F' is finite rank free. Hence, $C_G(g)$ is a free-by-cyclic group with finite order monodromy. □

We will need the following result due to G. Bartlett:

Theorem 7.5 (Bartlett [Bar25, Proposition 2.7])**.** *Let G be free-by-cyclic with finite order monodromy. Then G is conjugacy separable.*

We are now ready to prove our main result.

Theorem A. *Let G be a free-by-cyclic group with polynomially growing monodromy. Then G is conjugacy separable.*

Proof. Every free-by-cyclic group with polynomially growing monodromy admits a finite index normal subgroup that is free-by-cyclic with UPG monodromy [BFH05, Corollary 5.7.6]. Thus, G admits a finite index normal subgroup that is conjugacy separable by Theorem 7.2.

The centraliser of every non-trivial element is cyclic or free-by-cyclic with finite order monodromy by Theorem 7.4 and thus conjugacy separable by Theorem 7.5. Moreover, every free-by-cyclic or abelian subgroup H of G is fully separable by [HK25, Proposition 2.9], and thus we have that $\overline{C_G(g)} = \widehat{C_G(g)}$.

The result now follows from Theorem 7.3. □

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