



# On the higher topological complexity of manifolds with abelian fundamental group

N. Cadavid-Aguilar<sup>1</sup> · D. Cohen<sup>2</sup> · J. González<sup>1</sup> · S. Hughes<sup>3</sup> · L. Vandembroucq<sup>4</sup>

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## Abstract

We study the higher (or sequential) topological complexity  $TC_s$  of manifolds with abelian fundamental group. We give sufficient conditions for  $TC_s$  to be non-maximal in both the orientable and non-orientable cases. In combination with cohomological lower bounds, we also obtain some exact values for certain families of manifolds.

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✉ J. González  
jesus.glz-espino@cinvestav.mx

N. Cadavid-Aguilar  
ncadavia@gmail.com

D. Cohen  
cohen@math.lsu.edu  
www.math.lsu.edu/~cohen

S. Hughes  
sam.hughes.maths@gmail.com ; hughes@math.uni-bonn.de  
samhughesmaths.github.io

L. Vandembroucq  
lucile@math.uminho.pt

- <sup>1</sup> Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Av. Instituto Politécnico Nacional 2508, San Pedro Zacatenco, Ciudad de México 07000, Mexico
- <sup>2</sup> Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803, USA
- <sup>3</sup> Rheinische Friedrich-Wilhelms-Universität Bonn, Mathematical Institute, Endenicher Allee 60, 53115 Bonn, Germany
- <sup>4</sup> Centro de Matemática, Universidade do Minho, Braga, Portugal

## Introduction

For a path-connected space  $X$ , the  $s$ -th *higher topological complexity*  $\text{TC}_s(X)$  is the sectional category of the fibration  $e_s: PX \rightarrow X^s$ , that is

$$\text{TC}_s(X) = \text{secat}(e_s: PX \rightarrow X^s),$$

where  $PX$  denotes the space of paths in  $X$  and

$$e_s(\gamma) = \left( \gamma(0), \gamma\left(\frac{1}{s-1}\right), \dots, \gamma\left(\frac{s-2}{s-1}\right), \gamma(1) \right)$$

is the usual  $s$ -th evaluation map. Explicitly, in the reduced version used here,  $\text{TC}_s(X)$  is one less than the minimal number of open sets covering  $X^s$ , over each of which the fibration  $e_s$  admits a section.

Topological complexity  $\text{TC}(X) = \text{TC}_2(X)$  was introduced in Farber (2003) and the ‘higher’ invariants were introduced in Rudyak (2010). These invariants were developed and motivated by applications for motion planning problems in robotics. More precisely, viewing  $X$  as the space of configurations, or states, of a mechanical system, the integer  $\text{TC}_s(X)$  provides a topological measure of the complexity of planning motion in  $X$  from an initial configuration to a terminal configuration, passing through  $s - 2$  specified intermediate configurations.

Despite a huge body of research into these invariants, there are very few complete computations of  $\text{TC}_s(X)$ . Examples for which the full spectrum of invariants is known include products of spheres, surfaces, path-connected topological groups whose Lusternik–Schnirelmann category is known, closed simply-connected symplectic manifolds, Fadell–Neuwirth configuration spaces based on (possibly punctured) Euclidean spaces, classifying spaces of hyperbolic groups and some (additional) polyhedral product type spaces; see Basabe et al. (2014); González et al. (2016); González and Grant (2015); Hughes and Li (2022); Aguilar et al. (2022). In a number of these examples, the higher topological complexities attain the maximal values possible.

If  $X$  is not simply connected, this maximal value is  $\text{TC}_s(X) \leq s \dim(X)$ , where  $\dim(X)$  is the homotopy dimension of  $X$ , see Basabe et al. (2014). Cohen and Vandembroucq (2021) explored the non-maximality of  $\text{TC}_2(M)$  when  $M$  is a manifold with abelian fundamental group. In this paper we extend these ideas to  $\text{TC}_s(M)$  for  $s \geq 2$ .

Espinosa Baro, Farber, Mescher, and Oprea (Espinosa Baro et al. 2025) have recently characterized the maximality of  $\text{TC}_s$  of a finite-dimensional CW-complex  $X$  in terms of a canonical cohomology class generalizing the ‘Costa–Farber class’ introduced in Costa and Farber (2010) (see Section 1). Restricting our attention to a manifold  $M$  with an abelian fundamental group  $\pi$  and following the strategy of Cohen and Vandembroucq (2021), we first express this characterization in terms of a homology class of the group  $\pi^{s-1}$  (see Proposition 2.3 and Corollary 2.4). This permits us to establish the non-maximality of  $\text{TC}_s(M)$  in some cases. For example, when  $M$  is orientable, we obtain the following result (see Section 3):

**Theorem A** *Let  $M$  be an orientable  $n$ -dimensional connected closed manifold. In each of the following cases, we have  $\text{TC}_s(M) < sn$ :*

- (1)  $\pi_1(M) = \mathbb{Z}^r$  with  $(s - 1)r < s \dim(M)$ ;
- (2)  $\pi_1(M) = \mathbb{Z}_q$ ;
- (3)  $\pi_1(M) = \mathbb{Z}^r \times \mathbb{Z}_q$  with  $r < \dim(M)$ .

Limiting examples discussed in Section 3 show that these conditions are sharp.

Computations of cohomological lower bounds of the  $s$ -th topological complexity of the real projective spaces and lens spaces have attracted much interest (Cadavid-Aguilar et al. 2018; Davis 2018; Farber and Grant 2008; Daundkar 2024). In Section 4, we show how these results provide lower bounds on  $\text{TC}_s$  for larger families of manifolds (see Proposition 4.2 and Theorem 4.7). Then Theorem A enables us to obtain the following exact values:

**Theorem B** *Let  $M$  be an essential orientable  $n$ -dimensional connected closed manifold.*

- (1) *If  $n \equiv 1 \pmod 4$  and  $\pi_1(M) = \mathbb{Z}_2$ , then  $\text{TC}_s(M) = sn - 1$  for  $s$  sufficiently large.*
- (2) *If  $n \equiv 1 \pmod 2$  and  $\pi_1(M) = \mathbb{Z}_p$  where  $p \geq 3$  is a prime that does not divide  $\binom{2n}{n}$ , then  $\text{TC}_s(M) = sn - 1$ .*

See Corollary 4.4 for an explicit description of the condition “ $s$  sufficiently large” in (1). We recall that closed essential manifolds were first studied by Gromov (1983) in the context of systolic geometry. For our purposes, the key property is that closed essential manifolds  $M$  have maximal possible Lusternik–Schnirelmann category, indeed, they can be characterized by the equality  $\text{cat}(M) = \dim(M)$ —see for instance (Katz and Rudyak 2006, Theorem 4.1).

The case of non-orientable manifolds is much more complicated. By (Cohen and Vandembroucq 2021, Theorem 1.2(1)), the topological complexity of a non-orientable manifold with abelian fundamental group is always non-maximal. However, it is well known that there exist such non-orientable manifolds with maximal  $\text{TC}_s$  for  $s \geq 3$ . For instance, for the real projective plane  $P^2$ , we have  $\text{TC}_3(P^2) = 6 = 3 \dim(P^2)$ , see González et al. (2016). Furthermore, it has been shown in Cadavid-Aguilar et al. (2018) and Davis (2018) that, when  $n$  is even, the real projective space  $P^n$  satisfies  $\text{TC}_s(P^n) = sn$  for  $s$  sufficiently large. For a fixed even integer  $n$ , the sequence  $(\text{TC}_s(P^n))_{s \geq 2}$  forms an increasing sequence starting at  $\text{TC}_2(P^n)$ , equal to the immersion dimension of  $P^n$  as shown in Farber et al. (2003), and stabilizing to  $sn$  when  $s$  is sufficiently large. As explained in Cadavid-Aguilar et al. (2018), it would be very interesting to better understand this sequence. Our methods, developed in Section 5, permit us to obtain new information in this direction. In particular, in combination with results from Davis (2018), when  $n = 2^r - 2$ , we have:

$$\text{TC}_s(P^n) \leq sn - 1 \text{ for even } s \leq n, \quad \text{TC}_n(P^n) = n^2 - 1, \quad \text{and} \quad \text{TC}_s(P^n) = sn \text{ for } s > n.$$

As before, this result can be extended to a larger family of manifolds:

**Theorem C** *Let  $M$  be a non-orientable  $n$ -dimensional connected closed manifold with  $\pi_1(M) = \mathbb{Z}_2$  and  $n = 2^r - 2$  with  $r \geq 3$ . Then, for any even  $s \leq n$ , we have*

$\text{TC}_s(M) < sn$ . If in addition  $M$  is essential, then  $\text{TC}_n(M) = n^2 - 1$  and  $\text{TC}_s(M) = sn$  for  $s > n$ .

## Notation and conventions

For a topological space  $Y$ , we use  $\dim(Y)$  to denote the homotopy dimension of  $Y$ . The integral homology of  $Y$  is denoted by  $H_*(Y)$ , and the reduced homology by  $\tilde{H}_*(Y)$ . If  $\pi = \pi_1(Y)$ , we denote the cohomology of  $Y$  with coefficients in the local system determined by the  $\mathbb{Z}[\pi]$ -module  $V$  by  $H^*(Y; V)$ .

For an element  $a$  of a group  $\pi$ , we often denote the inverse of  $a$  by  $\bar{a}$ .

We use the reduced version of sectional category throughout, so that for a fibration  $p: E \rightarrow B$ , when finite,  $\text{secat}(p: E \rightarrow B)$  is one less than the minimal number of open sets covering  $B$ , over each of which the fibration admits a section.

## 1 A $\text{TC}_s$ canonical class

Canonical cohomology classes for higher topological complexity were recently introduced and studied by in Espinosa Baro et al. (2025). In this brief preliminary section, with this work as a general reference (see (Espinosa Baro et al. 2025, §§5–6) in particular), we recall and discuss aspects of these classes which will be of subsequent use.

Let  $X$  be a CW-complex. The standard dimensional upper bound for higher topological complexity is

$$\text{TC}_s(X) \leq s \dim(X). \quad (1.1)$$

Although (1.1) can be improved in terms of the connectivity of  $X$ , we are interested in the improvements coming from obstruction-theory techniques in cases where  $X$  is not simply connected. A fundamental concept in this context is the notion of homological obstruction as considered in Schwarz (1966). Recall that the fiber of  $e_s: P X \rightarrow X^s$  is  $\Omega X^{s-1} = (\Omega X)^{s-1}$ . In Espinosa Baro et al. (2025), the homological obstruction for sectioning  $e_s$  over the 1-dimensional skeleton of  $X^s$  is identified with a canonical twisted class,

$$\mathfrak{v}_{X,s} \in H^1(X^s; I_s(\pi^{s-1})) = H^1(X^s; \tilde{H}_0(\Omega X^{s-1})), \quad (1.2)$$

where  $\pi := \pi_1(X)$  and  $I_s(\pi^{s-1})$  denotes the augmentation ideal of  $\pi^{s-1}$ , viewed as a  $\mathbb{Z}[\pi^s]$ -submodule of  $\mathbb{Z}[\pi^{s-1}]$ . Here the action of  $\pi^s$  on  $I_s(\pi^{s-1})$ , which corresponds to the monodromy associated with the fibration  $e_s$ , is given by

$$(a_1, \dots, a_s) \cdot (b_1, \dots, b_{s-1}) = (a_1 b_1 \bar{a}_2, a_2 b_2 \bar{a}_3, \dots, a_{s-1} b_{s-1} \bar{a}_s).$$

The class  $\mathfrak{v}_{X,s}$  can also be described as the cohomology class induced by the crossed homomorphism  $\nu_{X,s}: \pi^s \rightarrow I_s(\pi^{s-1})$  given by

$$v_{X,s}(a_1, \dots, a_s) = (a_1\bar{a}_2, a_2\bar{a}_3, \dots, a_{s-1}\bar{a}_s) - 1_{s-1},$$

where  $1_{s-1}$  is the unit element of  $\pi^{s-1}$ . Obstruction-theoretic arguments lead then to the following result:

**Theorem 1.1** (Espinosa Baro et al. (2025)) *Let  $X$  be a CW-complex of dimension  $n \geq 2$ . Then  $TC_s(X) < sn$  if and only if the  $sn$ -th cup-power  $v_{X,s}^{sn} = 0$ .*

Here,  $v_{X,s}^{sn}$  lies in the cohomology of  $X^s$  with coefficients in the  $sn$ -th tensor power of  $I_s(\pi^{s-1})$  endowed with the diagonal action of  $\pi_s$ , denoted by  $I_s^{sn}(\pi^{s-1})$ .

The construction of the class  $v_{X,s}$  generalizes the TC canonical class of Costa and Farber (2010) and provides for  $TC_s$  an analogue of the classical Berstein–Schwarz class  $b_X \in H^1(X; I(\pi))$ . Note that in this case,  $I(\pi)$  is the augmentation ideal of  $\pi$  endowed with the left  $\mathbb{Z}[\pi]$ -module structure induced by the multiplication of  $\pi$ . As is well known, the Lusternik–Schnirelmann category of  $X$ ,  $cat(X)$ , satisfies  $cat(X) = \dim(X)$  if and only if  $b_X^{\dim(X)} \neq 0$  (see Berstein (1976), Schwarz (1966) and Dranishnikov and Rudyak (2009) for a proof including the case  $\dim(X) = 2$ ).

**Remark 1.2** We conclude this section with a brief remark regarding the functoriality of these classes. For  $\pi = \pi_1(X)$ , applying the above result to the classifying space  $B\pi$  yields a crossed homomorphism and associated cohomology class, which we denote by  $v_{\pi,s}$  and  $v_{\pi,s}$  respectively.

Recall that if  $f : X \rightarrow Y$  is a map, and  $A$  is a  $\mathbb{Z}[\pi_1(Y)]$ -module, then  $f^*(A)$  denotes the  $\mathbb{Z}[\pi_1(X)]$ -module whose underlying abelian group is  $A$  and the action of  $g \in \pi_1(X)$  on  $a \in A$  is given by  $g \cdot a := \pi_1(f)(g) \cdot a$ . Taking  $f : X \rightarrow Y = B\pi$  to be a classifying map (i.e., a map inducing a  $\pi_1$ -isomorphism, so that it classifies the  $\pi$ -covering projection  $\tilde{X} \rightarrow X$  from the universal cover of  $X$ ), the isomorphism  $I_s(\pi^{s-1}) \cong (f^s)^* I_s(\pi^{s-1})$  yields

$$v_{X,s} = (f^s)^* v_{\pi,s}.$$

Similar considerations apply to the Berstein–Schwarz class  $b_\pi \in H^1(\pi; I(\pi))$  (resp.,  $b_X \in H^1(X; I(\pi))$ ), induced by the crossed homomorphism  $\beta_\pi : \pi \rightarrow I(\pi)$ ,  $\alpha \mapsto \alpha - 1$ . Namely, the isomorphism  $I(\pi) \cong f^* I(\pi)$  yields  $b_X = f^* b_\pi$ .

## 2 Abelian fundamental group

In this section we extend to higher topological complexity some results of Cohen and Vandembroucq (2021) which will be useful for our computations. The arguments are essentially similar to those of Cohen and Vandembroucq (2021) as well as some of Dranishnikov (2016). In Remark 2.5, we explain one difficulty which arises in the non-orientable case. This difficulty will be addressed in Section 5.

Assume from now on that  $\pi = \pi_1(X)$  is abelian. We consider the group homomorphism  ${}^s\chi : \pi^s \rightarrow \pi^{s-1}$  given by

$${}^s\chi(a_1, \dots, a_s) = (a_1\bar{a}_2, a_2\bar{a}_3, \dots, a_{s-1}\bar{a}_s).$$

Note that the  $\mathbb{Z}[\pi^s]$ -module  ${}^s\chi^*(I(\pi^{s-1}))$  is exactly the  $\mathbb{Z}[\pi^s]$ -module  $I_s(\pi^{s-1})$ . With the notation regarding canonical classes, Bernstein–Schwarz classes, and crossed homomorphisms of the previous section, we also have, for any  $(a_1, \dots, a_s) \in \pi^s$ ,

$$v_{\pi,s}(a_1, \dots, a_s) = \beta_{\pi^{s-1}}(a_1\bar{a}_2, a_2\bar{a}_3, \dots, a_{s-1}\bar{a}_s) = \beta_{\pi^{s-1}}({}^s\chi(a_1, \dots, a_s)).$$

We then have  $v_{\pi,s} = {}^s\chi^*\mathfrak{b}_{\pi^{s-1}}$  in  $H^1(\pi^s; I_s(\pi^{s-1})) = H^1(B\pi^s; I_s(\pi^{s-1}))$ , and, for any  $k$ ,

$$v_{X,s}^k = (\gamma^s)^*\mathfrak{b}_{\pi,s}^k = (\gamma^s)^*({}^s\chi)^*\mathfrak{b}_{\pi^{s-1}}^k \text{ in } H^k(X^s; I_s^k(\pi^{s-1}))$$

where  $\gamma : X \rightarrow B\pi$  is a classifying map.

In order to establish our results, it is useful to consider the cofiber of the diagonal map  $\Delta_s = \Delta_s^X : X \rightarrow X^s$ . We denote it by  $C_{\Delta_s}(X)$ . We will more generally use the notation  $\Delta_s^Z : Z \rightarrow Z^s$  to denote the  $s$ -diagonal of a set  $Z$  and suppress the superscript when the context is clear.

**Proposition 2.1** *Let  $X$  be an  $n$ -dimensional CW-complex with  $n \geq 2$ . Suppose that  $\pi = \pi_1(X)$  is abelian and let  $\gamma : X \rightarrow B\pi$  be a classifying map. Then for any  $s \geq 2$  we have*

- (1)  $v_{X,s} = q^*\mathfrak{b}_{C_{\Delta_s}(X)}$  in  $H^1(X^s; I_s(\pi^{s-1}))$  where  $q : X^s \rightarrow C_{\Delta_s}(X)$  is the identification map.
- (2)  $\text{TC}_s(X) < sn$  if and only if  $\text{cat}(C_{\Delta_s}(X)) < sn$ .
- (3) If  $\text{TC}_s(X) < sn$  then, for any  $\mathbb{Z}[\pi^{s-1}]$ -module  $A$  and for any homology class  $\mathfrak{c} \in H_{sn}(X^s; ({}^s\chi\gamma^s)^*A)$ , the class  $\mathfrak{c} = \gamma_*^s(\mathfrak{c}) \in H_{sn}(\pi^s; ({}^s\chi)^*A)$  satisfies  ${}^s\chi_*(\mathfrak{c}) = 0$ .

**Proof** First observe that the homomorphism  ${}^s\chi \circ \Delta_s^\pi$  is trivial. Consequently, the map  $B^s\chi \circ B\Delta_s^\pi$  obtained after applying the functor  $B$  is also trivial. By identifying  $B\pi^s$  with  $(B\pi)^s$  and  $B\Delta_s^\pi$  with  $\Delta_s^{B\pi}$ , we have a (homotopy) commutative diagram of the following form

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_s^X} & X^s & \xrightarrow{q} & C_{\Delta_s}(X) \\ \gamma \downarrow & & \gamma^s \downarrow & & \xi \downarrow \\ B\pi & \xrightarrow{\Delta_s^{B\pi}} & (B\pi)^s & \xrightarrow{B^s\chi} & B\pi^{s-1} \end{array}$$

where  $\xi$  is induced by the quotient property. Since  $\pi$  is abelian, we have an exact sequence  $1 \rightarrow \pi \xrightarrow{\Delta_s^\pi} \pi^s \xrightarrow{{}^s\chi} \pi^{s-1} \rightarrow 1$  and, using the Van Kampen theorem, we can see that  $\pi_1(C_{\Delta_s}(X)) = \pi^{s-1}$  and that  $\pi_1(\xi)$  is an isomorphism. Consequently  $\xi$  is a classifying map and the Bernstein–Schwarz class of  $C_{\Delta_s}(X)$  is given by  $\mathfrak{b}_{C_{\Delta_s}(X)} = \xi^*\mathfrak{b}_{\pi^{s-1}}$ . By the commutativity of the diagram we then get  $q^*(I(\pi^{s-1})) \cong {}^s\chi^*(I(\pi^{s-1})) \cong I_s(\pi^{s-1})$  and  $q^*\mathfrak{b}_{C_{\Delta_s}(X)} = v_{X,s}$  as claimed in the first item.

The equality established above implies that  $q^*\mathfrak{b}_{C_{\Delta_s}(X)}^{sn} = v_{X,s}^{sn}$ . For dimensional reasons, the map  $q^* : H^{sn}(C_{\Delta_s}(X); I(\pi^{s-1})) \rightarrow H^{sn}(X^s; I_s(\pi^{s-1}))$  is an isomorphism. We therefore have  $\mathfrak{b}_{C_{\Delta_s}(X)}^{sn} = 0$  if and only if  $v_{X,s}^{sn} = 0$ , which implies the second item.

We now prove the last item. Let  $c \in H_{sn}(X^s; ({}^s\chi\gamma^s)^*A)$  be a nonzero class and let  $c = \gamma_*^s(c)$ . We have  ${}^s\chi_*(c) = \xi_*q_*(c)$ . Note that  $q_*(c)$  is a homology class of degree  $sn$ . Since  $TC_s(X) < sn$  we have  $\text{cat}(C_{\Delta_s}(X)) < sn$ . Therefore the classifying map  $\xi$  factors up to homotopy through an  $(sn - 1)$ -dimensional space. Consequently,  $\xi_*q_*(c) = 0$  and the result follows.  $\square$

**Remark 2.2** Notice that, if  $A$  is a trivial  $\mathbb{Z}[\pi^{s-1}]$ -module, then the modules  $({}^s\chi)^*A$  and  $({}^s\chi\gamma^s)^*A$  are just the abelian group  $A$ . A special case of Proposition 2.1 (3) can then be expressed as follows: if  $A$  is a trivial  $\mathbb{Z}[\pi^{s-1}]$ -module and  $c \in H_{sn}(X^s; A)$  is an element such that the class  $c = \gamma_*^s(c) \in H_{sn}(\pi^s; A)$  satisfies  ${}^s\chi_*(c) \neq 0$  then  $TC_s(X) = sn$ .

Item (3) of Proposition 2.1 is sharp under reasonably general conditions. Let  $M$  be an  $n$ -dimensional connected closed manifold with fundamental group  $\pi = \pi_1(M)$  and let  $\omega = \omega_M : \pi \rightarrow \{\pm 1\}$  be the homomorphism determined by the first Stiefel-Whitney class of  $M$ . Recall that the orientation module of  $M$ , denoted by  $\tilde{\mathbb{Z}} = \tilde{\mathbb{Z}}_M$ , is the abelian group  $\mathbb{Z}$  given with a structure of  $\mathbb{Z}[\pi]$ -module determined by  $a \cdot t = \omega(a)t$  for  $a \in \pi, t \in \mathbb{Z}$ . We denote by  $[M]$  the fundamental class of  $M$ , that is, a generator of  $H_n(M; \tilde{\mathbb{Z}}) = \mathbb{Z}$ . Note that  $\tilde{\mathbb{Z}}_M^s = \tilde{\mathbb{Z}}_M^{\otimes s}$ , which additively is  $\mathbb{Z}$  with  $\pi^s$  action given by  $(a_1, a_2, \dots, a_s)t = \omega(a_1)\omega(a_2) \cdots \omega(a_s)t$ .

**Proposition 2.3** *Let  $M$  be an  $n$ -dimensional connected closed manifold with  $n \geq 2$  and  $\pi = \pi_1(M)$  abelian. Assume there is a  $\mathbb{Z}[\pi^{s-1}]$ -module  $A$  such that the  $\mathbb{Z}[\pi^s]$ -modules  ${}^s\chi^*(A)$  and  $\tilde{\mathbb{Z}}^{\otimes s}$  are isomorphic. Then the following two conditions are equivalent:*

- (1) *The class  $m := \gamma_*([M]) \in H_n(\pi; \tilde{\mathbb{Z}})$  satisfies  ${}^s\chi_*(m^{\times s}) = 0$  in  $H_{sn}(\pi^{s-1}; A)$ .*
- (2)  *$TC_s(M) < sn$ .*

Here we denote by  $m^{\times s} \in H_{sn}(\pi^s; \tilde{\mathbb{Z}}^{\otimes s})$  the image of the fundamental class of  $M^s$  under the homomorphism induced by  $\gamma^s : M^s \rightarrow B\pi^s$ . Note that we also denote by  $\tilde{\mathbb{Z}}$  the local system over  $B\pi$  arising from the isomorphism  $\pi_1(\gamma)$  induced by the classifying map  $\gamma : M \rightarrow B\pi$ .

**Proof** From the naturality of the cap-product and the assumption that  $A$  is a  $\mathbb{Z}[\pi^{s-1}]$ -module satisfying  ${}^s\chi^*(A) \cong \tilde{\mathbb{Z}}^{\otimes s}$  we get the following diagram.

$$\begin{array}{ccc}
 H_{sn}(M^s; \tilde{\mathbb{Z}}^{\otimes s}) \otimes H^{sn}(M^s; I_s^{sn}(\pi^{s-1})) & \xrightarrow[\cong]{\cap} & I_s^{sn}(\pi^{s-1}) \otimes_{\pi^s} \tilde{\mathbb{Z}}^{\otimes s} \\
 \downarrow (\gamma^s)_* & \uparrow (\gamma^s)^* & \downarrow \\
 H_{sn}(B\pi^s; \tilde{\mathbb{Z}}^{\otimes s}) \otimes H^{sn}(B\pi^s; I_s^{sn}(\pi^{s-1})) & \xrightarrow{\cap} & I_s^{sn}(\pi^{s-1}) \otimes_{\pi^s} \tilde{\mathbb{Z}}^{\otimes s} \\
 \downarrow ({}^s\chi)_* & \uparrow ({}^s\chi)^* & \cong \downarrow ({}^s\chi)_* \\
 H_{sn}(B\pi^{s-1}; A) \otimes H^{sn}(B\pi^{s-1}; I_s^{sn}(\pi^{s-1})) & \xrightarrow{\cap} & I_s^{sn}(\pi^{s-1}) \otimes_{\pi^{s-1}} A
 \end{array}$$

The cap-product on the first line is an isomorphism by Poincaré duality. The bottom vertical map in the third column corresponds to the morphism

$${}^s\chi_* : H_*(B\pi^s; I_s^{sn}(\pi^{s-1}) \otimes \widetilde{\mathbb{Z}}^{\otimes s}) \rightarrow H_*(B\pi^{s-1}; I^{sn}(\pi^{s-1}) \otimes A)$$

in degree 0. It is induced by the obvious isomorphism between the underlying  $\mathbb{Z}$ -modules  $I^{sn}(\pi^{s-1}) \otimes \widetilde{\mathbb{Z}}^{\otimes s}$  and  $I^{sn}(\pi^{s-1}) \otimes A$  and is an isomorphism on the coinvariants because  ${}^s\chi$  is surjective.

Consider the fundamental class  $[M^s] \in H_{sn}(M^s; \widetilde{\mathbb{Z}}^{\otimes s})$  of  $M^s$ . Since the third column of the diagram is comprised of isomorphisms, we have

$$({}^s\chi)_*(\gamma^s)_*([M^s]) \cap \mathfrak{b}_{\pi^{s-1}}^{sn} = 0 \text{ if and only if } [M^s] \cap (\gamma^s)^*({}^s\chi)^*\mathfrak{b}_{\pi^{s-1}}^{sn} = 0.$$

This is equivalent to saying

$$({}^s\chi)_*(\mathfrak{m}^{\times s}) \cap \mathfrak{b}_{\pi^{s-1}}^{sn} = 0 \text{ if and only if } [M^s] \cap \mathfrak{v}_{M^s}^{sn} = 0.$$

The hypothesis  ${}^s\chi_*(\mathfrak{m}^{\times s}) = 0$  yields  $[M^s] \cap \mathfrak{v}_{M^s}^{sn} = 0$ . By Poincaré duality, we can then conclude that  $\mathfrak{v}_{M^s}^{sn} = 0$  and consequently  $\text{TC}_s(M) < sn$ . This proves (1)  $\Rightarrow$  (2). The converse follows from Proposition 2.1 (3). □

**Corollary 2.4** *Let  $M$  be an orientable  $n$ -dimensional manifold with  $n \geq 2$  and abelian fundamental group  $\pi = \pi_1(M)$ . The class  $\mathfrak{m} = \gamma_*([M]) \in H_n(\pi; \mathbb{Z})$  satisfies  ${}^s\chi_*(\mathfrak{m}^{\times s}) = 0$  in  $H_{sn}(\pi^{s-1}; \mathbb{Z})$  if and only if  $\text{TC}_s(X) < sn$ .*

**Proof** Since  $M$  is orientable, the orientation module  $\widetilde{\mathbb{Z}}$  is just  $\mathbb{Z}$  with trivial action. Taking  $A = \mathbb{Z}$  also with trivial action, we have  ${}^s\chi^*(A) \cong \widetilde{\mathbb{Z}}^{\otimes s}$  and the result follows from Proposition 2.3. □

**Remark 2.5** When  $M$  is non-orientable, the  $\mathbb{Z}[\pi^{s-1}]$ -module  $A = \widetilde{\mathbb{Z}}^{\otimes s-1}$  satisfies the assumptions of Proposition 2.3 for  $s = 2$  (see Cohen and Vandembroucq (2021)) but fails to do so for all  $s > 2$ . For instance, set  $s = 3$  and suppose that  $b \in \pi$  is an element for which the orientation character  $\omega : \pi \rightarrow \{\pm 1\}$  satisfies  $\omega(b) = -1$ . Then, for  $a, c \in \pi$  and  $t \in \mathbb{Z}$  we have

$$(a, b, c) \cdot t = \omega(a)\omega(b)\omega(c)t = -\omega(a)\omega(c)t$$

while

$${}^3\chi(a, b, c) \cdot t = (a\bar{b}, b\bar{c}) \cdot t = \omega(a\bar{b})\omega(b\bar{c})t = \omega(a)\omega(c)t.$$

This shows that the map  ${}^3\chi$  does not induce a homomorphism from  $H_*(\pi^3; \widetilde{\mathbb{Z}}^{\otimes 3})$  to  $H_*(\pi^2; \widetilde{\mathbb{Z}}^{\otimes 2})$ . Note that, in Proposition 2.3,  $A$  must be, as an abelian group, isomorphic to  $\mathbb{Z}$ . Furthermore, since  ${}^s\chi$  is surjective, the  $\mathbb{Z}[\pi^{s-1}]$ -module structure on  $A$  is forced by the hypothesis  ${}^s\chi^*(A) \cong \widetilde{\mathbb{Z}}^{\otimes s}$  and this condition is impossible when  $s$  is odd. For

instance, again set  $s = 3$ , choose  $b \in \pi$  as above and assume  ${}^3\chi^*(A) \cong \widetilde{\mathbb{Z}}^{\otimes 3}$ . The equalities  ${}^3\chi(b, 1, 1) = {}^3\chi(1, \bar{b}, \bar{b}) = (b, 1)$  then lead to the impossible

$$t = \omega(\bar{b})\omega(\bar{b})t = {}^3\chi(1, \bar{b}, \bar{b}) \cdot t = (b, 1) \cdot t = {}^3\chi(b, 1, 1) \cdot t = \omega(b)t = -t.$$

Nonetheless, when  $s = 2\sigma$ ,  $\sigma \geq 1$ , the  $\mathbb{Z}[\pi^{s-1}]$ -module  $A = \widetilde{\mathbb{Z}} \otimes (\mathbb{Z} \otimes \widetilde{\mathbb{Z}})^{\sigma-1}$  does satisfy  ${}^s\chi^*(A) \cong \widetilde{\mathbb{Z}}^{\otimes s}$ , and we explore its usage in Section 5.

### 3 Some calculations in the orientable case

Let  $M$  be an orientable connected closed manifold with  $\pi = \pi_1(M)$  abelian. In this section we will use Corollary 2.4 to establish the non-maximality  $\text{TC}_s(M) < s \dim(M)$  for some families of manifolds with abelian fundamental groups. The results of this section establish Theorem A stated in the Introduction.

Let  $\gamma : M \rightarrow B\pi$  be a classifying map and let  $\mathfrak{m} = \gamma_*([M]) \in H_n(\pi; \mathbb{Z})$ . Since  $M$  is orientable, we will suppress the  $\mathbb{Z}$ -coefficients from the notation. In all cases, we will see that  ${}^s\chi_*(\mathfrak{m}^{\times s}) = 0$  in  $H_{sn}(\pi^{s-1})$ .

In our first result, we suppose that  $\pi$  is a free abelian group. This case has already been considered in Espinosa Baro et al. (2025) in the more general context of finite CW-complexes. Here, restricting to closed manifolds, we obtain a slightly stronger statement than (Espinosa Baro et al. 2025, Corollary 7.14).

**Proposition 3.1** *Let  $M$  be an orientable  $n$ -dimensional connected closed manifold with  $\pi_1(M) = \mathbb{Z}^r$  and let  $s \geq 2$ . If  $sn > (s - 1)r$  then  $\text{TC}_s(M) < sn$ .*

**Proof** Let  $\pi = \mathbb{Z}^r$ .

For degree reasons, we can see that  ${}^s\chi_*(\mathfrak{m}^{\times s}) = 0$  in  $H_{sn}(\pi^{s-1})$ . Indeed  $B\pi = (S^1)^r$  and  $H_k(\pi) = 0$  for  $k > r$ . Consequently  $H_{sn}(\pi^{s-1}) = 0$  if  $sn > (s - 1)r$ .  $\square$

In general, observe that the homomorphism  ${}^s\chi : \pi^s \rightarrow \pi^{s-1}$  given by

$${}^s\chi(a_1, \dots, a_s) = (a_1\bar{a}_2, a_2\bar{a}_3, \dots, a_{s-1}\bar{a}_s)$$

can be decomposed as

$${}^s\chi = \underbrace{(\chi \times \dots \times \chi)}_{s-1} \circ (\text{Id} \times \underbrace{\Delta \times \dots \times \Delta}_{s-2} \times \text{Id}) \tag{3.1}$$

where  $\Delta = \Delta_2^\pi : \pi \rightarrow \pi \times \pi$  is the diagonal map and  $\chi = {}^2\chi$ . Denote by  $j : \pi \rightarrow \pi$  the inversion. Since  $\chi$  can be seen as the multiplication of  $\pi$ ,  $\mu : \pi \times \pi \rightarrow \pi$ , precomposed with  $\text{Id} \times j$ , we have, for classes  $\mathfrak{a}, \mathfrak{b} \in H_*(\pi)$ ,

$$\chi_*(\mathfrak{a} \times \mathfrak{b}) = \mathfrak{a} \wedge j_*(\mathfrak{b})$$

where  $\wedge$  is the Pontryagin product, that is, the product induced by  $\mu$  in homology, see (Brown 1982, V.5).

In the results below, we consider the cyclic group  $\mathbb{Z}_q = \langle v \mid v^q = 1 \rangle$  and work at the chain level. Recall the classical resolution of  $\mathbb{Z}$  as a trivial  $\mathbb{Z}[\mathbb{Z}_q]$ -module given by

$$\cdots \longrightarrow \mathbb{Z}[\mathbb{Z}_q] \xrightarrow{N_q(v)} \mathbb{Z}[\mathbb{Z}_q] \xrightarrow{v-1} \mathbb{Z}[\mathbb{Z}_q] \xrightarrow{N_q(v)} \mathbb{Z}[\mathbb{Z}_q] \xrightarrow{v-1} \mathbb{Z}[\mathbb{Z}_q] \xrightarrow{\varepsilon} \mathbb{Z} \tag{3.2}$$

where  $N_q(v) = 1 + v + \cdots + v^{q-1}$ .

In the following lemma, we recall the morphisms induced by the diagonal  $\Delta$ , the multiplication  $\mu$  and the inversion  $j$  on the level of resolutions (see (Brown 1982, page 108) and (Cohen and Vandembroucq 2021, §3.2)). Let  $[k]$  denote the generator of degree  $k$  in (3.2), and write  $B_{i,j}$  for the binomial coefficient  $\binom{i+j}{i}$ .

**Lemma 3.2** *At the level of the resolution (3.2),*

(a)  $\Delta$  is given on generators by  $[p] \mapsto \sum_{k+l=p} \Delta_{kl}[p]$  where

$$\Delta_{kl}[p] = \begin{cases} [k] \otimes [l] & k \text{ even}; \\ [k] \otimes v[l] & k \text{ odd}, l \text{ even}; \\ \sum_{0 \leq i < j \leq q-1} v^i [k] \otimes v^j [l] & k \text{ odd}, l \text{ odd}. \end{cases}$$

(b)  $\mu$  is given on generators by the formulae

$$\begin{aligned} [2i] \otimes [2j] &\mapsto B_{i,j} [2(i+j)]; \\ [2i] \otimes [2j+1] &\mapsto B_{i,j} [2(i+j)+1]; \\ [2i+1] \otimes [2j] &\mapsto B_{i,j} [2(i+j)+1]; \\ [2i+1] \otimes [2j+1] &\mapsto 0. \end{aligned}$$

(c)  $j$  is given on generators by

$$[i] \mapsto N_{q-1}^k(v)[i] \quad \text{if } i \in \{2k, 2k-1\}.$$

where  $N_{q-1}^k(v)$  is the  $k$ -th power of  $N_{q-1}(v)$ .

We denote by  $C_\bullet(\mathbb{Z}_q)$  the  $\mathbb{Z}$ -chain complex obtained by tensoring the resolution (3.2) with  $\mathbb{Z}$  over  $\mathbb{Z}_q$ .

$$C_\bullet(\mathbb{Z}_q) : \cdots \xrightarrow{0} C_{2k}(\mathbb{Z}_q) \xrightarrow{q} C_{2k-1}(\mathbb{Z}_q) \xrightarrow{0} \cdots \xrightarrow{q} C_1(\mathbb{Z}_q) \xrightarrow{0} C_0(\mathbb{Z}_q), \tag{3.3}$$

where  $C_k(\mathbb{Z}_q) = \mathbb{Z}$  for each  $k$ . Abusing notation, we denote the generator of  $C_k(\mathbb{Z}_q)$  by  $[k]$ . Recall that the homology of this chain complex gives  $H_*(\mathbb{Z}_q) = H_*(\mathbb{Z}_q; \mathbb{Z})$ . In positive degrees,  $H_+(\mathbb{Z}_q)$  is concentrated in odd degrees.

We will use the diagonal approximation of  $C_\bullet(\mathbb{Z}_q)$ , obtained from Lemma 3.2 (a):

$$\begin{aligned} \Delta_\bullet : C_\bullet(\mathbb{Z}_q) &\rightarrow C_\bullet(\mathbb{Z}_q) \otimes C_\bullet(\mathbb{Z}_q) \\ [p] &\mapsto \sum_{k+l=p} \alpha_{kl}[k] \otimes [l]. \end{aligned} \tag{3.4}$$

Here  $\alpha_{kl} = 1$  if  $kl$  is even and  $\alpha_{kl} = (q - 1)q/2$  if  $kl$  is odd. As in Cohen and Vandembroucq (2021), we denote by  $\wedge : C_\bullet(\mathbb{Z}_q) \otimes C_\bullet(\mathbb{Z}_q) \rightarrow C_\bullet(\mathbb{Z}_q)$  the Pontryagin product, which is given by the formulæ (b) of Lemma 3.2:

$$\begin{cases} [2i] \wedge [2k] = B_{i,k}[2i + 2k], & [2i + 1] \wedge [2k + 1] = 0, \\ [2i] \wedge [2k + 1] = [2k + 1] \wedge [2i] = B_{i,k}[2i + 2k + 1]. \end{cases} \tag{3.5}$$

Let  $\mathbf{j} : C_\bullet(\mathbb{Z}_q) \rightarrow C_\bullet(\mathbb{Z}_q)$  denote the morphism induced by the inversion. From Lemma 3.2 (c), this morphism is given by

$$\mathbf{j}([i]) = (q - 1)^k [i] \quad \text{if } i \in \{2k, 2k - 1\}. \tag{3.6}$$

In these terms, the chain map  $\chi_\bullet = {}^2\chi_\bullet$  induced by  $\chi = {}^2\chi$  can be described as the composite

$$C_\bullet(\mathbb{Z}_q) \otimes C_\bullet(\mathbb{Z}_q) \xrightarrow{\text{Id} \otimes \mathbf{j}} C_\bullet(\mathbb{Z}_q) \otimes C_\bullet(\mathbb{Z}_q) \xrightarrow{\wedge} C_\bullet(\mathbb{Z}_q).$$

**Proposition 3.3** *Let  $M$  be an orientable  $n$ -dimensional connected closed manifold with  $\pi_1(M) = \mathbb{Z}_q$ . Then, for any  $s \geq 2$ , we have  $\text{TC}_s(M) < sn$ .*

**Proof** Let  $\gamma : M \rightarrow B\pi$  be a classifying map, where  $\pi = \mathbb{Z}_q$ , and let  $\mathbf{m} = \gamma_*([M]) \in H_n(\pi; \mathbb{Z})$ . We will see that  ${}^s\chi_*(\mathbf{m}^{\otimes s}) = 0$  in  $H_{sn}(\pi^{s-1}; \mathbb{Z})$ . If  $\dim(M)$  is even, this is immediate since  $H_+(\pi)$  is concentrated in odd degrees, which implies  $\mathbf{m} = 0$ . We then suppose that  $\dim(M) = 2p + 1$ . A cycle  $\mathbf{m} \in C_{2p+1}(\mathbb{Z}_q)$  representing the class  $\mathbf{m}$  is of the form  $\mathbf{m} = \lambda[2p + 1]$  for some  $\lambda \in \mathbb{Z}$ . In order to compute  ${}^s\chi_*(\mathbf{m}^{\otimes s})$  we use the decomposition (3.1) and analyze the element  ${}^s\chi_\bullet(\mathbf{m}^{\otimes s})$  which is given by

$$(\chi_\bullet)^{\otimes s-1}(\mathbf{m} \otimes \underbrace{\Delta_\bullet \mathbf{m} \otimes \cdots \otimes \Delta_\bullet \mathbf{m}}_{s-2} \otimes \mathbf{m}).$$

The element  $\mathbf{m} \otimes \Delta_\bullet \mathbf{m} \otimes \cdots \otimes \Delta_\bullet \mathbf{m} \otimes \mathbf{m}$  is given by a  $\mathbb{Z}$ -linear combination of elements of the form

$$[2p + 1] \otimes [k_1] \otimes [l_1] \otimes \cdots \otimes [k_{s-2}] \otimes [l_{s-2}] \otimes [2p + 1]$$

where  $k_i + l_i = 2p + 1$  for any  $1 \leq i \leq s - 2$ . Setting  $l_0 = k_{s-1} = 2p + 1$ , there will be necessarily some  $i \in \{0, \dots, s - 2\}$  such that  $l_i$  and  $k_{i+1}$  are both odd. Applying  $(\chi_\bullet)^{\otimes s-1}$  to the element above yields

$$([2p + 1] \wedge \mathbf{j}[k_1]) \otimes ([l_1] \wedge \mathbf{j}[k_2]) \otimes \cdots \otimes ([l_{s-2}] \wedge \mathbf{j}[k_{s-1}]).$$

If  $l_i$  and  $k_{i+1}$  are both odd, the corresponding factor  $([l_i] \wedge \mathbf{j}[k_{i+1}])$  vanishes since  $\mathbf{j}[k_{i+1}]$  is a multiple of  $[k_{i+1}]$  and the Pontryagin product of two odd degree elements is zero. Consequently, we obtain  ${}^s\chi_\bullet(\mathbf{m}^{\otimes s}) = 0$  and  ${}^s\chi_*(\mathbf{m}^{\times s}) = 0$ .  $\square$

**Proposition 3.4** *Let  $M$  be an orientable  $n$ -dimensional connected closed manifold with  $\pi_1(M) = \mathbb{Z}^r \times \mathbb{Z}_q$  such that  $r < n$ . Then, for any  $s \geq 2$ , we have  $\text{TC}_s(M) < sn$ .*

**Proof** Let  $\gamma : M \rightarrow B\pi$  be a classifying map, where  $\pi = \mathbb{Z}^r \times \mathbb{Z}_q$ , and let  $\mathbf{m} = \gamma_*([M]) \in H_n(\pi; \mathbb{Z})$ . By the Künneth formula, we have  $H_*(\mathbb{Z}^r \times \mathbb{Z}_q) = H_*(\mathbb{Z}^r) \otimes H_*(\mathbb{Z}_q)$ . Since  $n > r$ , we can write  $\mathbf{m} = \sum \sigma_i \otimes \alpha_i$  where  $\alpha_i \in H_+(\mathbb{Z}_q)$  and  $\sigma_i \in H_*(\mathbb{Z}^r) = \bigwedge(x_1, \dots, x_r)$  with each  $x_j$  of degree 1. Since  $H_+(\mathbb{Z}_q)$  is concentrated in odd degrees, a cycle  $\mathbf{m}$  representing  $\mathbf{m}$  can be described as a sum of terms of the form  $\lambda \sigma \otimes [2p + 1]$  where  $\lambda \in \mathbb{Z}$ ,  $p \geq 0$  and  $\sigma \in \bigwedge(x_1, \dots, x_r)$  is a class regarded as a cycle. The element  $\mathbf{m} \otimes \Delta_\bullet \mathbf{m} \otimes \dots \otimes \Delta_\bullet \mathbf{m} \otimes \mathbf{m}$  is therefore given by a  $\mathbb{Z}$ -linear combination of elements of the form

$$\sigma_0 \otimes [l_0] \otimes \sigma_1 \otimes [k_1] \otimes \tilde{\sigma}_1 \otimes [l_1] \otimes \dots \otimes \sigma_{s-2} \otimes [k_{s-2}] \otimes \tilde{\sigma}_{s-2} \otimes [l_{s-2}] \otimes \sigma_{s-1} [k_{s-1}] \tag{3.7}$$

where  $l_0, k_i + l_i$  for  $1 \leq i \leq s - 2$ , and  $k_{s-1}$  are all odd and the elements  $\sigma_i, \tilde{\sigma}_i$  belong to  $\bigwedge(x_1, \dots, x_r)$ . The calculation of  $\chi_\bullet$  on (say)  $\sigma_0 \otimes [l_0] \otimes \sigma_1 \otimes [k_1]$  is made componentwise and gives rise to factors of the form

$$\pm(\sigma_0 \wedge \sigma_1) \otimes ([l_0] \wedge \mathbf{j}[k_1]).$$

As in the proof of Proposition 3.3, there will be necessarily, in the expression (3.7), some  $i \in \{0, \dots, s - 2\}$  such that  $l_i$  and  $k_{i+1}$  are both odd. After applying  $\chi_\bullet$ , the corresponding factor will be 0. Consequently, we obtain  ${}^s\chi_\bullet(\mathbf{m}^{\otimes s}) = 0$  and  ${}^s\chi_*(\mathbf{m}^{\times s}) = 0$ . We can hence conclude that  $\text{TC}_s(M) < sn$ .  $\square$

**Limiting examples**

Examples 4.1 and 4.2 from Cohen and Vandembroucq (2021) show that the conditions in Propositions 3.1 and 3.4 are sharp. We now show that Proposition 3.3 cannot be extended to manifolds whose fundamental group is of the form  $\mathbb{Z}_p \times \mathbb{Z}_p$  where  $p$  is a prime.

**Example 3.5** *A manifold  $N$  with  $\pi_1(N) = \mathbb{Z}_3 \times \mathbb{Z}_3$  and  $\text{TC}_3(N) = 3 \dim(N)$ .*

Set  $\pi = \mathbb{Z}_3 \times \mathbb{Z}_3$  and consider  $C_\bullet(\pi) = C_\bullet(\mathbb{Z}_3) \otimes C_\bullet(\mathbb{Z}_3)$ . We will write  $[ik]$  instead of  $[i] \otimes [k]$ . We first consider the cycle  $\mathbf{m} = [05] + [50]$  and denote by  $\mathbf{m}$  its homology class. We will see that  ${}^3\chi_*(\mathbf{m}^{\times 3}) \neq 0$ . By the Universal Coefficient Theorem, it is actually sufficient to see that  ${}^3\chi_*(\mathbf{m}_{\mathbb{Z}_3}^{\times 3}) \neq 0$  where  $\mathbf{m}_{\mathbb{Z}_3}$  corresponds to  $\mathbf{m}$  in  $H_5(\pi; \mathbb{Z}_3)$ . As  $H_*(\pi; \mathbb{Z}_3) \cong H_*(\mathbb{Z}_3; \mathbb{Z}_3) \otimes H_*(\mathbb{Z}_3; \mathbb{Z}_3)$  and  $H_k(\mathbb{Z}_3; \mathbb{Z}_3) = \mathbb{Z}_3[k]$  for each  $k \geq 0$  ( $\mathbb{Z}_3[k]$  denoting a copy of  $\mathbb{Z}_3$  with generator  $[k]$ ), we will continue to write  $\mathbf{m}_{\mathbb{Z}_3} = [05] + [50]$ .

Using the diagonal approximation associated to the resolution (3.2) described in Lemma 3.2 (or, tensoring the diagonal (3.4) by  $\mathbb{Z}_q$ ) we can check that the homology diagonal of  $H_*(\mathbb{Z}_3; \mathbb{Z}_3)$  satisfies

$$\Delta_*[0] = [0] \otimes [0] \quad \Delta_*[5] = \sum_{k+l=5} [k] \otimes [l].$$

Consequently, the homology diagonal of  $H_*(\pi; \mathbb{Z}_3)$  satisfies:

$$\Delta_*[05] = \sum_{k+l=5} [0k] \otimes [0l] \quad \Delta_*[50] = \sum_{k+l=5} [k0] \otimes [l0].$$

We have to compute:

$$(\chi_* \otimes \chi_*) (([05] + [50]) \otimes (\Delta_*[05] + \Delta_*[50]) \otimes ([05] + [50])).$$

A term of the form  $\chi_*([kl] \otimes [k'l'])$  is given in  $H_*(\pi; \mathbb{Z}_3) = H_*(\mathbb{Z}_3; \mathbb{Z}_3) \otimes H_*(\mathbb{Z}_3; \mathbb{Z}_3)$  by a componentwise calculation:

$$\chi_*([kl] \otimes [k'l']) = (-1)^{lk'} ([k] \wedge j_*[k']) \otimes ([l] \wedge j_*[l']).$$

Taking into account the formulas for the inversion and for the Pontryagin product (induced in  $\mathbb{Z}_3$ -homology by the formulas (3.6) and (3.5) given above) we have

$$\chi_*([04] \otimes [05]) = ([0] \wedge j_*[0]) \otimes ([4] \wedge j_*[5]) = [0] \otimes ([4] \wedge (-[5])) = -[0] \otimes (6[9]) = -6[09]$$

which vanishes since we are working with coefficients in  $\mathbb{Z}_3$ . We can thus check that

$$(\chi_* \otimes \chi_*) ([50] \otimes ([01] \otimes [04]) \otimes ([05] + [50])) = [51] \otimes [54]$$

and that this is the only term belonging to  $\mathbb{Z}_3[51] \otimes H_*(\pi; \mathbb{Z}_3)$  in the expansion of  ${}^3\chi_*(\mathfrak{m}_{\mathbb{Z}_3}^{\times 3})$ . Since  $[51] \otimes [54]$  does not vanish in  $\mathbb{Z}_3[51] \otimes H_*(\pi; \mathbb{Z}_3)$ , we can conclude that  ${}^3\chi_*(\mathfrak{m}_{\mathbb{Z}_3}^{\times 3}) \neq 0$ . Consequently  ${}^3\chi_*(\mathfrak{m}^{\times 3}) \neq 0$ .

We can next follow the same strategy as in Cohen and Vandembroucq (2021) to show that there exists a manifold  $N$  with fundamental group  $\pi = \mathbb{Z}_3 \times \mathbb{Z}_3$  and maximal  $\text{TC}_3$ . More precisely, considering the lens spaces  $L_3^5 = S^5/\mathbb{Z}_3$  and  $L_3^\infty = S^\infty/\mathbb{Z}_3 = B\mathbb{Z}_3$ , we can realize the class  $[05] + [50] \in H_*(\pi)$  as the image of the fundamental class of  $M = L_3^5 \# L_3^5$  under the map induced by

$$f: M \xrightarrow{\text{pinch}} L_3^5 \vee L_3^5 \hookrightarrow L_3^\infty \times L_3^\infty = B(\pi).$$

We can then use surgery to replace  $M$  by a manifold  $N$  with  $\pi_1(N) = \pi$  and  $f$  by a classifying map  $\gamma: N \rightarrow B\pi$ . In this way,  $\mathfrak{m} = \gamma_*([N])$  and, from  ${}^3\chi_*(\mathfrak{m}^{\times 3}) \neq 0$  and Proposition 2.1 (3), we can deduce that  $\text{TC}_3(N) = 3n$ .

In Cohen and Vandembroucq (2021), it has been shown that the regular topological complexity  $TC = TC_2$  of a non-orientable manifold with abelian fundamental group is never maximal. This is no longer true for  $TC_s$  with  $s \geq 3$ . For instance, for the real projective plane  $P^2$ ,  $s$ -zero-divisor cup length considerations imply that  $TC_3(P^2) = 6$ , see Davis (2018) and the discussion in §4 below. With the approach of this paper, we pursue more general maximality results of this nature next.

### 4 Cohomological lower bounds

In this section, we obtain lower bounds on the higher topological complexity of families of manifolds with finite cyclic fundamental group and maximal LS-category. These bounds arise from cohomological considerations, namely,  $s$ -zero-divisor cup lengths and  $TC_s$ -weights, as well as specific calculations from Davis (2018); Daundkar (2024). In some cases, these considerations together with our results from Section 3 yield exact values. The results of this section establish Theorem B stated in the Introduction.

Let  $\mathbb{k}$  be a field. Recall that, for a space  $X$ , the ( $\mathbb{k}$ -coefficients)  $s$ -zero-divisor cup length,  $zcl_s(X) = zcl_s(X; \mathbb{k})$ , is the maximum of the set

$$\{\ell \mid u_1 \dots u_\ell \neq 0, u_i \in \mathbf{Z}_s(X; \mathbb{k})\}$$

where

$$\mathbf{Z}_s(X; \mathbb{k}) = \ker \left( \bigotimes_{i=1}^s H^*(X; \mathbb{k}) \xrightarrow{\cup} H^*(X; \mathbb{k}) \right).$$

We have  $zcl_s(X) \leq TC_s(X)$ , see Basabe et al. (2014).

In some cases, a better lower bound can be obtained through the notion of  $TC_s$ -weight. Recall (see (Farber and Grant 2008, §2)) that if  $p : E \rightarrow B$  is a fibration and  $u \in \tilde{H}^*(B; \mathbb{k})$  is a nontrivial class, the weight of  $u$  associated to  $p$ ,  $wgt_p(u)$ , is the largest integer  $k$  such that  $f^*(u) = 0$  for any map  $f : Y \rightarrow B$  satisfying  $\text{secat}(f^*(p)) < k$ . If  $u \neq 0$ , then  $wgt_p(u) > 0$  if and only if  $p^*(u) = 0$  and  $\text{secat}(p) \geq wgt_p(u)$ . Moreover, if  $u_1, \dots, u_\ell \in \tilde{H}^*(B; \mathbb{k})$  satisfy  $u_1 \cup \dots \cup u_\ell \neq 0$  then

$$wgt_p(u_1 \cup \dots \cup u_\ell) \geq wgt_p(u_1) + \dots + wgt_p(u_\ell).$$

For a space  $X$ , the  $TC_s$ -weight, denoted by  $wgt_s$ , is the weight associated to the fibration  $e_s : PX \rightarrow X^s$ . Taking coefficients in  $\mathbb{k}$ , the morphism  $e_s^*$  can be identified with the  $s$ -fold cup product and we can define the ( $\mathbb{k}$ -coefficients) *weighted  $s$ -zero-divisor cup length*,  $zcl_s^w(X) = zcl_s^w(X; \mathbb{k})$ , to be the maximum of the set

$$\left\{ \sum_{i=1}^{\ell} wgt_s(u_i) \mid u_1 \dots u_\ell \neq 0, u_i \in \mathbf{Z}_s(X; \mathbb{k}) \right\}.$$

We have  $TC_s(X) \geq zcl_s^w(X)$ . We also note that if  $f : Y \rightarrow X$  is a map and  $u \in H^*(X^s; \mathbb{k})$  satisfies  $(f^s)^*(u) \neq 0$  then  $wgt_s((f^s)^*(u)) \geq wgt_s(u)$ .

### 4.1 Manifolds with $\pi_1(M) = \mathbb{Z}_2$ and $\text{cat}(M) = \dim(M)$

The  $\mathbb{Z}_2$ -coefficient  $s$ -zero-divisor cup length of the real projective space  $P^n$  has been studied extensively, see Cadavid-Aguilar et al. (2018) and Davis (2018). We will see in Proposition 4.2 below how to use these results to obtain information on  $\text{TC}_s(M)$  when  $\pi_1(M) = \mathbb{Z}_2$  and  $\text{cat}(M) = \dim(M)$ . We first recall some results from Davis (2018).

For an integer  $n > 0$  with binary expansion  $\dots d_2d_1d_0$ , i.e.,  $n = \sum_{i \geq 0} d_i 2^i$ , with digits  $d_i \in \{0, 1\}$ , let

- $\nu(n)$  denote the exponent in the maximal 2-power dividing  $n$ , i.e.,  $\nu(n)$  is the minimal  $i$  with  $d_i = 1$ ;
- $S(n) := \{i > 0 : \dots d_{i+1}d_i d_{i-1} \dots = \dots 011 \dots\}$ , the set of binary positions  $i$  starting (from left to right) a block of consecutive 1's of length at least 2;
- $Z_i(n) := \sum_{j=0}^i (1 - d_j)2^j$ , the complement of the binary expansion of  $n$  mod  $2^{i+1}$ .

Building on Cadavid-Aguilar et al. (2018), Davis (2018) proves that, for  $s \geq 3$ , the  $\mathbb{Z}_2$ -coefficient  $s$ -zero-divisor cup length of the  $n$ -dimensional real projective space  $P^n$  is given by

$$\text{zcl}_s(P^n) = sn - m_{n,s}$$

where  $m_{n,s} = \max\{2^{\nu(n+1)} - 1, 2^{i+1} - 1 - sZ_i(n) : i \in S(n)\}$ . In particular, for even  $n$  (so that  $P^n$  is non-orientable),  $P^n$  has maximal possible  $\text{TC}_s(P^n)$ , that is,  $\text{TC}_s(P^n) = sn$ , whenever

$$s \geq \max \left\{ 3, \left\lceil \frac{2^{i+1} - 1}{Z_i(n)} \right\rceil : i \in S(n) \right\}. \tag{4.1}$$

We specialize two cases of the condition (4.1):

- Example 4.1** (a) When  $n$  is even and its binary expansion has no blocks of two or more consecutive 1's, we have  $S(n) = \emptyset$  and the inequality (4.1) reduces to  $s \geq 3$ . Note that this condition for the maximality of  $\text{TC}_s(P^n)$  is sharp, since  $\text{TC}_2(P^n) < 2n$  (see Farber et al. (2003) and (Costa and Farber 2010, Theorem 1)).
- (b) For  $n = 2^{r+1} - 2$ , we have  $S(n) = \{r\}$ ,  $m_{n,s} = 2^{r+1} - 1 - s$  and (4.1) becomes  $s \geq 2^{r+1} - 1$ . Note that, when  $s = n = 2^{r+1} - 2$ , we have  $\text{TC}_n(P^n) \in \{sn, sn - 1\}$ . We will see in Section 5 that  $\text{TC}_n(P^n) = sn - 1$  so that the condition  $s \geq 2^{r+1} - 1$  for the maximality of  $\text{TC}_s(P^n)$  is sharp again.

Thanks to the following result, Davis' computations of  $\text{zcl}_s(P^n)$  have impact on more general manifolds.

**Proposition 4.2** *Let  $M$  be an  $n$ -dimensional connected closed manifold with  $\text{cat}(M) = n$  and  $\pi_1(M) = \mathbb{Z}_2$ . Then, for any  $s \geq 2$ ,  $\text{TC}_s(M) \geq \text{zcl}_s(P^n; \mathbb{Z}_2)$ .*

**Proof** Let  $\gamma : M \rightarrow P^\infty = B\mathbb{Z}_2$  be a classifying map and let  $x \in H_1(B\mathbb{Z}_2; \mathbb{Z}_2) = \mathbb{Z}_2$  be the generator. For dimensional reasons,  $\gamma$  factors as  $M \xrightarrow{c} P^n \hookrightarrow P^\infty$ . Let  $x_M = \gamma^*(x) = c^*(x) \in H^1(M; \mathbb{Z}_2)$ . The hypothesis that  $\text{cat}(M) = n$  implies that

$x_M^n \neq 0$ , see Bernstein (1976). Consequently  $c^* : H^*(P^n; \mathbb{Z}_2) \rightarrow H^*(M; \mathbb{Z}_2)$  as well as  $(c^s)^* : H^*((P^n)^s; \mathbb{Z}_2) \rightarrow H^*(M^s; \mathbb{Z}_2)$  are monomorphisms. As the image under  $(c^s)^*$  of a  $s$ -zero-divisor of  $P^n$  over  $\mathbb{Z}_2$  gives rise to a  $s$ -zero-divisor of  $M$  over  $\mathbb{Z}_2$ , we can conclude that  $\text{zcl}_s(M; \mathbb{Z}_2) \geq \text{zcl}_s(P^n; \mathbb{Z}_2)$  and the result follows.  $\square$

From Example 4.1 and the discussion above, we directly obtain:

**Corollary 4.3** *Let  $M$  be an  $n$ -dimensional connected closed manifold with  $\text{cat}(M) = n$  and  $\pi_1(M) = \mathbb{Z}_2$ . If  $n$  is even and  $s$  satisfies the inequality (4.1), then  $\text{TC}_s(M) = sn$ . In particular:*

- (a) *If  $n$  is even and its binary expansion of  $n$  contains no consecutive digits equal to 1, then  $\text{TC}_s(M) = sn$  for any  $s \geq 3$ .*
- (b) *If  $n = 2^{r+1} - 2$ , then  $\text{TC}_s(M) = sn$  for any  $s \geq 2^{r+1} - 1$ .*

By using Davis' computations in combination with Proposition 3.3, we can also state:

**Corollary 4.4** *Let  $M$  be an orientable connected closed manifold satisfying the conditions  $n = \dim(M) = \text{cat}(M)$  and  $\pi_1(M) = \mathbb{Z}_2$ . If  $n \equiv 1 \pmod{4}$ , then  $\text{TC}_s(M) = sn - 1$  for  $s$  satisfying the inequality (4.1).*

**Proof** In this case  $m_{n,s} = 1$  so that  $\text{TC}_s(M) \geq sn - 1$  for  $s$  satisfying the inequality (4.1). The other direction follows from Proposition 3.3.  $\square$

**Remark 4.5** Observe that the orientability hypothesis, together with the condition  $\dim(M) = \text{cat}(M)$ , implies that  $n$  is odd. Indeed, if  $n$  were even, the image of the fundamental class of  $M$  by  $\gamma_* : H_n(M) \rightarrow H_n(B\mathbb{Z}_2) = 0$  would vanish. But this fact would force, by Poincaré duality, the  $n$ th power of the Bernstein–Schwarz class of  $M$  to vanish, contradicting the equality  $\dim(M) = \text{cat}(M)$ .

## 4.2 Odd dimensional manifolds with $\pi_1(M) = \mathbb{Z}_p$ and $\text{cat}(M) = \dim(M)$

Throughout this section we consider a prime  $p \geq 3$ . Recall that the classifying space  $B\mathbb{Z}_p$  can be identified with the infinite dimensional lens space  $L_p^\infty$ . In order to have an analogue of Proposition 4.2, we first note the following result:

**Lemma 4.6** *Let  $M$  be an orientable connected closed  $(2n + 1)$ -manifold satisfying the conditions  $\text{cat}(M) = 2n + 1$  and  $\pi_1(M) = \mathbb{Z}_p$ . If  $\gamma : M \rightarrow B\mathbb{Z}_p$  is a classifying map, then  $\gamma_*([M]) \in H_{2n+1}(\mathbb{Z}_p; \mathbb{Z}_p)$  is non-zero.*

**Proof** Considering the Bernstein–Schwarz class  $\mathfrak{b}_M \in H^1(M; I(\pi))$ , we have  $\text{cat}(M) = \dim(M) = 2n + 1$  if and only if  $\mathfrak{b}_M^{2n+1} \neq 0$ . By Poincaré duality, the second statement is equivalent to  $\mathfrak{b}_M^{2n+1}([M]) \neq 0$ . Taking cap products we obtain

$$[M] \cap \mathfrak{b}_M^{2n+1} = [M] \cap \gamma^*(\mathfrak{b}_{\mathbb{Z}_p}^{2n+1}) \neq 0.$$

Since  $\gamma$  induces an isomorphism at the level of fundamental groups, naturality of the cap-products yields

$$\gamma_*([M]) \cap \mathfrak{b}_{\mathbb{Z}_p}^{2n+1} \neq 0.$$

Hence,  $\gamma_*([M]) \neq 0$  in  $H_{2n+1}(\mathbb{Z}_p; \mathbb{Z}) = \mathbb{Z}_p$ . Since  $H_{2n+1}(\mathbb{Z}_p; \mathbb{Z}) = H_{2n+1}(\mathbb{Z}_p; \mathbb{Z}_p) = \mathbb{Z}_p$  we can conclude that  $\gamma_*([M]) \neq 0$  in  $H_{2n+1}(\mathbb{Z}_p; \mathbb{Z}_p)$ .  $\square$

**Theorem 4.7** *Let  $M$  be a closed orientable  $(2n + 1)$ -manifold with  $\text{cat}(M) = 2n + 1$  and  $\pi_1(M) = \mathbb{Z}_p$  where  $p \geq 3$  is a prime. Then,  $\text{TC}_s(M) \geq \text{zcl}_s^w(M; \mathbb{Z}_p) \geq \text{zcl}_s^w(L_p^{2n+1}; \mathbb{Z}_p)$ .*

**Proof** Let  $\gamma : M \rightarrow B\mathbb{Z}_p$  be a classifying map. We have a commutative triangle

$$\begin{array}{ccc}
 M & \xrightarrow{\phi} & L_p^{2n+1} \\
 & \searrow \gamma & \downarrow \\
 & & L_p^\infty
 \end{array} \tag{4.2}$$

where the inclusion is simply the  $(2n + 1)$ -skeleton of the infinite dimensional lens space  $L_p^\infty \simeq B\mathbb{Z}_p$ . Since  $\text{cat}(M) = 2n + 1$ , we know by the previous lemma that  $\gamma_*([M]) \neq 0$  in  $H_{2n+1}(\mathbb{Z}_p; \mathbb{Z}_p)$ . Consequently  $\phi_*([M]) \neq 0$  in  $H_{2n+1}(L_p^{2n+1}; \mathbb{Z}_p)$ . Recall that

$$H^*(L_p^{2n+1}; \mathbb{Z}_p) = \mathbb{Z}_p[x, y]/(y^{n+1}, x^2)$$

where  $|x| = 1, |y| = 2$ . In particular,  $H^{2n+1}(L_p^{2n+1}; \mathbb{Z}_p) = \mathbb{Z}_p$ , generated by  $xy^n$ . We first check that  $\phi^* : H^*(L_p^{2n+1}; \mathbb{Z}_p) \rightarrow H^*(M; \mathbb{Z}_p)$  is a monomorphism. It suffices to show that  $\phi^*(xy^n)$  is non-trivial in  $H^*(M; \mathbb{Z}_p)$ . Consider the following cap-product diagram:

$$\begin{array}{ccc}
 H_{2n+1}(M; \mathbb{Z}) \otimes H^{2n+1}(M; \mathbb{Z}_p) & \xrightarrow[\cong]{\cap} & H_0(M; \mathbb{Z}_p) = \mathbb{Z}_p \\
 \downarrow \phi_* & & \downarrow = \\
 H_{2n+1}(L_p^{2n+1}; \mathbb{Z}_p) \otimes H^{2n+1}(L_p^{2n+1}; \mathbb{Z}_p) & \xrightarrow[\cong]{\cap} & H_0(L_p^{2n+1}; \mathbb{Z}_p) = \mathbb{Z}_p \\
 & \uparrow \phi^* & \\
 & & H^{2n+1}(M; \mathbb{Z}_p)
 \end{array}$$

Both horizontal morphisms are isomorphisms by Poincaré duality. As  $\phi_*([M]) \neq 0$  in  $H_{2n+1}(L_p^{2n+1}; \mathbb{Z}_p)$ ,  $\phi_*([M])$  is not divisible by  $p$  in  $H_{2n+1}(L_p^{2n+1}; \mathbb{Z}) = \mathbb{Z}$ . Consequently,  $\phi_*([M]) \cap xy^n \neq 0$  and using the diagram we have that  $[M] \cap \phi^*(xy^n) \neq 0$ . We thus have  $\phi^*(xy^n) \neq 0$  in  $H^*(M; \mathbb{Z}_p)$ . Therefore  $\phi^*$  is a monomorphism. As  $p$  is prime, we can see, using the Künneth formula, that  $(\phi^s)^*$  is also a monomorphism. Let  $u \neq 0 \in \mathbf{Z}_s(L_p^{2n+1}; \mathbb{Z}_p)$ . Then  $(\phi^s)^*(u) \in \mathbf{Z}_s(M; \mathbb{Z}_p)$  and  $(\phi^s)^*(u) \neq 0$ . Consequently  $\text{wgt}_s((\phi^s)^*(u)) \geq \text{wgt}_s(u)$  and the results follows by the definition of  $\text{zcl}_s^w$ .  $\square$

There are extensive computations of  $\text{zcl}_s^w(L_p^n; \mathbb{Z}_p)$ , see (Farber and Grant 2008, §5) and (Daundkar 2024, Section 5). By using the previous theorem, we can use the information coming from these computations for a larger class of manifolds.

**Corollary 4.8** *Let  $s \geq 2$  and let  $M$  be a closed orientable  $(2n + 1)$ -manifold with  $\text{cat}(M) = 2n + 1$  and  $\pi_1(M) = \mathbb{Z}_p$  where  $p \geq 3$  is prime. Then,*

$$\text{TC}_s(M) \geq \begin{cases} s \cdot (\ell + \ell' + 1) - 1 & \text{if } s \text{ is even;} \\ (s - 1) \cdot (\ell + \ell') + s + 2n - 1 & \text{if } s \text{ is odd} \end{cases}$$

where  $0 \leq \ell, \ell' \leq n$  are any integers such that  $p$  does not divide  $\binom{\ell + \ell'}{\ell}$ .

**Proof** Here we apply the computation (Daundkar 2024, Theorem 5.2) and Theorem 4.7.  $\square$

In many situations, for example if  $p$  is much larger than the dimension of  $M$ , we obtain an exact computation.

**Corollary 4.9** *Let  $s \geq 2$  and let  $M$  be a closed orientable  $(2n + 1)$ -manifold with  $\text{cat}(M) = 2n + 1$  and  $\pi_1(M) = \mathbb{Z}_p$  where  $p \geq 3$  is a prime. If  $p$  does not divide  $\binom{2n}{n}$ , then*

$$\text{TC}_s(M) = s(2n + 1) - 1.$$

**Proof** The lower bound follows from Corollary 4.8 (see also (Daundkar 2024, Theorem 5.3)) and the upper bound is Proposition 3.3.  $\square$

**Remark 4.10** The results of Daundkar (2024) used in the proof of Corollary 4.8 also hold when  $p$  is not a prime, under the condition  $p$  does not divide  $\binom{\ell + \ell'}{\ell}^{\lfloor s/2 \rfloor}$ , the analogous condition in Corollary 4.9 being  $p$  does not divide  $\binom{2n}{n}^{\lfloor s/2 \rfloor}$ . It is possible that the results of this section could be extended to this more general setting. However, for the simplicity of some arguments in Theorem 4.7, we have restricted the discussion of this section to coefficients in a field.

## 5 Some calculations in the non-orientable case

We now address the (non-)maximality of  $\text{TC}_s(M)$  for non-orientable manifolds having  $\pi_1(M) = \mathbb{Z}_2$ . The results of this section establish Theorem C stated in the Introduction.

Since the case  $s = 2$  is well understood by (Costa and Farber 2010, Theorem 1), we assume  $s \geq 3$  throughout. For such cases, the non-maximality of  $\text{TC}_s(M)$  demands further restrictions on  $s$ . The aim of this section is to establish the following result.

**Theorem 5.1** *Let  $M$  be a non-orientable  $n$ -dimensional manifold with  $\pi_1(M) = \mathbb{Z}_2$  and  $n = 2^{r+1} - 2$ . Then, for any even  $s$  no greater than  $2^{r+1} - 2$ , we have  $\text{TC}_s(M) < sn$ .*

By Corollary 4.3 (b), we know that  $\text{TC}_s(M) = sn$  when  $n = 2^{r+1} - 2$  and  $s \geq 2^{r+1} - 1$ , so that in this case the upper limiting restriction on  $s$  in Theorem 5.1 is in fact sharp and we have:

**Corollary 5.2** *If the manifold  $M$  in Theorem 5.1 has  $\text{cat}(M) = n$ , then  $\text{TC}_s(M) = sn - 1$  for  $s = 2^{r+1} - 2$ , and  $\text{TC}_s(M) = sn$  for  $s \geq 2^{r+1} - 1$ .*

**Proof** The equality  $TC_s(M) = sn - 1$  for  $s = 2^{r+1} - 2$  follows from  $m_{2^{r+1}-2, 2^{r+1}-2} = 1$  (see Example 4.1 (b)), Proposition 4.2 and Theorem 5.1.  $\square$

Corollary 5.2 should be compared to the fact that  $TC_s(P^{2^r})$  is maximal for  $s \geq 3$ , while, as shown in Farber et al. (2003),  $TC_2(P^{2^r}) = \text{Imm}(P^{2^r}) = 2^{r+1} - 1$ . Worth noting is the fact that the case  $M = P^6$  in Corollary 5.2 (with  $r = 2$ ) upgrades the observation in (Cadavid-Aguilar et al. 2018, (7.4)) that  $\delta_6(6) \leq 1$  to an equality, giving evidence for what would be regular behavior of the higher topological complexity of projective spaces  $P^m$  with  $m = 2^a + 2^{a+1}$ .

Suitable analogues of Theorem 5.1 should hold for more general values of  $n$ , but the complexity of calculations seems to be a major obstacle towards obtaining corresponding proofs.

We now start working towards the proof of Theorem 5.1. From now on  $\pi := \mathbb{Z}_2$  and  $s = 2\sigma$  with  $1 \leq \sigma \leq 2^r - 1 = n/2 =: m$ . Set  $\widehat{\mathbb{Z}} := \widetilde{\mathbb{Z}} \otimes (\mathbb{Z} \otimes \widetilde{\mathbb{Z}})^{\sigma-1}$ , the  $\mathbb{Z}[\pi^{s-1}]$ -module of Remark 2.5. By Proposition 2.3, it suffices to establish the triviality of

$${}^s\chi_*(\mathfrak{m}^{\times s}) \in H_{sn}(\pi^{s-1}; \widehat{\mathbb{Z}}) \tag{5.1}$$

where

$${}^s\chi_* : H_*(\pi^s; \widetilde{\mathbb{Z}}^{\otimes s}) \rightarrow H_*(\pi^{s-1}; \widehat{\mathbb{Z}}). \tag{5.2}$$

As in §3, our starting point is the free  $\mathbb{Z}[\mathbb{Z}_q]$ -resolution (3.2) of  $\mathbb{Z}$  with  $q = 2$ . Recall that  $[k]$  denotes the generator of degree  $k$ . In addition to the chain complex  $C_\bullet(\pi)$  of (3.3), we will also need the complex  $\widetilde{C}_\bullet(\pi)$ ,

$$\xrightarrow{-2} \widetilde{C}_{2k}(\pi) \xrightarrow{0} \widetilde{C}_{2k-1}(\pi) \xrightarrow{-2} \dots \xrightarrow{0} \widetilde{C}_1(\pi) \xrightarrow{-2} \widetilde{C}_0(\pi), \tag{5.3}$$

where  $\widetilde{C}_k(\pi) = \mathbb{Z}$  for each  $k$ , obtained by tensoring (3.2) with  $\widetilde{\mathbb{Z}}$  over  $\pi$ . Abusing notation, we continue using  $[k]$  for the generators of both  $C_k(\pi)$  and  $\widetilde{C}_k(\pi)$ .

The homology groups in (5.2) can be computed from the complexes  $C_\bullet(\pi)^{\otimes s}$  and

$$\mathcal{D}_\bullet := \widetilde{C}_\bullet(\pi) \otimes (C_\bullet(\pi) \otimes \widetilde{C}_\bullet(\pi))^{\sigma-1}. \tag{5.4}$$

In both cases, we will use the shorthand  $[i_1, \dots, i_\ell]$  for a tensor product  $[i_1] \otimes \dots \otimes [i_\ell]$ . The Künneth formula and the fact that the homology of  $\widetilde{C}_\bullet(\pi)$  is 2-torsion (in all degrees) gives:

**Lemma 5.3** *The element  ${}^s\chi_*(\mathfrak{m}^{\times s})$  in (5.1) is torsion. Indeed, both groups in (5.2) are torsion.*

Let  $\mathcal{H}_\bullet$  denote the quotient of  $\mathcal{D}_\bullet$  resulting from killing all boundaries, and consider the obvious monomorphism  $\iota : H_\bullet(\pi^{s-1}; \widehat{\mathbb{Z}}) \hookrightarrow \mathcal{H}_\bullet$ . The triviality of the element in (5.1) follows from Lemma 5.3 and the following key result, whose proof is addressed in the rest of the section through a direct analysis of (5.1) and (5.2).

**Proposition 5.4** *The class  $\iota^s \chi_*(\mathfrak{m}^{\times s})$  lies in the torsion-free (graded) subgroup of  $\mathcal{H}_\bullet$ .*

We first describe a cycle representing the element  ${}^s \chi_*(\mathfrak{m}^{\times s})$ . We start with some notational considerations that allow us to use (a suitable analogue of) the factorization (3.1) in the current context. Since  $\Delta^*(\tilde{\mathbb{Z}} \otimes \mathbb{Z}) = \Delta^*(\mathbb{Z} \otimes \tilde{\mathbb{Z}}) = \tilde{\mathbb{Z}}$ , the map  $\Delta : \pi \rightarrow \pi \times \pi$  induces two well-defined chain maps:  $\tilde{C}_\bullet(\pi) \rightarrow \tilde{C}_\bullet(\pi) \otimes C_\bullet(\pi)$  and  $\tilde{C}_\bullet(\pi) \rightarrow C_\bullet(\pi) \otimes \tilde{C}_\bullet(\pi)$ . We will say that the former is induced by the map  $\Delta : \tilde{\pi} \rightarrow \tilde{\pi} \times \pi$  while the latter is induced by the map  $\Delta : \tilde{\pi} \rightarrow \pi \times \tilde{\pi}$ . Likewise, since the product operation  $\mu : \pi \times \pi \rightarrow \pi$  satisfies  $\mu^*(\mathbb{Z}) = \mathbb{Z} \otimes \mathbb{Z}$  and  $\mu^*(\tilde{\mathbb{Z}}) = \tilde{\mathbb{Z}} \otimes \tilde{\mathbb{Z}}$ , we get two morphisms  $C_\bullet(\pi) \otimes C_\bullet(\pi) \rightarrow C_\bullet(\pi)$  and  $\tilde{C}_\bullet(\pi) \otimes \tilde{C}_\bullet(\pi) \rightarrow \tilde{C}_\bullet(\pi)$ , which will be said to be induced by the maps  $\mu : \pi \times \pi \rightarrow \pi$  and  $\mu : \tilde{\pi} \times \tilde{\pi} \rightarrow \tilde{\pi}$ , respectively. In this context a tilde is thus used to indicate that the non-trivial action of  $\pi$  on the abelian group  $\mathbb{Z}$  must be considered. This abuse of notation will be particularly convenient in what follows. Observe that, since  $\pi = \mathbb{Z}_2 = \langle v \mid v^2 = 1 \rangle$ , the inversion morphism induces the identity in both complexes  $C_\bullet(\pi)$  and  $\tilde{C}_\bullet(\pi)$ . In these terms, we can see the map  ${}^s \chi_\bullet : \tilde{C}_\bullet(\pi)^{\otimes s} \rightarrow \mathcal{D}_\bullet$  as induced by the following composite:

$$\begin{aligned} \tilde{\pi}^s &\xrightarrow{1 \times (\Delta \times \Delta)^{\sigma-1} \times 1} \tilde{\pi} \times (\tilde{\pi} \times \pi \times \pi \times \tilde{\pi})^{\sigma-1} \times \tilde{\pi} \\ &= (\tilde{\pi} \times \tilde{\pi}) \times (\pi \times \pi \times \tilde{\pi} \times \tilde{\pi})^{\sigma-1} \xrightarrow{\mu \times (\mu \times \mu)^{\sigma-1}} \tilde{\pi} \times (\pi \times \tilde{\pi})^{\sigma-1}. \end{aligned} \tag{5.5}$$

Recalling that  $B_{i,j}$  denotes the binomial coefficient  $\binom{i+j}{i}$ , the formulæ of Lemma 3.2 (b) written with the shorthand in use in this section read

$$\begin{aligned} [2i, 2j] &\mapsto B_{i,j} [2(i+j)]; \\ [2i, 2j+1] &\mapsto B_{i,j} [2(i+j)+1]; \\ [2i+1, 2j] &\mapsto B_{i,j} [2(i+j)+1]; \\ [2i+1, 2j+1] &\mapsto 0. \end{aligned} \tag{5.6}$$

We note also that, since  $\pi = \mathbb{Z}_2$ , the formula of Lemma 3.2(a) giving  $\Delta$  on generators at the level of resolutions can be written as

$$[p] \mapsto \sum_{k+l=p} [k] \otimes v^{\text{odd}(k)} \cdot [l], \tag{5.7}$$

where  $v$  generates  $\pi$  and  $\text{odd}(k) = 1$  if  $k$  is odd and 0 otherwise.

The class  $\mathfrak{m} \in H_*(\pi; \tilde{\mathbb{Z}})$  is either trivial or, else, represented by the cycle  $[n]$  in (5.3) —recall  $n$  is even. For the purposes of proving Theorem 5.1, we may safely assume the latter possibility. Then,  $\mathfrak{m}^{\times s}$  is represented in  $\tilde{C}_\bullet(\pi)^{\otimes s}$  by the corresponding tensor product  $[n, n, \dots, n]$ . We chase the latter element under the first map of the composite (5.5) to get, in view of (5.7),

$$[\tilde{n}, \tilde{n}, \dots, \tilde{n}] \mapsto [\tilde{n}] \otimes \left( \left( \sum_{p+q=n} [\tilde{p}] \otimes v^{\text{odd}(p)} \cdot [q] \right) \otimes \left( \sum_{p+q=n} [p] \otimes v^{\text{odd}(p)} \cdot [\tilde{q}] \right) \right)^{\otimes \sigma-1} \otimes [\tilde{n}].$$

Note that in the latter expression we are extending in the obvious way the mnemonic convention above regarding the use of a tilde to indicate that the nontrivial action of  $\pi$  on  $\mathbb{Z}$  is to be taken in the corresponding tensor factor. Then, after tensoring with the needed coefficients (thus applying the  $\sim$  indicators), this becomes

$$\begin{aligned} [n, n, \dots, n] &\mapsto [n] \otimes \left( \left( \sum_{p+q=n} [p, q] \right) \otimes \left( \sum_{p+q=n} (-1)^p [p, q] \right) \right)^{\otimes \sigma-1} \otimes [n] \\ &= \sum_{\substack{p_i+q_i=n \\ 1 \leq i \leq s-2}} (-1)^{\sum_{j=1}^{\sigma-1} p_{2j}} [n, p_1, q_1, p_2, q_2, \dots, p_{s-2}, q_{s-2}, n]. \end{aligned}$$

Since  $n = 2m$  is even, the parity of each  $p_i$  agrees with the one of the corresponding  $q_i$ , so the last expression can be rewritten as

$$\sum (-1)^{\sum_{j=1}^{\sigma-1} \delta_{2j}} [n, 2p_1 + \delta_1, 2q_1 + \delta_1, 2p_2 + \delta_2, 2q_2 + \delta_2, \dots, 2p_{s-2} + \delta_{s-2}, 2q_{s-2} + \delta_{s-2}, n],$$

where the sum now runs over  $1 \leq i \leq s - 2$ ,  $\delta_i \in \{0, 1\}$  and  $p_i + q_i = m - \delta_i$ . Using formulae (5.6), we finally obtain the expression

$$[n, \dots, n] \mapsto \sum (-1)^{\sum_{j=1}^{\sigma-1} \delta_{2j}} B_{m, p_1} B_{q_1, p_2} \cdots B_{q_{s-3}, p_{s-2}} B_{q_{s-2}, m} \cdot \mathbf{G} \tag{5.8}$$

for the image of  $[n, \dots, n]$  under the composition in (5.5), where

$$\mathbf{G} = [n + 2p_1 + \delta_1, 2(q_1 + p_2) + \delta_1 + \delta_2, \dots, 2(q_{s-3} + p_{s-2}) + \delta_{s-3} + \delta_{s-2}, n + 2q_{s-2} + \delta_{s-2}]$$

and the sum runs over the same indices as above, except that now no two consecutive  $\delta_j$  and  $\delta_{j+1}$  can simultaneously be equal to 1, in view of (5.6).

Having described a cycle representing the obstruction in (5.1), we next spell out the complex (5.4) where it lies. Degreewise,  $\mathcal{D}_\bullet$  is  $\mathbb{Z}$ -free with basis given by elements  $[u_1, v_1, \dots, u_{\sigma-1}, v_{\sigma-1}, u_\sigma]$  for non-negative integers  $u_i$  and  $v_i$ , and with differential

$$\begin{aligned} \partial[u_1, v_1, \dots, u_{\sigma-1}, v_{\sigma-1}, u_\sigma] &= \\ &= -2 \text{ odd}(u_1)[u_1 - 1, v_1, u_2, v_2, \dots, u_{\sigma-1}, v_{\sigma-1}, u_\sigma] \\ &\quad + (-1)^{u_1} 2 \text{ even}(v_1)[u_1, v_1 - 1, u_2, v_2, \dots, u_{\sigma-1}, v_{\sigma-1}, u_\sigma] \\ &\quad - (-1)^{u_1+v_1} 2 \text{ odd}(u_2)[u_1, v_1, u_2 - 1, v_2, \dots, u_{\sigma-1}, v_{\sigma-1}, u_\sigma] \\ &\quad + (-1)^{u_1+v_1+u_2} 2 \text{ even}(v_2)[u_1, v_1, u_2, v_2 - 1, \dots, u_{\sigma-1}, v_{\sigma-1}, u_\sigma] \\ &\quad \pm \dots \end{aligned} \tag{5.9}$$

$$\begin{aligned}
 & - (-1)^{u_1+v_1+\dots+u_{\sigma-2}+v_{\sigma-2}} 2 \text{ odd } (u_{\sigma-1})[u_1, v_1, u_2, v_2, \dots, u_{\sigma-1} - 1, v_{\sigma-1}, u_{\sigma}] \\
 & + (-1)^{u_1+v_1+\dots+u_{\sigma-2}+v_{\sigma-2}+u_{\sigma-1}} 2 \text{ even } (v_{\sigma-1})[u_1, v_1, u_2, v_2, \dots, u_{\sigma-1}, v_{\sigma-1} - 1, u_{\sigma}] \\
 & - (-1)^{u_1+v_1+\dots+u_{\sigma-1}+v_{\sigma-1}} 2 \text{ odd } (u_{\sigma})[u_1, v_1, u_2, v_2, \dots, u_{\sigma-1}, v_{\sigma-1}, u_{\sigma} - 1],
 \end{aligned}$$

where a basis element with a negative entry is meant to be interpreted as zero and, as in (5.7),  $\text{even}(k) = 1$  if  $k$  is even and 0 otherwise.

All the information needed to prove Proposition 5.4 is of course contained in (5.8) and (5.9). Explicitly, Proposition 5.7 below isolates the input required from the form (5.8) of the obstruction (5.1), while Proposition 5.9 below allows us to organize the relations in  $\mathcal{H}_\bullet$  coming from the differential (5.9).

**Definition 5.5** For a positive integer  $k$ , let  $p(k)$  denote the set of binary positions where the binary expansion of  $k$  has digit 1. For instance,  $p(5 = 4 + 1) = \{0, 2\}$  and  $p(42 = 32 + 8 + 2) = \{1, 3, 5\}$ .

A standard well known fact is:

**Lemma 5.6** A binomial coefficient  $B_{a,b}$  is even if and only if  $p(a) \cap p(b) \neq \emptyset$ .

The following result is the only place where the special assumptions in Theorem 5.1 (i.e.,  $s = 2\sigma \leq n = 2m = 2^{r+1} - 2$  with  $r \geq 1$ ) are needed.

**Proposition 5.7** Any coefficient

$$B_{m,p_1} B_{q_1,p_2} \cdots B_{q_{s-3},p_{s-2}} B_{q_{s-2},m} \tag{5.10}$$

in (5.8) with  $p_1 + q_1 = m$  (i.e., with  $\delta_1 = 0$ ) is even.

**Proof** We use without further notice the fact coming from Lemma 5.6 that any binomial coefficient  $B_{m-j,i}$  is even whenever  $0 \leq j < i \leq m$ . Recall that  $m = 2^r - 1$  and

$$s = 2\sigma \leq 2^{r+1} - 2 \tag{5.11}$$

with  $r \geq 1$ . Assume for a contradiction that some coefficient (5.10) is odd (i.e., that all of its binomial-coefficient factors are odd) and has  $\delta_1 = 0$ . Recall the forced conditions

- (1)  $p_i + q_i = m - \delta_i$  with  $p_i, q_i \geq 0$  and  $\delta_i \in \{0, 1\}$ ;
- (2)  $1 \leq i < s - 2$  and  $\delta_i = 1$  implies  $\delta_{i+1} = 0$ ,

for  $1 \leq i \leq s - 2$ . Let  $2 \leq i_1 < i_2 < \dots < i_k \leq s - 2$  be all the indices  $j$  (if any) with  $\delta_j = 1$ . Note that

$$0 \leq k \leq \sigma - 1, \tag{5.12}$$

in view of (2).

The coefficient  $B_{m,p_1}$  is odd by hypothesis, so  $p_1 = 0$  and  $q_1 = m$  —the latter equality holds in view of (1) since  $\delta_1 = 0$ . Actually, the same argument can be used

iteratively for  $1 \leq j < i_1$  (so  $\delta_j = 0$ ) with the binomial coefficients  $B_{q_{j-1}, p_j}$  (e.g.,  $q_0 := m$ ) to show that

$$p_j = 0 \text{ and } q_j = m.$$

Next, since  $m = q_{i_1-1}$ ,  $B_{q_{i_1-1}, p_{i_1}}$  is odd and  $\delta_{i_1} = 1$ , we get

$$p_{i_1} = 0 \text{ and } q_{i_1} = m - 1,$$

and now the process repeats with a slight adjustment. For starters,  $q_{i_1} = m - 1$ ,  $B_{q_{i_1}, p_{i_1+1}}$  is odd and  $\delta_{i_1+1} = 0$  is forced by (2), so that  $p_{i_1+1} \leq 1$  and  $q_{i_1+1} \geq m - 1$ . We then iterate the latter argument: For  $i_1 < j < i_2$ , the assumption  $\delta_j = 0$  and the fact that  $B_{q_{j-1}, p_j}$  is odd with  $q_{j-1} \geq m - 1$  yield

$$p_j \leq 1 \text{ and } q_j \geq m - 1.$$

Of course, the last two inequalities now hold for all  $1 \leq j < i_2$ . The next round of iterations start with the fact that  $B_{q_{i_2-1}, p_{i_2}}$  is odd with  $q_{i_2-1} \geq m - 1$  and  $\delta_{i_2} = 1$ , to get

$$p_{i_2} \leq 1 \text{ and } q_{i_2} \geq m - 2,$$

and the process has a corresponding new obvious adjustment to yield

$$p_j \leq 2 \text{ and } q_j \geq m - 2,$$

for  $1 \leq j < i_3$ , whereas

$$p_{i_3} \leq 2 \text{ and } q_{i_3} \geq m - 3.$$

Just before the last adjustment we get

$$p_j \leq k - 1 \text{ and } q_j \geq m - k + 1,$$

for  $1 \leq j < i_k$ , whereas

$$p_{i_k} \leq k - 1 \text{ and } q_{i_k} \geq m - k.$$

However, after this point the conditions  $p_j \leq k$  and  $q_j \geq m - k$  are kept for all  $j \leq s - 2$ . In particular,  $q_{s-2} \geq m - k = 2^r - 1 - k \geq 1$ , in view of (5.11) and (5.12). But then the final factor  $B_{q_{s-2}, m}$  of (5.10) is even, a contradiction.  $\square$

**Definition 5.8** (a) Let  $\mathcal{D}_\bullet^{\text{even}}$  (respectively,  $\mathcal{D}_\bullet^{\text{odd}}$ ) denote the (graded) subgroup of  $\mathcal{D}_\bullet$  generated by the basis elements  $[u_1, v_1, \dots, u_{\sigma-1}, v_{\sigma-1}, u_\sigma]$  with  $u_1$  even (respectively, odd).

(b) Let  $\mathcal{H}_\bullet^{\text{even}}$  stand for the quotient of  $\mathcal{D}_\bullet$  by the graded subgroup  $\partial(\mathcal{D}_\bullet^{\text{odd}})$ .

**Proposition 5.9** *The canonical projection  $\mathcal{H}_\bullet^{\text{even}} \rightarrow \mathcal{H}_\bullet$  is an isomorphism.*

**Proof** Write (5.9) in the form

$$\begin{aligned} \partial[a, x_1, x_2, \dots, x_{s-2}] &= -2 \operatorname{odd}(a)[a - 1, x_1, x_2, \dots, x_{s-2}] \\ &\quad + \sum_{\ell=1}^{s-2} (-1)^{\ell+1+a+x_1+\dots+x_{\ell-1}} \delta(\ell, x_\ell) \cdot 2[a, x_1^{(\ell)}, \dots, x_{s-2}^{(\ell)}] \end{aligned}$$

where

$$\delta(j, \ell) = \begin{cases} 0, & \text{if } j \equiv \ell \pmod{2}; \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad x_j^{(\ell)} = \begin{cases} x_j - 1, & \text{if } j = \ell; \\ x_j, & \text{otherwise.} \end{cases}$$

For instance, for  $[y_1, \dots, y_{s-2}] \in \mathcal{D}_\bullet^{\operatorname{odd}}$ , we have

$$2[-1, y_1, \dots, y_{s-2}] \equiv \sum_{j=1}^{s-2} (-1)^{j+y_1+\dots+y_{j-1}} \delta(j, y_j) \cdot 2[y_1^{(j)}, \dots, y_{s-2}^{(j)}], \tag{5.13}$$

where  $\equiv$  stands for congruence mod  $\partial(\mathcal{D}_\bullet^{\operatorname{odd}})$ . In these terms the proof follows from the fact that  $\partial[e, x_1, \dots, x_{s-2}] \equiv 0$  for any  $[e, x_1, \dots, x_{s-2}] \in \mathcal{D}_\bullet^{\operatorname{even}}$ . In turn, the asserted congruence follows from

$$\begin{aligned} \partial[e, x_1, \dots, x_{s-2}] &= \sum_{\ell=1}^{s-2} (-1)^{\ell+1+x_1+\dots+x_{\ell-1}} \delta(\ell, x_\ell) \cdot 2[e, x_1^{(\ell)}, \dots, x_{s-2}^{(\ell)}] \\ &\equiv \sum_{\ell=1}^{s-2} (-1)^{\ell+1+x_1+\dots+x_{\ell-1}} \delta(\ell, x_\ell) \sum_{j=1}^{s-2} (-1)^{j+x_1^{(\ell)}+\dots+x_{j-1}^{(\ell)}} \delta(j, x_j^{(\ell)}) \cdot 2[e + 1, (x_1^{(\ell)})^{(j)}, \dots, (x_{s-2}^{(\ell)})^{(j)}] \\ &= \sum_{1 \leq \ell, j \leq s-2} (-1)^{\ell+j+1+x_1+\dots+x_{\ell-1}+x_1^{(\ell)}+\dots+x_{j-1}^{(\ell)}} \delta(\ell, x_\ell) \delta(j, x_j^{(\ell)}) \cdot 2[e + 1, (x_1^{(\ell)})^{(j)}, \dots, (x_{s-2}^{(\ell)})^{(j)}], \end{aligned}$$

which vanishes since a summand with  $\ell = j$  is trivial due to the equality  $\delta(i, x_i) \delta(i, x_i^{(i)}) = 0$  whereas, for  $1 \leq a < b \leq s - 2$ , the summand with  $(\ell, j) = (a, b)$  cancels out the summand with  $(\ell, j) = (b, a)$ . Indeed, both summands are multiples of

$$\delta(a, x_a) \delta(b, x_b) \cdot 2[e + 1, x_1, \dots, x_{a-1}, x_a - 1, x_{a+1}, \dots, x_{b-1}, x_b - 1, x_{b+1}, \dots, x_{s-2}];$$

the former one with coefficient  $(-1)^{a+b+1+x_a^{(a)}+\dots+x_{b-1}^{(a)}} = (-1)^{a+b+x_a+\dots+x_{b-1}}$ , and the latter one with coefficient  $(-1)^{a+b+1+x_1+\dots+x_{b-1}+x_1+\dots+x_{a-1}} = (-1)^{a+b+1+x_a+\dots+x_{b-1}}$ . □

**Proof of Proposition 5.4** Proposition 5.9 shows that  $\mathcal{H}_\bullet$  is the quotient of  $\mathcal{D}_\bullet$  by relations of the form

$$2\langle e, x_1, x_2, \dots, x_{s-2} \rangle = \sum_{\ell=1}^{s-2} \pm 2\langle e+1, x_1^{(\ell)}, x_2^{(\ell)}, \dots, x_{s-2}^{(\ell)} \rangle$$

with  $e$  even, where  $\langle a_1, \dots, a_{s-1} \rangle$  stands for the image in  $\mathcal{H}_\bullet$  of the corresponding basis element  $[a_1, \dots, a_{s-1}]$ . It follows that the image of  $\mathcal{D}_\bullet^{\text{odd}}$  in  $\mathcal{H}_\bullet$  lies in the torsion-free part of  $\mathcal{H}_\bullet$ . The proof is then complete in view of (5.13) and Proposition 5.7.  $\square$

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**Data Availability** No datasets were generated or analysed during the current study.

## Declarations

**Competing interests** The authors declare no competing interests.

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