FINITE QUOTIENTS OF SPHERICAL ARTIN GROUPS

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Abstract. We show the smallest non-abelian quotients of spherical and affine Artin groups are isomorphic to the smallest non-abelian quotients of the corresponding Coxeter groups. We deduce irreducible spherical Artin groups are determined by their finite quotient groups.

1. Introduction

Artin introduced braid groups in [Art47], and later these were generalised by Tits [Tit66] to the class of Artin groups. For any Coxeter graph S, Tits calls an Artin group A_S the extended Coxeter group of the Coxeter group W_S and equips A_S with a natural epimorphism to W_S. The systematic study of Artin groups began with the works [Bri71, Del72, BS72, Bri73] and [Sal87] with much research focusing on the *spherical Artin groups*: the Artin groups A_S where W_S is a finite group. The Coxeter diagrams of finite Coxeter groups were classified by Coxeter [Cox35] and are given in Figure 1. For notational clarity, the classical n-strand braid group is denoted by $\mathcal{A}_{n-1}$ to be distinguished from the Artin group $\mathcal{B}_n$, the dihedral Artin group is denoted $\mathcal{I}_2(n)$, and the dihedral Coxeter group is denoted Dh_n to distinguish it from the Artin group $\mathcal{D}_n$. We often denote the canonical Coxeter quotient of an Artin group G by $W(G)$.

Much is known about the spherical Artin groups. The $K(\pi, 1)$ conjecture holds [CD95, BW02], the groups are linear (hence residually finite [Mal40]); see [Kra12, Big01] for the braid group and [CW02, Dig03] for the other groups; and they are classified up to isomorphism [Par04]. Moreover, the geometry of these groups has been extensively studied but remains mysterious [Tit66, Del72, Cha95, CGGMW17, HO21, CW24, Rag25].

The braid group arises naturally in many different fields of mathematics, as a result the representation theory of braid groups has been extensively studied from different perspectives; see for instance [Jon87, For96, ACC03, Mar13, PS25], or [Mar19] for a survey. One important avenue has been to use local representations of the braid group to give topological models for quantum computation [FKLW01, Mar13]; see [DRW16] for a survey. Even the simplest representations—those factoring through finite quotient groups—have been intensively studied [Cox59, BW86, Waj91, ERW08, LR08, Cap23, KLP21, CKLP20, SV23, BPS24]. This culminated with Kolay's [Kol23] positive resolution of a longstanding question of Margalit asking if the symmetric group $\mathrm{Sym}(n)$ is the smallest non-abelian quotient of the braid group $\mathcal{A}_{n-1}$ for $n \geqslant 5$. Similar results were earlier established for $\mathrm{Aut}(F_N)$ [BKP19], mapping class groups [KP20], and surface braid groups [Tan24].

Our first main theorem is Theorem A which gives an analogous result to Kolay's Theorem for all spherical Artin groups. The techniques used in proving Theorem A are similar in spirit to [BKP19, KP20], but the implementation and specific tricks we require are rather different. We also highlight that many of our results in the body of the paper give much more structural information about quotients of spherical Artin groups than the theorem below (we refer the reader to Section 1.A for more information).

Theorem A. *Let G be an irreducible spherical Artin group. The following conclusions hold:*

1. *If G is isomorphic to $\mathcal{A}_{n-1}$, $\mathcal{B}_n$, or $\mathcal{D}_n$, for $n \geqslant 5$, then the smallest non-abelian quotient of G is* $\mathrm{Sym}(n)$ *and it is unique up to automorphisms of the image. ($\mathcal{A}_{n-1}$ is due to Kolay* [Kol23]*)*
2. *If G is isomorphic to one of $\mathcal{A}_2$ $\mathcal{A}_3$, $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{D}_4$, $\mathcal{F}_4$, or $\mathcal{I}_2(m)$ with $4|m$, then the smallest non-abelian quotient of G is* $\mathrm{Sym}(3)$.
3. *If G is isomorphic to $\mathcal{I}_2(m)$ with $4 \nmid m$ then the smallest non-abelian quotient of G is the dihedral group Dh_p, where p is the smallest odd prime divisor of m, and it is unique up to automorphisms of the image.*
4. *If G is isomorphic to $\mathcal{H}_3$, then the smallest non-abelian quotient of G is* $\mathrm{Alt}(5)$*. Up to automorphisms of the image, there are two such homomorphisms.*

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(5) *If G is isomorphic to $\mathcal{H}_4$, $\mathcal{E}_6$, $\mathcal{E}_7$, or $\mathcal{E}_8$, then the smallest non-abelian quotient of G is isomorphic to $W(G)/Z(W(G))$ and is unique up to automorphisms of the image.*

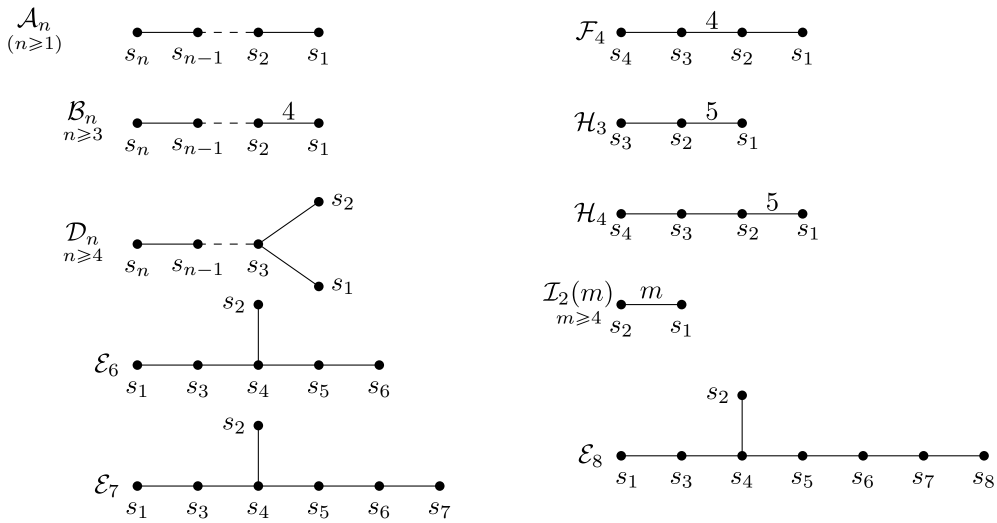

FIGURE 1. Irreducible Coxeter graphs of spherical type X_n, where X_n has n vertices.

The *affine Artin groups*, i.e., Artin groups where the corresponding Coxeter group is affine, are much more mysterious. The Coxeter diagrams of affine Coxeter groups were classified by Coxeter [Cox34] and are given in Figure 2. The $K(\pi, 1)$ conjecture for these groups was resolved recently [PS21]. It appears to be unknown if every affine Artin group is linear or even residually finite.

Theorem B. *Let G be an irreducible affine Artin group. The following conclusions hold:*

(1) *If G is isomorphic to $\widetilde{\mathcal{A}}_{n-1}$, $\widetilde{\mathcal{B}}_n$, $\widetilde{\mathcal{C}}_n$ or $\widetilde{\mathcal{D}}_n$, for $n \geqslant 5$, then the smallest non-abelian quotient of G is* $\mathrm{Sym}(n)$.

(2) *If G is isomorphic to one of $\widetilde{\mathcal{A}}_1$, $\widetilde{\mathcal{A}}_2$ $\widetilde{\mathcal{A}}_3$, $\widetilde{\mathcal{B}}_2$, $\widetilde{\mathcal{B}}_3$, $\widetilde{\mathcal{B}}_4$, $\widetilde{\mathcal{D}}_4$, $\widetilde{\mathcal{F}}_4$, or $\widetilde{\mathcal{G}}_2$, then the smallest non-abelian quotient of G is* $\mathrm{Sym}(3)$.

(3) *If G is isomorphic to $\widetilde{\mathcal{E}}_n$ for $n = 6, 7, 8$ then the smallest non-abelian quotient of G is isomorphic to $W(\mathcal{E}_n)/Z(W(\mathcal{E}_n))$.*

Our result thus far suggest the following natural conjecture:

Conjecture 1.1. *The smallest non-abelian quotient of an Artin group A_Γ is isomorphic to the smallest non-abelian quotient of the Coxeter group W_Γ.*

Profinite rigidity, the study of groups via their finite quotients, has attracted a lot of interest in recent years. See, for example, the ICM lecture notes of Alan Reid [Rei18] or the book of Wilkes [Wil24]. One of the main goals is to distinguish finitely generated residually finite groups up to isomorphism by their finite quotients. There are only a few examples where such a result has been achieved: some hyperbolic 3-manifold groups [BMRS20], some hyperbolic triangle groups [BMRS21], crystallographic groups in dimension at most 4 [PPW21], affine Coxeter groups [CHMV25, PS24], and certain lamplighter groups [Bla25, NW26]. There are also many negative results, see for example [BG04, Pyb04, Bri16, Bri24, CHMV25]. The relative problem of determining certain groups up to isomorphism within a fixed class of groups is much more widely studied, with most attention being paid to geometrically meaningful groups such as 3-manifold groups [BRW17, WZ17, Wil17, WZ19, Liu23, Xu25, Xu26], free-by-cyclic groups [HK25, BP25, AHLP25, NB26b], Coxeter groups [CHMV25, CHMV26, HMV26], and right-angled Artin groups [KW16, CHMV26, Liu26]. Both the braid group [NB26a] and certain right-angled Artin groups [CHMV26] are not profinitely rigid amongst all finitely generated residually finite groups.

Beyond the intrinsic appeal of profinite rigidity theorems, the profinite completions of Artin groups and specifically of braid groups are closely related to Grothendieck's vision of anabelian geometry [Gro97] and Drinfeld's construction of the group Grothendieck–Teichmüller group $\widehat{GT}$ [Dri91]. The latter group

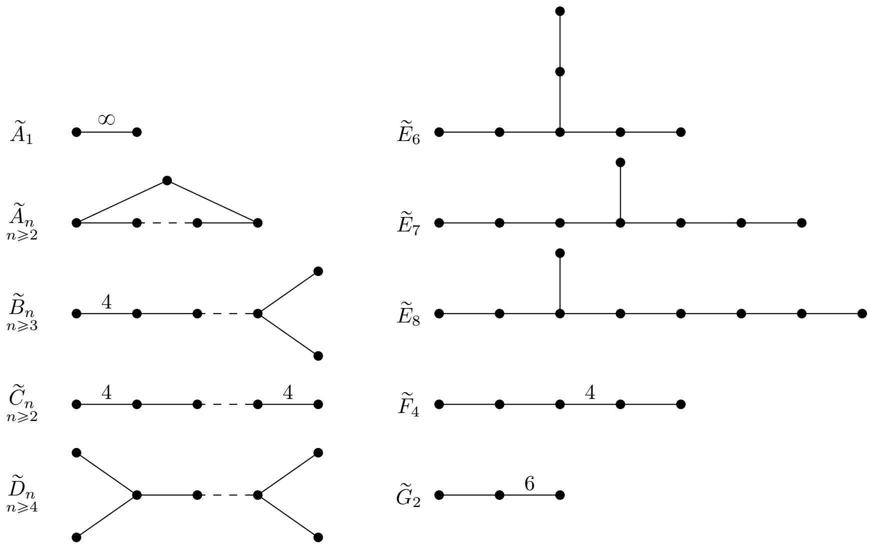

FIGURE 2. Coxeter graphs of affine type $\widetilde{X}_n$, where $\widetilde{X}_n$ has $n+1$ vertices.

is intimately related with the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and conjecturally isomorphic to it. In [MN22, Theorem B], it is shown for $n \geqslant 4$ that $\mathrm{Aut}(\widehat{\mathcal{A}}_n/Z(\widehat{\mathcal{A}}_n))$ is isomorphic to $\widehat{GT}$. For more information and recent developments we refer the reader to the surveys [Sch97, Loc12]. We remark that in [Mar12, Question 3] it was asked whether the braid groups of complex reflection groups, a class which includes spherical Artin groups, can be distinguished by their profinite completions. Our final result makes some new progress in this direction. In the sequel we will denote by $\widehat{G}$ the profinite completion of a group G.

Theorem C. *Let G and H be irreducible spherical Artin groups. If $\widehat{G} \cong \widehat{H}$, then $G \cong H$.*

We would like to recognize the concurrent and independent work *Smallest quotients and profinite rigidity of irreducible spherical type Artin groups* by Gavazzi–Haladijan–Paris who proved versions of Theorem A and Theorem C. Our approaches for Theorem A in the cases of $\mathcal{B}_n, \mathcal{D}_n, \mathcal{F}_4$ and $\mathcal{I}_2(m)$ are similar. Our approaches to Theorem A for the cases of $\mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8, \mathcal{H}_3$ and $\mathcal{H}_4$ are very different. Our approaches for Theorem C share the same broad strokes but the specific implementations are different.

1.A. **Proof strategy.** We first describe the proof strategy of Theorem A in the infinite families $\mathcal{B}_n$, $\mathcal{D}_n$ and $\mathcal{I}_2(n)$. The case of $\mathcal{B}_n$ and $\mathcal{D}_n$ are applications of Kolay's Theorem [Kol23] and some general facts about spherical Artin groups. The remaining case of $\mathcal{I}_2(n)$ involves consideration of the centre and the Orbit-Stabiliser Theorem.

The proof strategy of Theorem A in the case of $\mathcal{E}_n$ for $n = 6, 7, 8$ is very different. The first key observation is that the commutator subgroup of $\mathcal{E}_n'$ is perfect [Zin75] so every finite quotient is perfect-by-abelian. Using this and facts about the socle of a finite group we constrain the shapes of possible finite quotient groups that can occur (Lemma 2.4). Combining this structure description with the classification of finite simple groups of order less than $|W(\mathcal{E}_n)/Z(\mathcal{E}_n)|$ gives a list of candidate groups to rule out. In the case of $\mathcal{E}_8$, there are around 200 simple groups, the majority of which are isomorphic to $\mathrm{PSL}_2(q)$ for some natural number q.

The second key observation is that we can analyse the structure of totally symmetric sets in $\mathcal{E}_n$ and $\mathcal{E}_n'$; see Lemma 4.6 and Lemma 4.9. This analysis allows us to exclude all quotients with socle isomorphic to a product of $\mathrm{PSL}_2(q)$ for $\mathcal{E}_7'$, $\mathcal{E}_7$, and $\mathcal{E}_8$. We also introduce a new inductive method for excluding quotients isomorphic to subgroups of $\mathrm{GL}_n(q)$ using work of De Franceschi–Liebeck–O'Brien [DFLO25]; see Corollary 6.4.

The third key observation is that in a finite quotient there must be elements of certain orders; see Lemmas 4.16 and Lemma 4.17. The proof of this uses work of Soroko [Sor21], who identified generators of these torsion elements in $\mathcal{E}_n/Z(\mathcal{E})$. Our proof is essentially a direct computation but we introduce some new notation for working with equations in Artin groups which greatly simplifies the exposition.

With these reductions in hand, we are left with a tractable list of finite groups. Our general strategy is then to look at possible images for the generator s_n of $\mathcal{E}_n$ and inductively study their centralisers. In

$\mathcal{E}_6$, this amounts to the theorem of Kolay [Kol23] since $\mathcal{A}_4 \cong \langle s_1, s_3, s_2, s_4 \rangle \leqslant \mathcal{E}_6$ centralises s_6. In $\mathcal{E}_8$, we apply our results for $\mathcal{E}_6$, since $\mathcal{E}_6 \cong \langle s_1, s_2, s_3, s_4, s_5, s_6 \rangle$ centralizes s_8.

The verification for $\mathcal{H}_3$ is by direct computation. The strategy for $\mathcal{H}_4$ is similar to $\mathcal{E}_n$ but considerably easier. The remaining cases all admit Sym(3) as a quotient, so there is nothing to check.

1.B. **Paper outline.** In Section 2, we recall some notions and notation from finite group theory, prove Lemma 2.4 about the shapes of quotients, recall some generalities about totally symmetric sets, and state some results about parabolic subgroups in Artin groups.

In Section 3, we handle the cases of $\mathcal{B}_n$, $\mathcal{D}_n$, and $\mathcal{I}_2(n)$.

In Section 4, we recall a number of properties of spherical Artin groups. We construct totally symmetric sets in $\mathcal{E}_n$ and $\mathcal{E}'_n$ for all n; see Lemma 4.6. We prove a general result about collapsing sets in Artin groups; see Lemma 4.11. We also state the element order lemmas mentioned in Section 1.A; the computations which comprise the majority of their proofs are provided in Appendix A.

In Section 5 we handle the cases of $\mathcal{F}_4$, $\mathcal{H}_3$, and $\mathcal{H}_4$. In Section 6 we handle the case of $\mathcal{E}_6$. In Section 7 we handle the case of $\mathcal{E}_7$. In Section 8 we handle the case of $\mathcal{E}_8$. In Section 9 we prove Theorem A. In Theorem B we prove Theorem B. In Section 11 we prove Theorem C.

Contents

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2. Preliminaries

2.A. **Finite groups.** We first establish some standard notation from finite group theory.

Notation 2.1. A finite cyclic group of order k is denoted by k. A direct product of l copies of the cyclic group of order k is denoted by k^l.

A short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

of finite groups with unspecified extension will be denoted $N.Q$. If the extension is split, we write $N : Q$. If N is abelian and the extension is central, we write $N \cdot Q$. The wreath product $N \wr Q$ with head Q and base N is the semidirect product $\left(\prod_Q N\right) : Q$ where Q acts by permuting the copies of N.

Specific finite groups that will show up repeatedly throughout this article include $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$, the symmetric and alternating groups on n objects respectively, and Dh_n, the dihedral group of order $2n$.

2.A.1. *Groups of Lie type.* Let p be a prime, let $q = p^k$, and let $\mathbb{F}$ be an algebraically closed field of characteristic p. The simple algebraic groups are classified by the Dynkin diagrams $\mathcal{A}_n$ for $n \geqslant 1$, $\mathcal{B}_n$ for $n \geqslant 3$, $\mathcal{C}_n$ for $n \geqslant 2$, $\mathcal{D}_n$ for $n \geqslant 4$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$, $\mathcal{F}_4$, and $\mathcal{G}_2$. To each Dynkin diagram, we associate a finite number of simple algebraic groups; two such groups associated to the same diagram are called *versions*. All versions become isomorphic upon taking their quotients by their respective centres. There is a version (the *simply-connected* one) which maps onto every other version with a finite central kernel and there is a version (the *adjoint* one) which is a quotient of every other version with a finite central kernel.

Any *finite group of Lie type* G is obtained as the fixed point set of a Steinberg endomorphism of a connected simple algebraic group $\mathbf{G}$ defined over an algebraically closed field of characteristic p. The groups of Lie type fall into two families: the types $\mathtt{A}_n$, ${}^2\mathtt{A}_n$, $\mathtt{B}_n$, $\mathtt{C}_n$, $\mathtt{D}_n$, ${}^2\mathtt{D}_n$ are the *classical types*; the types ${}^2\mathtt{B}_2$, ${}^3\mathtt{D}_4$, $\mathtt{E}_6$, ${}^2\mathtt{E}_6$, $\mathtt{E}_7$. $\mathtt{E}_8$, $\mathtt{F}_4$, ${}^2\mathtt{F}_4$, $\mathtt{G}_2$, and ${}^2\mathtt{G}_2$ are the *exceptional types.* Note that groups of type ${}^2\mathtt{B}_2$ and ${}^2\mathtt{F}_4$ are only defined over fields of order 2^{2m+1} and groups of type ${}^2\mathtt{G}_2$ are defined over fields of order 3^{2m+1}.

Each type of group has a finite number of *versions*, the smallest one is again called the *adjoint.* The adjoint version is a simple group with the following exceptions:

$$\begin{array}{cccc} \mathtt{A}_1(2) \cong \mathrm{Sym}(3), & \mathtt{A}_1(3) \cong \mathrm{Alt}(4), & \mathtt{C}_2(2) \cong \mathrm{Sym}(6), & {}^2\mathtt{A}_2(2) \cong 3^2 : Q_8 \\ \mathtt{G}_2(2) \cong^2 \mathtt{A}_2(3) : 2, & {}^2\mathtt{B}_2(2) \cong 5 : 4, & {}^2\mathtt{G}_2(3) \cong \mathtt{A}_1(8) : 3, & \text{and } {}^2\mathtt{F}_4(2), \end{array}$$

where Q_8 is the quaternion group, and the commutator subgroup of ${}^2\mathtt{F}_4(2)$, which we denote by ${}^2\mathtt{F}_4'(2)$ is a simple group known as the *Tits group.* As is often the convention, we will treat ${}^2\mathtt{F}_4'(2)$ as an exceptional group of Lie type. We also record a number of exceptional isomorphisms between the groups: $\mathtt{A}_1(4) \cong \mathtt{A}_1(5) \cong \mathrm{Alt}(5)$, $\mathtt{A}_1(7) \cong \mathtt{A}_2(2)$, $\mathtt{A}_1(9) \cong \mathrm{Alt}(6)$, $\mathtt{A}_3(2) \cong \mathrm{Alt}(8)$, ${}^2\mathtt{A}_3(2) \cong \mathtt{C}_2(3)$, and $\mathtt{B}_n(2^\ell) \cong \mathtt{C}_n(2^\ell)$ for all $\ell \geqslant 1$ and $n \geqslant 3$.

The adjoint classical groups admit the following descriptions: $\mathtt{A}_n(q) \cong \mathrm{PSL}_{n+1}(q)$, $\mathtt{B}_n(q) \cong \mathrm{P}\Omega_{2n+1}(q)$, $\mathtt{C}_n(q) \cong \mathrm{PSp}_{2n}(q)$, $\mathtt{D}_n(q) \cong \mathrm{P}\Omega^+_{2n}(q)$, ${}^2\mathtt{A}_n(q^2) \cong \mathrm{PSU}_{n+1}(q)$, and ${}^2\mathtt{D}_n(q^2) \cong \mathrm{P}\Omega^-_{2n}(q)$. When working with a specific group, we will tend to use the right-hand-side notation, e.g, $\mathrm{PSL}_{n+1}(q)$. However, when referring to a family of groups, we will tend to use the Chevalley notation, e.g., $\mathtt{A}_n(q)$.

2.A.2. *The classification of finite simple groups.*

Theorem 2.2. *Every finite simple group is, up to isomorphism, one of the following groups:*

- *a cyclic group of prime order;*
- *an alternating group of degree at least 5;*
- *a finite simple group of Lie type;*
- *one the 26 sporadic groups.*

Note here we view the Tits group ${}^2\mathtt{F}_4'(2)$ *as a group of Lie type.*

Of the 26 sporadic groups, the relevant ones for our purposes will be the Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, and M_{24}, the Janko groups J_1, J_2, and J_3, and the Higman–Sims group HS. Constructions of the sporadic groups as well as all of the relevant facts we will need are contained in the 'ATLAS of finite simple groups' [CCN+85]. From here on we will often refer to [CCN+85] as *the ATLAS.*

2.A.3. *The socle.* A key tool in our studies will be an important characteristic subgroup of a group G.

The *socle*, denoted $\mathrm{Soc}(G)$, of a group G is the subgroup generated by all minimal normal subgroups of G. We briefly recall a number of elementary facts, see [DM96, Theorem 4.3A]. If the group G is finite, then the socle is isomorphic to the direct product of the minimal normal subgroups of G. Moreover, the socle of a finite group G is isomorphic to a direct product of finite simple groups and a finite abelian group.

2.B. **The shape of a smallest quotient of perfect-by-cyclic group.** The following lemma describes the structure of a smallest non-cyclic quotient of a perfect-by-cyclic group. This will be of central importance in our study of the exceptional Artin groups. We begin with a definition.

Definition 2.3. A group G is *almost simple* if there is a non-abelian simple group S such that $S \leqslant A \leqslant \mathrm{Aut}(S)$.

Lemma 2.4. *If G is a group such that G^{ab} is cyclic and $[G,G]$ is perfect, then a smallest non-cyclic quotient Q of G fits into an exact sequence $1 \to S^n \to Q \to k \to 1$ where S is a finite simple group and k is cyclic. Moreover, $\mathrm{Soc}(Q) = S^n$ and*

(1) *if $n = 1$, then Q is almost simple;*
(2) *if $n > 1$, then k divides $|\mathrm{Out}(S^n)|$ and G is a subgroup of $S^n \wr k$.*

Proof. Any finite quotient Q of G fits into an exact sequence $1 \to N \to Q \to C_k \to 1$ with N perfect. The result follows if we can show that $\mathrm{Soc}(Q) = N$ and that N is a product of simple groups. If $\mathrm{Soc}(Q)$ is not equal to N, then there is a normal subgroup $L \lhd Q$ such that the quotient Q/L is non-abelian, contradicting minimality of Q. Thus, we may suppose $\mathrm{Soc}(N) = Q$. The socle of any finite group is the product of finitely many simple groups and an abelian group. Indeed, a minimal normal subgroup is characteristically simple and so since Q is finite any minimal normal subgroup is a direct product of isomorphic simple groups.

If Q had a non-trivial abelian normal subgroup A, we would be able to take a further smaller quotient Q/A. Thus $\mathrm{Soc}(Q) = \prod_{i=1}^{\ell} S_i$ with each S_i simple. Next, we show that each of the S_i are isomorphic. Indeed, if they were not, then we would have a proper subset I of $[\ell]$ such that $L = \prod_{i \in I} S_i$ is a normal subgroup of Q with Q/L non-abelian. This contradicts minimality of Q. Thus, $S_i \cong S_j \cong S$ for all i, j and $\mathrm{Soc}(Q) = S^n$ as required.

It follows immediately that if $n = 1$, then Q is almost simple. Otherwise, Q would admit a further non-abelian quotient with kernel isomorphic to the kernel of map $k \to \mathrm{Out}(S^n)$. If $n > 1$, then by the Kaloujinine–Krasner Universal Embedding Theorem [KK51], every extension of S^n and k is a subgroup of the wreath product $S^n \wr k$. Clearly $n|k$; otherwise, k would fix some copies of S. It remains to show that k divides $|\mathrm{Out}(S^n)|$. Note that $\mathrm{Sym}(n) \leqslant \mathrm{Out}(S^n)$, so in particular, n divides $|\mathrm{Out}(S^n)|$. Suppose that k does not divide n. The kernel of the homomorphism $k \to \mathrm{Out}(S^n)$ must be trivial because if it were not, Q would admit non-trivial central elements and hence a smaller non-abelian quotient. Hence, k divides $|\mathrm{Out}(S^n)|$ as required. ☐

2.C. **Bounding the of size non-cyclic quotients using totally symmetric sets.** In this section, we introduce one of the tools we will use to bound the size of non-cyclic quotients. For a group G, a subset $T = \{t_1, t_2, \dots\} \subseteq G$ is called a *totally symmetric set* (TSS) if for any permutation $\sigma \in \mathrm{Sym}(T)$ there exists $g_\sigma \in G$ such that $g_\sigma t_i g_\sigma^{-1} = t_{\sigma(i)}$. This notion was first introduced by Kordek and Margalit [KM22] who worked under an additional pairwise commuting hypothesis. This definition was later generalized by Caplinger–Salter to the version which appears in this paper; we refer to totally symmetric sets consisting of pairwise commuting elements as *commuting totally symmetric sets*. One can immediately observe that homomorphic images of totally symmetric sets remain totally symmetric.

A key property of TSS's is the following "collision implies collapse" result. A version of this result was first proven by Kordek–Margalit with the additional pairwise commuting hypothesis [KM22, Lemma 2.1], and the version which appears in this paper was proven by Caplinger [Cap23, Lemma 4]:

Lemma 2.5. *Let G be a group and $T \subseteq G$ be any totally symmetric set. For any homomorphism $f : G \to H$, either f restricts to an injection on T or $|f(T)| = 1$.*

This "collision implies collapse" property is weaker than the property of being a totally symmetric set, and we will sometimes require only that a given set satisfies the former.

Definition 2.6. Let G be a group. A subset X is a *collapsing set* if for any quotient map $\varphi : G \to Q$, either φ restricts to an injection on X or $|\varphi(X)| = 1$.

The following is a convenient rephrasing of [KM22, Lemma 2.2] in light of Lemma 2.5.

Lemma 2.7. *The standard generators of braid groups $\mathcal{A}_n$ for $n \geqslant 4$ form a collapsing set.*

2.D. **Artin groups.** One obtains a presentation for an Artin group A_Γ from a labelled defining graph Γ as follows. For each vertex v_i of Γ, let s_i be a generator. For each pair i and j, m_{ij} is the label of the edge connecting vertices v_i and v_j in Γ, if it exists. If v_i and v_j are connected via an unlabelled edge, $m_{ij} = 3$. If v_i and v_j are not adjacent, $m_{ij} = 2$. The group is then given via the relations below.

$$A_\Gamma = \langle s_1, ... s_n | \underbrace{s_i s_j s_i ...}_{m_{ij}} = \underbrace{s_j s_i s_j ...}_{m_{ij}} \rangle$$

Relations of the type in the presentation above will often be referred to as *Artin relations.* We will use the notation $[x, y]_\ell$ to denote the alternating word $xyxy\ldots$ of length ℓ. So, an Artin relation can be rewritten as $[x, y]_\ell = [y, x]_\ell$ where $m_{i,j} = m_{j,i} = \ell$.

In general, Artin groups may have generators which do not admit any relation, but no such groups will arise in this paper. An Artin group is called *reducible* if it can be decomposed as the direct product of two Artin groups where neither factor is trivial. The irreducible finite-type Artin groups are exactly those Artin groups whose defining graphs appear in Figure 1 [Cox34].

Throughout the paper, we will use the labellings of Artin generators given in Figure 1 unless otherwise specified.

2.D.1. *Parabolic subgroups control finite quotients.* In addition to totally symmetric sets, parity of the length of Artin relations furnish additional means of bounding finite quotients. We record the following elementary facts:

A first fact is that pairs of generators with an odd relation are conjugate. The next fact, recorded in the following lemma, is a straightforward application of Bezout's identity.

Lemma 2.8. *Let x and y be elements of any group G. If x and y satisfy Artin relations of length m and n then they satisfy the Artin relation of length* $\gcd(m, n)$.

Proof. By Bezout's indentity there are integers a and b such that $k = \gcd(m, n) = am + bn$. Up to reordering, assume that $a > 0 > b$. Let $[x, y]_\ell$ denote the alternating word $xyxy\ldots$ of length ℓ. It is easy to see that Artin relations of a fixed length imply those for all multiples and that:

$$[x,y]_{bn}^{-1}[x,y]_{am} = \begin{cases} [x,y]_k & \text{if } bm \text{ is odd} \\ [y,x]_k & \text{if } bm \text{ is even} \end{cases} \qquad [y,x]_{bn}^{-1}[y,x]_{am} = \begin{cases} [y,x]_k & \text{if } bm \text{ is odd} \\ [x,y]_k & \text{if } bm \text{ is even.} \end{cases}$$

An immediate consequence is that $[x, y]_k = [y, x]_k$. □

Lemma 2.9. *Suppose that A_Γ is an Artin group and x, y are a pair of standard generators whose corresponding vertices in the defining graph are joined by an edge labelled m. If d is any divisor of m then there exists a quotient $A_\Gamma \to A_{\overline{\Gamma}}$ where $\overline{\Gamma}$ is obtained by replacing the edge labelled by m with an edge labelled by d.*

Proof. This can be seen immediately from the presentation. By adding the relation $[x, y]_d$ the prior relation $[x, y]_m$ becomes redundant using the same reasoning in Lemma 2.8. □

Recall that if Γ is the defining graph for an Artin group A_Γ and Λ is a proper subgraph of Γ, then the inclusion $\Lambda \subset \Gamma$ induces an embedding of A_Λ whose image is called a *standard parabolic subgroup* of A_Γ. The third fact allows us to show that in many cases where the standard parabolic subgroup of an Artin group is a braid group, one can find a retraction map from an Artin group to this braid group.

Lemma 2.10. *Let A_Γ be an Artin group and suppose that the standard parabolic subgroup corresponding to $\Lambda \subset \Gamma$ is a braid group. If every $x \in \Gamma \smallsetminus \Lambda$ has even labelled relations with all generators in Λ, then there is a retraction $A_\Gamma \twoheadrightarrow A_\Lambda$.*

Proof. It is a general fact that given a group $G = \langle S \mid R \rangle$ and another group H, a map φ which assigns an element of H to each generator $s \in S$ extends to a homomorphism $\varphi : G \to H$ if and only if $\varphi(r) = 1$ for all $r \in R$, where $\varphi(r)$ is the natural extension $\varphi(s_i) \cdots \varphi(s_n)$ when $r = s_i \cdots s_n$.

Set $\varphi(s) = s$ if $s \in V(\Lambda)$ and $\varphi(s) = 1$ if $s \in V(\Gamma) - V(\Lambda)$. Taking R to be the standard Artin relations on A_Γ, it is clear that any $r \in R$ satisfies the above condition if the two Artin generators which appear in it are either both in Λ or both in $\Gamma - \Lambda$. Any Artin relation involving $s \in \Lambda$ and $x \in \Gamma - \Lambda$ must be even, so the relation r is of the form $(sx)^{m/2}(s^{-1}x^{-1})^{m/2}$. Applying φ, we obtain precisely $\varphi(s)^{m/2}\varphi(s)^{-m/2} = 1$. Thus the map φ extends to a homomorphism $A_\Gamma \to A_\Lambda$ in which every generator of $A_\Lambda \leqslant A_\Gamma$ is fixed. □

3. Smallest quotients of non-exceptional Artin groups

In this subsection, we will determine the smallest quotients of the non-exceptional Artin groups $\mathcal{A}_n, \mathcal{I}_2(n), \mathcal{B}_n$ and $\mathcal{D}_n$, in this order.

3.A. Smallest quotient of $\mathcal{A}_n$.

Since $\mathcal{A}_n \cong Br_{n+1}$ this case is resolved by Kolay.

Theorem 3.1. [Kol23, Claim 7 and Theorem 1] *The smallest non-abelian quotient of $\mathcal{A}_2 \cong Br_3$ is $Sym(3)$.*

The smallest non-abelian quotient of $\mathcal{A}_n \cong Br_{n+1}$ is $\mathrm{Sym}(n+1)$. *Moreover, the quotient map is unique up to an automorphism of* $\mathrm{Sym}(n+1)$.

Remark 3.2. The group $\mathcal{A}_3 \cong Br_4$ admits a quotient to $\mathrm{Sym}(3)$ because its associated Coxeter group is $\mathrm{Sym}(4)$, which has an exceptional homomorphism to $\mathrm{Sym}(3)$.

When $n = 4$, one can also classify the homomorphisms from $\mathcal{A}_4$ to $\mathrm{Sym}(n+2)$. This result is used in later sections to examine the possible images of $\mathcal{A}_4$ parabolic subgroups in quotients of other Artin groups.

Lemma 3.3. *Any homomorphism $\mathcal{A}_4 \to \mathrm{Sym}(6)$ must either have cyclic image or be equal to a copy of* $\mathrm{Sym}(5)$. *In the latter case, the homomorphism factors through the canonical Coxeter quotient, $W(\mathcal{A}_4)$.*

Proof. We claim any homomorphism $\alpha : \mathcal{A}_4 \to \mathrm{Sym}(6)$ must either have cyclic image or be equal to a copy of $\mathrm{Sym}(5)$. If the generator $\alpha(s_4)$ is trivial, then all of $\alpha(\mathcal{A}_4)$ is trivial because all Artin generators are conjugate. If $\alpha(s_4)$ has order 2, then all Artin generators must have order 2 for the same reason. In this case, α factors through the canonical Coxeter quotient $W(\mathcal{A}_4)$, so $\alpha(\mathcal{A}_4)$ must be a subgroup of $W(\mathcal{A}_4) = \mathrm{Sym}(5)$. The image cannot be a proper subgroup by Theorem 3.1.

Suppose the order of $\alpha(s_4)$ is larger than 2, and let $\mathcal{A}_2$ denote the standard parabolic subgroup $\langle s_1, s_2\rangle$ which commutes with s_4. We examine the possible images of $\alpha(s_4)$. A non-trivial permutation in $\mathrm{Sym}(6)$ with order not equal to 2 has one of the following 7 cycle types: $2^1 3^1$, $2^1 4^1$, 3^1, 3^2, 4^1, 5^1, 6^1. The centraliser of such an element is isomorphic to either 6, 2×4, $3 \times \mathrm{Sym}(3)$, $3 \wr 2$, 2×4, 5, and 6 respectively, written in the finite groups notation given at the beginning of Section 2.A.

If the image $\alpha(\mathcal{A}_2)$ is cyclic, then $\alpha(\mathcal{A}_4)$ must also be cyclic by Lemma 2.7, so it remains only to consider the possibility that the centraliser of $\alpha(s_4)$ is $\mathrm{Sym}(3)$ or a subgroup of $3 \wr 2$. If $\mathcal{A}_2$ maps to $\mathrm{Sym}(3)$, then this factors through the canonical Coxeter quotient $\mathcal{A}_2 \to \mathrm{Sym}(3)$. This forces all Artin generators to have order 2, so in fact, α itself factors through the canonical Coxeter quotient $W(\mathcal{A}_4)$.

The group $3 \wr 2$ is isomorphic $3 \times \mathrm{Sym}(3)$. If the image $\alpha(\mathcal{A}_2)$ is abelian, then $\alpha(s_1)$ and $\alpha(s_2)$ commute in the quotient, but they also satisfy a braid relation, so $\alpha(s_1) = \alpha(s_2)$ by Lemma 2.8. The entire image $\alpha(\mathcal{A}_4)$ is then cyclic by Lemma 2.7.

If $\alpha(\mathcal{A}_2)$ is a non-abelian subgroup of $3 \times \mathrm{Sym}(3)$, then in fact $\alpha(\mathcal{A}_2) \cong \mathrm{Sym}(3)$. Indeed, the previous paragraph shows that $\alpha(s_1)$ and $\alpha(s_2)$ must be in the same factor of the direct product if $\alpha(\mathcal{A}_2)$ is non-cyclic, and these elements generate $\alpha(\mathcal{A}_2)$. As before, $\alpha(\mathcal{A}_2)$ must factor though the canonical Coxeter quotient by Theorem 3.1, implying that $\alpha(\mathcal{A}_4)$ also factors through $W(\mathcal{A}_4)$. □

We will bootstrap off the control of finite quotients obtained in Theorem 3.1 to study quotients of other Artin groups using the following results.

Lemma 3.4. *Let $n \geqslant 2$. If $\mathcal{A}_n \to C$ is any map to an abelian group then the generators collapse*

Proof. This is immediate from Lemma 2.7 and Lemma 2.8 since $\gcd(2,3) = 1$. □

Proposition 3.5. *Let A_Γ be an Artin group with A_Λ an (irreducible) standard parabolic subgroup isomorphic to $\mathcal{A}_n = Br_{n+1}$ for $n \geqslant 4$.*
If $\varphi : A_\Gamma \to Q$ is any quotient onto a finite group whose restriction to A_Λ is noncyclic then

$$|Q| \geqslant n!,$$

Moreover if there is a generator $t \in \Gamma$ such that $\varphi(t) \notin \varphi(A_\Lambda)$ then the inequality is strict.

Proof. Consider the restriction of φ to the parabolic braid subgroup. By [Kol23], the order $|\varphi(H)| \geqslant n!$, so $|Q| \geqslant |\varphi(H)|$. The moreover statement follows by the Orbit-Stabiliser Theorem applied to cosets of $\varphi(H)$ by powers of $\varphi(t)$. □

3.B. **Smallest quotient of $\mathcal{I}_2(n)$.** We begin this subsection by proving that when n is divisible by 4, then $\mathcal{I}_2(n)$ has an exceptional quotient that does not factor through the associated Coxeter group, including the case $\mathcal{I}_2(4) = \mathcal{B}_2$.

Lemma 3.6. *Suppose that n is divisible by 4. The smallest nonabelian quotient of $\mathcal{I}_2(n)$ is* Sym(3).

Proof. By Lemma 2.9, there exists a quotient $\mathcal{I}_2(n) \to \mathcal{I}_2(4)$. Consider the presentation $I_2(4) = \langle x, y \mid (xy)^2 = (yx)^2 \rangle$. The assignment $x \mapsto (123)$ and $y \mapsto (23)$ defines the desired homomorphism. To see this, it suffices to check that the relation is satisfied. Observe that $xy \mapsto (12)$ and $yx \mapsto (13)$ both map to involutions. Thus, the relation $(xy)^2 = (yx)^2$ is also satisfied in Sym(3).

The smallest nonabelian finite group is $Sym(3)$, so it must be the smallest nonabelian quotient of $\mathcal{I}_2(n)$. □

Before considering values of $n \neq 4$, we note the following simple fact which will be used to see how the centre of $\mathcal{I}_2(n)$ behaves under quotients.

Lemma 3.7. *A central extension of a cyclic group is abelian.*

Proof. Suppose that

$$1 \to A \to G \to C \to 1,$$

is a short exact sequence where A is central in G and C is cyclic. Lift a generator of C to $z \in G$. Every pair of elements $g, h \in G$ can thus be rewritten as $g = az^k$ and $h = bz^j$ where $a, b \in A$. Since A is central we see that $gh = abz^{j+k} = hg$. □

Recall that if x, y are the stardard Artin generators of $\mathcal{I}_2(n)$, then the centre is generated by either $[x, y]_n = (xy)^{\frac{n}{2}}$ when n is even or $[x, y]_n^2 = ((xy)^{\frac{n-1}{2}}x)^2$ when n is odd [BS72]. In either case, these elements have a nontrivial root involving the element xy, which gives rise to convenient alternate presentations.

Lemma 3.8. *The Artin group $\mathcal{I}_2(n)$ admits one of the following presentations:*

$$\langle x, r \mid x^{-1}r^m x = r^m \rangle \quad \textit{for } n = 2m, \qquad \langle r, s \mid s^2 = r^n \rangle \quad \textit{for } n = 2m+1.$$

Proof. Suppose first that $n = 2m$ is even. Let $r = xy$, implying $y = x^{-1}r$. The following rewriting of the Artin relation gives the desired presentation:

$$\begin{aligned} [x, y]_{2m} &= [y, x]_{2m} \\ (xy)^m &= (yx)^m = y(xy)^{m-1}x \\ r^m &= x^{-1}r^m x. \end{aligned}$$

Now suppose $n = 2m + 1$ is odd. Let $r = xy$ and let $s = (xy)^m$. The following rewriting of the Artin relation gives the desired presentation:

$$\begin{aligned} [x, y]_{2m+1} &= [y, x]_{2m+1} \\ (xy)^m x &= yx(yx)^{m-1}y = y(xy)^m \\ r^m x &= x^{-1}r^{m+1} \\ (r^m x)(r^m x) &= r^m r^{m+1} \\ s^2 &= r^n. \end{aligned}$$

□

The presentations in Lemma 3.8 makes it clear to see that the quotient of $I_2(n)$ by its centre is virtually free which is also proved in [McC10, Theorem 6.2]. Indeed,

$$\mathcal{I}_2(n)\Big/Z(\mathcal{I}_2(n)) = \begin{cases} \langle x, r \mid r^m = 1\rangle & \text{for even } \ n, \\ \langle r, s \mid s^2 = r^m = 1\rangle & \text{for odd } \ n. \end{cases}$$

As we will see, similar analysis can be used to control the size of finite quotients.

Theorem 3.9. *If n is not divisible by 4 then the smallest nonabelian quotient of $I_2(n)$ is the dihedral group Dh_p where p is the smallest odd prime divisor of n. Otherwise the smallest nonabelian quotient is* $\mathrm{Sym}(3)$

Proof. The case when 4 divides n is handled in Lemma 3.6.

Suppose that $\varphi : \mathcal{I}_2(n) \twoheadrightarrow Q$ is any nonabelian finite quotient. Since n is not divisible by 4, we will consider the case where there exists an odd $m \geqslant 3$ such that $n = 2m$ and the case where $n = 2m + 1$ in turn. Let p be the smallest odd prime divisor of n (hence $p > 2$).

Suppose there exists an odd $m \geqslant 3$ such that $n = 2m$, that is, $n \cong 2 \pmod 4$. By Lemma 3.8, we have the following presentation:

$$\mathcal{I}_2(2m) = \langle x, r \mid xr^m = r^m x\rangle.$$

Let $\overline{r} = \varphi(r)$ and $\overline{x} = \varphi(x)$. The presentation implies that Q is a central extension of $\overline{Q} := Q\Big/\langle \overline{r}^m\rangle$. By Lemma 3.7, $\overline{Q}$ cannot be cyclic, implying that neither $\overline{r}$ nor $\overline{x}$ are trivial. Consider the proper subgroup $H = \langle \overline{r}\rangle$ of $\overline{Q}$. By construction, $\overline{r}^m = 1_{\overline{Q}}$, so $|H|$ divides m. In particular, $|H| \geqslant p$ because m is odd. Since $\overline{x} \notin H$, the cosets H and $\overline{x}H$ are disjoint. By the Orbit-Stabiliser Theorem, $|Q| \geqslant |\overline{Q}| \geqslant |H \sqcup \overline{x}H| \geqslant 2p$.

We now suppose $n = 2m + 1$ for $m \geqslant 1$. By Lemma 3.8 we have the following presentation

$$\mathcal{I}_2(2m + 1) = \langle r, s \mid s^2 = r^{2m+1}\rangle.$$

Let $\overline{r} = \varphi(r)$ and $\overline{s} = \varphi(s)$. The presentation implies that Q is a central extension of $\overline{Q} = Q\Big/\langle \overline{r}^{2m+1}\rangle$. By Lemma 3.7, $\overline{Q}$ cannot be cyclic implying that neither $\overline{r} \neq 1_{\overline{Q}} \neq \overline{s}$. Consider the proper subgroup $H = \langle \overline{r}\rangle$. By construction, $\overline{r}^{2m+1} = 1_{\overline{Q}}$, so $|H|$ divides $2m + 1$. In particular, H and $\overline{s}H$ are disjoint cosets where $|H| = |\overline{s}H| \geqslant p$. By the Orbit-Stabiliser Theorem, $|Q| \geqslant |\overline{Q}| \geqslant |H \sqcup \overline{s}H| \geqslant 2p$.

We conclude by noting that each of the previous cases are constructive. Indeed, if Q is smallest among all nonabelian finite quotients, then the subgroup H must have order $|H| = p$, otherwise the natural dihedral quotient obtained from the composition $\mathcal{I}_2(n) \twoheadrightarrow \mathcal{I}_2(p) \twoheadrightarrow Dh_p$ would be smaller. Similarly, the element $\overline{x}$ is necessarily 2-torsion. Thus, we may conclude that there is a unique quotient of $\mathcal{I}_2(2m)$ to a group of order $2p$ up to an automorphism. □

3.C. **Smallest quotient of $\mathcal{B}_n$.** Note that $\mathcal{B}_n$ is generated by a canonical maximal parabolic braid subgroup $P(\Lambda) \cong \mathcal{A}_{n-1}$ and one additional element s_1 having even Artin relations with every generator of Λ. By Lemma 2.9, there exists a retraction onto $P(\Lambda)$ coming from the first two maps in the following chain of quotients

$$\mathcal{B}_n \to \mathbb{Z} \times \mathcal{A}_{n-1} \to \mathcal{A}_{n-1}.$$

We will see that this is often the source of smallest quotient for this family.

Before we establish bounds on finite quotients we observe the following facts.

Lemma 3.10. *If $n = 2, 3, 4$, then the smallest nonabelian quotient of $\mathcal{B}_n$ is* $\mathrm{Sym}(3)$.

Proof. We handle each value of n separately. When $n = 2$, $\mathcal{B}_2 = \mathcal{I}_2(4)$ which has unique smallest quotient $\mathrm{Sym}(3)$ from Lemma 3.6. When $n = 3$, the Coxeter group $2 \wr \mathrm{Sym}(3)$ has a natural map to $\mathrm{Sym}(3)$. When $n = 4$, the parabolic subgroup $P(\Lambda) \cong \mathcal{A}_3$ admits an exotic quotients to $\mathrm{Sym}(3)$ as mentioned in Remark 3.2. □

Lemma 3.11. *Let $n \geqslant 3$. For any quotient map $\varphi : \mathcal{B}_n \to Q$, if $\varphi(P(\Lambda))$ is cyclic then Q is abelian.*

Proof. By Lemma 3.4, each of the generators $s_2, \ldots, s_n$ of $P(\Lambda)$ are mapped to a single element in Q, call it $\overline{t}$. Hence, Q is generated by $\overline{t}$ and $\overline{s} = \varphi(s_1)$. Since $n \geqslant 3$, we have relation $[\overline{s}, \overline{t}] = 1$. Thus, Q is abelian. □

Theorem 3.12. *Let $n \geqslant 5$. If $\varphi : \mathcal{B}_n \to Q$ is any quotient to a nonabelian finite group, then one of the following must hold:*

(1) The quotient map factors through the Coxeter group and $|Q| \geqslant n!$.
(2) The quotient map does not factor through the Coxeter group and $|Q| > n!$.

Proof. By Lemma 3.11 and Proposition 3.5, the image of the parabolic subgroup cannot be cyclic and has order $|Q| \geqslant n!$ with equality only if φ factors through the Coxeter quotient. In particular, if φ does not factor through the Coxeter quotient then $|Q| > n!$. □

3.D. **Smallest quotient of $\mathcal{D}_n$.** Recall that $\mathcal{D}_n$ is only defined for $n \geqslant 4$ and that the Coxeter group for $\mathcal{D}_n$ is the finite group $2^{n-1}.S_n$.

The Coxeter group $W(\mathcal{D}_4) \cong 2^3.S_4$ is isomorphic to $2^{1+4} : S_3$, so $\mathcal{D}_4$ has Sym(3) as a quotient.

Theorem 3.13. *Let $n \geqslant 5$. If $\varphi : \mathcal{D}_n \to Q$ is a nonabelian quotient map, then either $|Q| > n!$ or φ factors through the natural quotient map from $\mathcal{D}_n$ to its Coxeter group and $|Q| \geqslant n!$.*

Proof. Notice that $\mathcal{D}_n$ has two standard parabolic subgroups $P(\Gamma\smallsetminus\{s_1\})$ and $P(\Gamma\smallsetminus\{s_2\})$ both isomorphic to $\mathcal{A}_{n-1} \cong Br_n$. Applying Proposition 3.5, either $|Q| \geqslant n!$ or both braid subgroups have cyclic image in Q. The cyclic case is impossible because each of the parabolic subgroups share the generator s_n, so by Lemma 3.4 this would force Q to be cyclic contradicting it being nonabelian.

Suppose now that φ does not factor through the Coxeter group. All generators are conjugate in $\mathcal{D}_n$, so if φ does not factor through the Coxeter group on $\mathcal{D}_n$, then it also does not factor through the Coxeter group when restricted to either of them. Thus $|Q| \geqslant |\varphi(\Gamma\smallsetminus\{s_i\})| > n!$ for at least one value of $i = 1, 2$. □

The lower bound in Theorem 3.13 is realized by composing the quotient map which identifies x_n and x_{n-1} and the natural quotient map from $\mathcal{A}_{n-1}$ to S_n. One can check that the first map is indeed a homomorphism via the same method used in the proof of Lemma 2.10.

4. Properties of Artin groups

This section is devoted to structural properties of spherical Artin groups which will be used in later sections. We begin by recalling the following standard fact.

Lemma 4.1. *The abelianization of an Artin group A_Γ is free abelian, and the rank is equal to the number of connected components of the graph Γ^{odd} obtained by removing all even-labelled edges from Γ.*

In the case where Γ^{odd} is connected, this yields the following concise description of the commutator subgroup of A_Γ.

Lemma 4.2. *If $A_\Gamma^{ab} \cong \mathbb{Z}$, the map which sends each Artin generator to $1 \in \mathbb{Z}$ defines a homomorphism ℓ called the* length homomorphism*, and the commutator subgroup of A_Γ is precisely* $\ker(\ell)$.

Theorem 4.3. [Zin75] [MR06] [Ore12] *If G is isomorphic to one of $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$, $\mathcal{H}_3$, $\mathcal{H}_4$, or $\mathcal{A}_n$ with $n \geqslant 4$, then the commutator subgroup G' is perfect.*

We make the following observation, which will be used throughout.

Lemma 4.4. *If G is isomorphic to one of $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$, $\mathcal{H}_3$, $\mathcal{H}_4$, or $\mathcal{A}_n$ with $n \geqslant 4$, every homomorphism $G \to H$ where H is a solvable group must have cyclic image.*

Proof. The abelianization of G is cyclic, and the commutator subgroup of G is perfect. □

Two Artin generators are conjugate if they are connected by an odd-labelled edge path in Γ. This allows us to make the following observation.

Remark 4.5. If Γ is connected and odd labelled, then all Artin generators have the same order in any quotient Q of A_Γ. In particular, if the image of any Artin generator is trivial, Q is trivial.

4.A. **Totally symmetric sets in $\mathcal{E}_n$.** In this section, we construct totally symmetric sets in Artin groups of type $\mathcal{E}_n$ for all n. The results in this section are stronger than what is needed to determine the smallest quotients of $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{E}_8$, but they are interesting in their own right and may prove useful for readers interested in finite quotients of $\mathcal{E}_n$ for $n > 8$.

Lemma 4.6. *In an Artin group of type $\mathcal{E}_n$ where n is even, the sets*

$$\{s_1, s_4, s_{4+2}, \ldots, s_n\} \quad \textit{and} \quad \{s_2, s_3, s_{3+2}, \ldots, s_{n-1}\}$$

are commutative totally symmetric sets of size $\frac{1}{2}n$. In an Artin group of type $\mathcal{E}_n$ where n is odd, the sets

$$\{s_1, s_4, s_{4+2}, \ldots, s_{n-1}\} \quad \textit{and} \quad \{s_2, s_3, s_{3+2}, \ldots, s_{n-2}\}$$

are commutative totally symmetric sets of size $\frac{1}{2}(n-1)$.

Proof. It is immediate that all elements of the given sets pairwise commute. The sets $\{s_1, s_4, s_{4+2}, \ldots, s_n\}$ for even n and $\{s_1, s_4, s_{4+2}, \ldots, s_{n-1}\}$ for odd n are contained entirely in the standard parabolic subgroup $\langle s_1, s_3, s_4, s_5, \ldots, s_n\rangle$. This subgroup is isomorphic to an Artin group of type $\mathcal{A}_{n-1}$. It was shown in [CKLP20, KM22] that sets of this type in braid groups are commutative totally symmetric sets, and the elements of A_{n-1} which induce the desired permutations clearly still induce the desired permutations when included into $\mathcal{E}_n$.

The sets $\{s_2, s_3, s_{3+2}, \ldots, s_{n-1}\}$ for even n and $\{s_2, s_3, s_{3+2}, \ldots, s_{n-2}\}$ are not contained in a braid-type parabolic subgroup, so a slightly different strategy is required. First, observe that we can apply the analogous fact for braid groups to realize any permutation supported in $\{s_{3+2}, \ldots, s_{n-1}\}$ or $\{s_{3+2}, \ldots, s_{n-2}\}$ as conjugation *by an element of the parabolic subgroup* $\langle s_{3+2}, \ldots, s_n\rangle$. This subgroup commutes with both s_2 and s_3, so this element induces the desired permutation on the entire proposed set. Since the symmetric group is generated by the adjacent transpositions, it now suffices to show that we can realize the transpositions (s_2, s_5) and (s_2, s_3).

We begin with the transposition (s_2, s_5). Notice that there is a standard parabolic subgroup of type D_5, $\langle s_1, s_2, s_3, s_4, s_5\rangle$ which commutes with $\{s_7, \ldots, s_n\}$. Let $\Delta^L_{D_5}$ denote the inclusion into A_Γ of the Garside element of the Artin group D_5 with the obvious generating set. Conjugation by $\Delta^L_{D_5}$ permutes s_2 and s_5 while fixing all other Artin generators in the D_5 subgroup, including s_3, and while fixing anything which commutes with the subgroup, including $\{s_{3+4}, \ldots\}$.

A similar argument holds for any pair of Artin generators which can be realized as the "prong" vertices of a parabolic subgroup of type D_{2k+1} for any k. In particular, vertices s_2 and s_3 form the "prongs" of the type D_{n-1} subgroup $\langle s_2, s_3, s_4, \ldots, s_n\rangle$ when n is even, or of the type D_{n-2} subgroup $\langle s_2, s_3, s_4, \ldots, s_{n-1}\rangle$ when n is odd. In either case, this subgroup contains all of the generators in the proposed totally symmetric set, so conjugation by the inclusion of the Garside element permutes s_2 and s_3 while leaving the remainder of the set fixed. In particular, this element commutes with the entire braid-type parabolic subgroup $\langle s_{3+2}, \ldots\rangle$. ☐

In the $\mathcal{E}_7$ case, we also make use of totally symmetric sets in the commutator subgroup $\mathcal{E}_7'$. The following was proven by Zinde [Zin75], see also [MR06, Ore12].

Theorem 4.7. *For $n = 6, 7,$ or 8 the commutator subgroup $\mathcal{E}_n'$ of the Artin group $\mathcal{E}_n$ is a finitely presented perfect group. $\mathcal{E}_n'$ is the group generated by*

$$p_0 = a_3 a_1^{-1}, \quad p_1 = a_1 a_3 a_1^{-2}, \quad q_\ell = a_\ell a_1^{-1} \text{ for } \ell = 2, 4, \ldots, n, \quad b = a_3 a_1^{-1} a_4 a_3^{-1},$$

with defining relations

$$\begin{gathered}
b = p_0 q_4 p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q_4^{-1} b, \\
p_1 q_4 p_1^{-1} = q_4^{-1} b, \quad p_1 b p_1^{-1} = (q_4^{-1} b)^3 q_4^{-2} b, \\
p_0 q_j = q_j p_1, \quad p_1 q_j = q_j p_0^{-1} p_1 \quad (5 \leqslant j \leqslant n \text{ or } j = 2), \\
q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (4 \leqslant i \leqslant n-1), \\
q_2 q_4 q_2 = q_4 q_2 q_4, \\
q_i q_j = q_j q_i \quad (4 \leqslant i < j \leqslant n, |i-j| \geqslant 2), \\
q_i q_2 = q_2 q_i \quad (5 \leqslant i \leqslant n).
\end{gathered}$$

We will also need the following lemma.

Lemma 4.8. *Let $n \in \{6, 7, 8\}$. If two elements $g, h \in \mathcal{E}_n'$ are conjugate in $\mathcal{E}_n$ by an element x and $[g, s_i] = 1$ for some i, then g and h are conjugate in $\mathcal{E}_n'$ by $x s_i^\ell$ where ℓ is the image of x in $\mathcal{E}_n^{\mathrm{ab}}$.*

Proof. Clearly, $x s_i^\ell \in \mathcal{E}_n'$. Now, $x s_i^k g s_i^{-k} x^{-1} = x g x^{-1} = h$. ☐

Lemma 4.9. *The set $\{s_2 s_1^{-1}, s_5 s_1^{-1}, s_{5+2} s_1^{-1}, \ldots\}$ forms a commutative totally symmetric set of size $\frac{n-2}{2}$ in $\mathcal{E}_n'$ when n is even, and it forms a commutative totally symmetric set of size $\frac{n-1}{2}$ in $\mathcal{E}_n'$ if n is odd.*

Proof. It is clear that all elements in this set commute and that the set has the prescribed size. Observe that $\mathcal{E}_n$ has only odd labelled edges, and recall that this implies an element $g \in \mathcal{E}_n$ is in the commutator subgroup exactly when g is in the kernel of the word length map to $\mathbb{Z}$, i.e., when there is a word in the Artin generators representing g such that the sum of the powers of all Artin generators which appear in the word is 0. From this, we deduce that each element in the set $\{s_2s_1^{-1}, s_5s_1^{-1}, s_{5+2}s_1^{-1}, \dots\}$ is contained in $\mathcal{E}'_n$.

Notice that $\{s_2, s_5, s_{5+1}, \dots\}$ is contained in a braid-type standard parabolic subgroup $\langle s_2, s_4, s_5, s_6, \dots\rangle$, so there is an element *of the braid subgroup* that induces all possible permutations of this collection [CKLP20, KM22]. Choose some permutation σ, and let the corresponding conjugating element be g. The entire parabolic subgroup commutes with the Artin generator s_1, so by Lemma 4.8, the element gs_1^l for some l induces the desired permutation in $\mathcal{E}'_n$. □

We can build slightly larger totally symmetric sets in $\mathcal{E}'_n$ if we do not require commutativity.

Lemma 4.10. *When n is even, the set*

$$\{s_2s_4^{-1}, s_3s_4^{-1}, s_5s_4^{-1}, s_{5+2}s_4^{-1}, \dots, s_{n-1}s_4^{-1}\}$$

forms a (non-commutative) totally symmetric set of size $\frac{n}{2}$ in $\mathcal{E}'_n$. When n is odd, the set

$$\{s_2s_4^{-1}, s_3s_4^{-1}, s_5s_4^{-1}, s_{5+2}s_4^{-1}, \dots, s_{n-2}s_4^{-1}\}$$

forms a (non-commutative) totally symmetric set of size $\frac{n-2}{2}$ when n is odd.

Proof. It is clear that the sets have the correct size. When $n \geqslant 7$, we see that every element of the proposed set commutes with the Artin generator s_7, so by Lemma 4.8, it suffices to construct appropriate conjugating elements in $\mathcal{E}_7$. As in the previous cases, elements of the form $s_7s_4^{-1} = s_7$, $s_9s_4^{-1} = s_9$, etc. can be permuted via conjugation by an element in the braid-type standard parabolic subgroup $\langle s_7, s_8, \dots, s_n\rangle$. Such an element commutes with each of $s_2s_4^{-1}, s_3s_4^{-1}$, and $s_5s_4^{-1}$, so elements which realize transpositions of this subset within the braid-type subgroup realize the same transpositions on the entire set. Because the symmetric group is generated by adjacent transpositions, it then suffices to realize the transpositions $(s_2s_4^{-1}, s_3s_4^{-1})$ and $(s_2s_4^{-1}, s_5s_4^{-1})$ by conjugation.

Conjugation by the inclusion of the Garside element of the D_5 Artin group $\langle s_1, s_2, s_3, s_4, s_5\rangle$ fixes $s_3, s_4, s_7, s_{7+2}, \dots$, while permuting s_2 and s_5, so it induces the transposition $(s_2s_4^{-1}, s_5s_4^{-1})$. Similarly, conjugation by the inclusion of the Garside element of either $\langle s_2, s_3, s_4, \dots, s_n\rangle$ or $\langle s_2, s_3, s_4, \dots, s_{n-1}\rangle$ (whichever has an odd number of generators) fixes $s_4, s_7, s_{7+2}, \dots$, while permuting s_2 and s_3, so it induces the transposition $(s_2s_4^{-1}, s_3s_4^{-1})$.

Now we consider the case $\mathcal{E}_6$. Here, there is no Artin generator which commutes with all of the elements in the proposed set, so we must show that the desired conjugations can be realized in $\mathcal{E}'_6$ more directly. Let Δ_{D_4} denote the inclusion into $\mathcal{E}_6$ of the Garside element of $\langle s_3, s_4, s_2, s_5\rangle$. The Garside element of D_4 is central in D_4, so in particular, this element commutes with each element in $\{s_3s_4^{-1}, s_2s_4^{-1}, s_5s_4^{-1}\}$. By [BS72], $\ell(\Delta^L_{D_5}) = \ell(\Delta^R_{D_5}) = 20$ and $\ell(\Delta_{D_4}) = 12$.

To realize the transpositions $(s_2s_4^{-1}, s_3s_4^{-1})$ and $(s_2s_4^{-1}, s_5s_4^{-1})$ in $\mathcal{E}'_6$, we highlight that the square of the Garside element is always central in a spherical Artin group. The desired conjugacy relations in $\mathcal{E}_6$ are also realized by $(\Delta^R_{D_5})^3$ and $(\Delta^L_{D_5})^3$, each of which has length 60. The transpositions can then be realized in $\mathcal{E}'_6$ by $(\Delta^R_{D_5})^3(\Delta_{D_4})^{-5}$ and $(\Delta^L_{D_5})^3(\Delta_{D_4})^{-5}$. □

4.B. **Collapsing sets in non-braid Artin groups.** Recall that collapsing sets were defined in Definition 2.6. Thus far, we have only used that the standard generators of braid groups on 5 or more strands form a collapsing sets. In this section we will exhibit collapsing sets in more general Artin groups that will be particularly suited to studying $\mathcal{H}_4$ and $\mathcal{E}_n$

Lemma 4.11. *Let A_Γ be an irreducible Artin group satisfying the following conditions:*

(1) $|V(\Gamma)| \geqslant 4$,
(2) Γ has no cycles of length less than 5,
(3) All edge labels in Γ are odd, and
(4) No two valence 1 vertices of Γ are adjacent to the same vertex.

Then the Artin generators form a collapsing set.

Proof. Throughout the proof, we conflate an Artin generator s_i with the corresponding vertex in the defining Γ for ease of notation. In what follows, $\mathrm{lk}(s_i)$ denotes the collection of vertices which are adjacent to s_i in Γ. Since we have no 3-cycles, this is equivalent to the standard definiton. The star $\mathrm{st}(s_i)$ denotes $\mathrm{lk}(s_i) \cup \{v\}$, and for a subgraph Λ of Γ, $\mathrm{st}(\Lambda)$ is the union of $\mathrm{st}(s_i)$ for each $s_i \in \Lambda$. We first prove that all of the Artin groups of interest have the following property.

Claim: *For any two Artin generators s_i and s_j which are not adjacent, there is at least one Artin generator s_k such that $s_k \in \mathrm{lk}(s_i) \cup \mathrm{lk}(s_j)$ but $s_k \notin \mathrm{lk}(s_i) \cap \mathrm{lk}(s_j)$, where $\mathrm{lk}(s_i)$ is considered in Γ.*

Proof of claim: Choose a pair s_i and s_j which are not adjacent in Γ. First, suppose s_i and s_j each have valence 1. The unique vertex v adjacent to s_i is disjoint from s_j by assumption, so we are done.

Now suppose at least one of s_i and s_j has valence at least two. Without loss of generality, say the link of s_j contains vertices s_k and $s_{k'}$. The vertices s_k and $s_{k'}$ cannot both be adjacent to s_i because there would then be a 4-cycle in Γ of the form $s_i \to s_k \to s_j \to s_{k'}$, so at least one of s_k and $s_{k'}$ is adjacent to s_j but disjoint from s_i. ◆

We can now justify the following stronger claim.

Claim: *Let $\varphi : A_\Gamma \to G$ be a group homomorphism. If $\varphi(s_i) = \varphi(s_j)$ for any pair of Artin generators, then every $s_k \in \mathrm{st}(s_i) \cup \mathrm{st}(s_j)$ has $\varphi(s_k) = \varphi(s_i) = \varphi(s_j)$, where $\mathrm{st}(s_i)$ is considered in Γ.*

Proof of claim: If s_i and s_j are adjacent, then every vertex of $\mathrm{lk}(s_j) - \{s_i\}$ is disjoint from s_i and vice versa since Γ has no 3-cycles. In particular, for any vertex s_k of $\mathrm{lk}(s_i)$ or $\mathrm{lk}(s_j)$, the image $\varphi(s_k)$ satisfies Artin relations of length both 2 and an odd integer with $\varphi(s_i) = \varphi(s_j)$ by the assumption that edge labels are odd. By Lemma 2.8, this implies that $\varphi(s_k) = \varphi(s_i) = \varphi(s_j)$.

Now suppose s_i and s_j are not adjacent. For any vertex s_k of $\mathrm{lk}(s_j)$ which is disjoint from s_i (or vice versa), s_k satisfies an odd length Artin relation with s_j. It also satisfies a commuting relation with s_i, so by Lemma 2.8, this implies that $\varphi(s_k) = \varphi(s_i) = \varphi(s_j)$. Thus it remains only to consider the case of $s_k \in \mathrm{lk}(s_i) \cap \mathrm{lk}(s_j)$. By the first claim, there is some $s_{k'} \in \mathrm{lk}(s_i)$ or $\mathrm{lk}(s_j)$ which is disjoint from s_j or s_i respectively, so Lemma 2.8 shows that $\varphi(s_{k'}) = \varphi(s_i) = \varphi(s_j)$. Suppose without loss of generality that $s_{k'}$ is in $\mathrm{lk}(s_j)$. The vertices s_k and $s_{k'}$ cannot be adjacent because they are both contained in the link of s_j and Γ has no 3-cycles. This implies that s_k satisfies an Artin relation of odd length with s_j and of length 2 with $s_{k'}$. We conclude by Lemma 2.8 applied to $\varphi(s_k)$ and $\varphi(s_i) = \varphi(s_j) = \varphi(s_{k'})$. ◆

Finally, notice that since Γ is connected, either $\Lambda = \Gamma$ or $\Lambda \subsetneq \mathrm{st}(\Lambda)$ for any induced subgraph Λ of Γ. To complete the proof, suppose that $\varphi(s_i) = \varphi(s_j)$ for some s_i and s_j, and let Λ be the induced subgraph whose vertex set is the vertex set of $\mathrm{st}(s_i) \cup \mathrm{st}(s_j)$. By the second claim, every vertex in Λ has the same image under φ. We can now reapply the second claim to show that all vertices in $\mathrm{st}(\lambda)$ also have this image. Because this is a strictly increasing process and Γ is finite, we obtain that all vertices in Γ have the same image after finitely many repetitions. □

Corollary 4.12. *The Artin generators form a collapsing set in $\mathcal{H}_4$ and in $\mathcal{E}_n$ for all n.*

While the above proof requires at least 4 vertices, the desired statement is also true in $\mathcal{H}_3$.

Lemma 4.13. *The Artin generators in $\mathcal{H}_3$ form a collapsing set.*

Proof. Observe that for any pair of Artin generators s_i and s_j, the third generator s_k satisfies coprime length Artin relations m_{ik} and m_{jk}. The result then follows immediately from Lemma 2.8. □

We conclude this subsection with some applications which will be useful in the remainder of the paper.

Lemma 4.14. *Let A_Γ be an odd-labelled Artin group where the Artin generators form a collapsing set, and let A_X be an irreducible standard parabolic subgroup with $|X| > 1$. Let $\alpha\colon A_\Gamma \to Q$ be a homomorphism such that the restriction of α to A_X has cyclic image. The entire homomorphism α has cyclic image.*

Proof. Since the Artin generators $\{s_i\} = X$ form a generating set for the standard parabolic subgroup A_X, their images generate the cyclic subgroup $\alpha(A_X)$. In particular, there is one generator s_i in this set such that $\alpha(s_i)$ generates $\alpha(A_X)$. Let $\overline{s}$ denote this generator for $\alpha(A_X)$.

We claim that every Artin generator in the set X has image $\overline{s}$. To see this, it suffices to recall that because X is irreducible and all edge labels are odd, the generators in this set are pairwise conjugate *by elements of A_X*. Let $\alpha(s_j) = \overline{s}^k$ for some $j \neq i$, let g be an element of A_X which conjugates s_i to s_j, and let m be an integer such that $\alpha(g) = \overline{s}^m$. In the quotient, this relation becomes $s^m s s^{-m} = s^k$, which implies either $k = 1$ or the entire cyclic subgroup $\alpha(A_X)$ is trivial. The images of the generators coincide in either case. The statement then follows from Lemma 4.11. □

Lastly, we make the following observation.

Lemma 4.15. *Let A_Γ be an odd-labelled Artin group where the Artin generators form a collapsing set, and let $q : A_\Gamma \to Q$ be a non-cyclic quotient. For each generator s_i, the image of the standard parabolic subgroup generated by $\Gamma - St(s_i)$ is a proper subgroup of Q.*

Proof. Suppose that the image of the standard parabolic subgroup generated by $\Gamma - \mathrm{St}(s_i)$ is all of Q. This parabolic subgroup commutes with s_i in A_Γ, so we must have that $q(s_i) \in Z(Q)$. If the Artin generating set is collapsing, then the defining graph Γ is necessarily connected, so there is some $s_k \in \mathrm{Lk}(s_i)$. Since Γ is odd-labelled, the generators s_k and s_i satisfy an odd-length Artin relation. In the quotient, however, we then have that $q(s_k)$ and $q(s_i)$ satisfy both a commuting relation and an Artin relation of odd length. By Lemma 2.8, the group Q is cyclic. ☐

4.C. **Element orders.** In this section, we will show that every non-cyclic finite quotient of an Artin group of type $\mathcal{E}_6$, $\mathcal{E}_7$, or $\mathcal{H}_4$ must have torsion elements of certain orders.

Before we begin, we give an overview of the proof strategy. Elements of a spherical Artin group A_Γ whose images in $A_\Gamma/Z(A_\Gamma)$ are torsion of various orders were constructed in [Sor21] for Artin groups of type $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{H}_4$. The precise elements and their orders will be stated later in the section. We aim to show that these torsion elements survive in every non-cyclic finite quotient of A_Γ, and that suitable powers yield torsion elements of the same order.

To do this, we will consider an element which has order k in $A_\Gamma/Z(A_\Gamma)$, say ϵ of order 10. We will show that, in our example, $A_\Gamma/\langle \epsilon^2 \rangle$ and $A_\Gamma/\langle \epsilon^5 \rangle$ are cyclic. It will then follow that in any non-cyclic quotient of A_Γ where ϵ has finite order, its order must be divisible by $\mathrm{lcm}(2,5) = 10$. This will in turn imply that any non-cyclic finite quotient of A_Γ has an element of order 10 by taking the image of a suitable power of ϵ.

To show that the relevant quotients are trivial, we will show that in any quotient of A_Γ where the required powers of ϵ are trivial, the required powers being ϵ^2 and ϵ^5 in the example, a relation is imposed which forces two Artin generators to have the same image. We will then conclude by Lemma 4.11.

Primarily, we will do this by computing conjugates of a particular Artin generator by powers of ϵ and reducing the resulting word via the Artin relations. This process is necessarily fairly computation-heavy. As such, we provide the computations in an appendix. The appendix presents these computations in a compact notation which allows an interested reader to reconstruct each step.

Lemma 4.16. *Let $G = \mathcal{E}_6/Z(\mathcal{E}_6)$. The following conclusions hold:*

(1) The image of $\epsilon_{12} = s_4s_2s_3s_1s_5s_6$ has order 12.
(2) The image of $\epsilon_9 = s_4s_2s_5s_4s_3s_1s_5s_6$ has order 9.
(3) The image of $\epsilon_8 = s_4s_3s_1s_5s_4s_2s_3s_6s_5$ has order 8.
(4) The groups $\mathcal{E}_6/\langle\langle \epsilon_{12} \rangle\rangle$, $\mathcal{E}_6/\langle\langle \epsilon_9 \rangle\rangle$, $\mathcal{E}_6/\langle\langle \epsilon_8 \rangle\rangle$, $\mathcal{E}_6/\langle\langle \epsilon_{12}^4 \rangle\rangle = \mathcal{E}_6/\langle\langle \epsilon_9^3 \rangle\rangle$, $\mathcal{E}_6/\langle\langle \epsilon_{12}^3 \rangle\rangle = \mathcal{E}_6/\langle\langle \epsilon_8^2 \rangle\rangle$, $\mathcal{E}_6/\langle\langle \epsilon_8^4 \rangle\rangle$, $\mathcal{E}_6/\langle\langle \epsilon_{12}^2 \rangle\rangle$, and $\mathcal{E}_6/\langle\langle \epsilon_{12}^6 \rangle\rangle$ are all cyclic.

In particular, any finite non-cyclic quotient of $\mathcal{E}_6$ has elements of order 8*,* 9*, and* 12.

Proof. The first three conclusions and the equalities in the fourth follow from Theorem 7 of [Sor21]. Computations which demonstrate that the quotients $\mathcal{E}_6/\langle\langle \epsilon_{12}^6 \rangle\rangle$, $\mathcal{E}_6/\langle\langle \epsilon_9^3 \rangle\rangle$, and $\mathcal{E}_6/\langle\langle \epsilon_8^4 \rangle\rangle$ are all cyclic are given in the appendix sections A.B.1, A.B.2, and A.B.3. The other quotients must also be cyclic, as they are further quotients of these three. ☐

Lemma 4.17. *Let $G = \mathcal{E}_7/Z(\mathcal{E}_7)$. The following conclusions hold:*

(1) the image of $\epsilon_7 = s_4s_2s_7s_6s_5s_4s_2s_3s_1$ in G has order 7;
(2) the image of $\epsilon_9 = s_4s_2s_3s_1s_5s_6s_7$ in G has order 9;
(3) the groups $\mathcal{E}_7/\langle\langle \epsilon_7 \rangle\rangle$, $\mathcal{E}_7/\langle\langle \epsilon_9 \rangle\rangle$, and $\mathcal{E}_7/\langle\langle \epsilon_9^3 \rangle\rangle$ are cyclic.

In particular, any finite non-cyclic quotient of $\mathcal{E}_7$ contains elements of order 7 *and* 9.

Proof. The first and second parts are proved in [Sor21, Theorem 7]. Computations which demonstrate that $\mathcal{E}_7/\langle\langle \epsilon_7 \rangle\rangle$ and $\mathcal{E}_7/\langle\langle \epsilon_9^3 \rangle\rangle$ are cyclic are provided in appendix sections A.C.1 and A.C.2. The quotient $\mathcal{E}_7/\langle\langle \epsilon_9 \rangle\rangle$ must also be cyclic, because it is a further quotient. ☐

Lemma 4.18. *Let $G = \mathcal{H}_4/Z(\mathcal{H}_4)$. The following conclusions hold:*

(1) The image of $\epsilon_{10} = s_1s_2s_1s_2s_3s_4$ has order 10.
(2) The image of $\epsilon_{15} = s_1s_2s_3s_4$ has order 15.
(3) The groups $\mathcal{H}_4/\langle\langle \epsilon_{10} \rangle\rangle$, $\mathcal{H}_4/\langle\langle \epsilon_{10}^2 \rangle\rangle$, $\mathcal{H}_4/\langle\langle \epsilon_{10}^5 \rangle\rangle$, $\mathcal{H}_4/\langle\langle \epsilon_{15} \rangle\rangle$, $\mathcal{H}_4/\langle\langle \epsilon_{15}^3 \rangle\rangle$, and $\mathcal{H}_4/\langle\langle \epsilon_{15}^5 \rangle\rangle$ are all cyclic.

In particular, any finite non-cyclic quotient of $\mathcal{H}_4$ contains elements of order 10 and 15.

Proof. The first and second parts again follow from [Sor21, Theorem 7]. Computations which demonstrate that $\mathcal{H}_4/\langle\langle\epsilon_{10}^2\rangle\rangle$ and $\mathcal{H}_4/\langle\langle\epsilon_{10}^5\rangle\rangle$ are cyclic are provided in appendix section A.D.1. Computations which demonstrate that $\mathcal{H}_4/\langle\langle\epsilon_{15}^3\rangle\rangle$ and $\mathcal{H}_4/\langle\langle\epsilon_{15}^5\rangle\rangle$ are cyclic are provided in appendix section A.D.2. The other two quotients must also be cyclic because each is a further quotients of one of these four. □

5. The Artin groups $\mathcal{F}_4$, $\mathcal{H}_3$, $\mathcal{H}_4$

5.A. The Artin group $\mathcal{F}_4$.

Theorem 5.1. *The smallest non-abelian quotient of $\mathcal{F}_4$ is* Sym(3).

Proof. The group Sym(3) is the smallest non-abelian group, so it suffices to show that Sym(3) is a quotient of $\mathcal{F}_4$. The Coxeter group of $\mathcal{F}_4$ is the group $2^{1+4} : (\mathrm{Sym}(3) \times \mathrm{Sym}(3))$ [Wil09]. One obtains a quotient $q : \mathcal{F}_4 \to \mathrm{Sym}(3)$ by composing the canonical projective homomorphism $\mathcal{F}_4 \to (\mathrm{Sym}(3) \times \mathrm{Sym}(3))$ with a projection map to one of the coordinates of $(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$. □

Remark 5.2. The smallest non-solvable quotient of $\mathcal{F}_4$ is Alt(5). As in the theorem, it suffices to verify that Alt(5) is a quotient. To obtain a quotient map, one first retracts the Artin group $\mathcal{F}_4$ onto one of its $\mathcal{A}_2$ standard parabolic subgroups. The group $\mathcal{A}_2$ is isomorphic to $\tilde{SL}_2(\mathbb{Z})$. One can then map onto $SL_2(\mathbb{Z})$ and further onto $\mathrm{PSL}_2(5) \cong \mathrm{Alt}(5)$.

5.B. The Artin group $\mathcal{H}_3$.

We construct two epimorphisms $\mathcal{H}_3 \to \mathrm{Alt}(5)$. The first factors through the Coxeter group $W(\mathcal{H}_3)$ and sends a generator to an involution. The second sends generators to 5-cycles and can be given as follows

$$\psi\colon \mathcal{H}_3 \twoheadrightarrow \mathrm{Alt}(5) \quad \text{by} \quad s_1 \mapsto (1\ 2\ 3\ 4\ 5) \quad s_2 \mapsto (1\ 4\ 5\ 2\ 3) \quad s_3 \mapsto (1\ 5\ 4\ 3\ 2).$$

One easily checks that $\psi(s_1s_2s_1s_2s_1) = \psi(s_2s_1s_2s_1s_2)$, that $\psi(s_1s_3) = \psi(s_3s_1)$, and that $\psi(s_2s_3s_2) = \psi(s_2s_3s_2)$. Note that the homomorphisms are identified by Aut(Alt(5)) since the generators of $\mathcal{H}_3$ are sent to elements of different orders.

Theorem 5.3. *If G is isomorphic to $\mathcal{H}_3$, then the smallest non-abelian quotient of G is* Alt(5)*. Up to automorphisms of the image there are two such homomorphisms.*

Proof. Every non-cyclic quotient of $\mathcal{H}_3$ is non-solvable because $\mathcal{H}_3'$ is perfect. We have already exhibited two distinct quotients up to automorphisms of the image. The only other possible element order for an element in Alt(5) is 3. A long tedious and direct computation (that we omit for brevity) shows that the relations in $\mathcal{H}_3$ are not satisfied by any triple of 3-cycles in Alt(5). □

5.C. The Artin group $\mathcal{H}_4$.

In this section we will compute the smallest non-abelian quotient of $\mathcal{H}_4$. Throughout we will use Notation 2.1. Recall that $\mathcal{H}_4$ has cyclic abelianization, and its commutator subgroup is perfect by Theorem 4.3, so we are in the setting of Lemma 2.4.

Family	Group	Order	Out(G)	10s	15s
Alternating	Alt(5)	60	2	n	n
	Alt(6)	360	2^2	n	n
	Alt(7)	2,520	2	n	n
$\mathtt{A}_1$	$\mathrm{PSL}_2(7)$	168	2	n	n
	$\mathrm{PSL}_2(8)$	504	3	n	n
	$\mathrm{PSL}_2(11)$	660	2	n	n
	$\mathrm{PSL}_2(13)$	1,092	2	n	n
	$\mathrm{PSL}_2(17)$	2,488	2	n	n
	$\mathrm{PSL}_2(19)$	3,420	2	y	n
	$\mathrm{PSL}_2(16)$	4,080	4	n	y
$\mathtt{A}_2$	$\mathrm{PSL}_3(3)$	5,616	2	n	n
${}^2\mathtt{A}_2$	$\mathrm{SU}_3(3)$	6,048	2	n	n

Table 1. Simple groups with order at most 7200.

Lemma 5.4. *There is exactly one isomorphism class of groups* $\mathrm{Alt}(5)^2.2$ *with the property that every proper quotient is cyclic. The group is isomorphic to* $\mathrm{Alt}(5) \wr 2$.

Proof. We have that $\mathrm{Out}(\mathrm{Alt}(5))^2 \cong 2 \wr 2$. Write an element as $(a, b; c)$ where $(a, b) \in 2^2$ and $c \in 2$. We have that $(1, 0; 0)$ and $(0, 1; 0)$ correspond to the non-trivial element in $\mathrm{Out}(\mathrm{Alt}(5)) \cong 2$ for each factor. We call these outer classes *symmetric*. The non-trivial central element $(1, 1; 0)$ is a symmetric outer automorphism on each factor. We call this outer class *central*. The element $(0, 0; 1)$ is the outer automorphism that permutes the two factors of Alt(5). We call this outer automorphism *wreathing*. The final involution in $2 \wr 2$ is the element $(1, 1; 1)$. We call this outer class *diagonal*.

Since $Z(\mathrm{Alt}(5)^2)) = 1$, an equivalence class of group extensions

$$1 \longrightarrow \mathrm{Alt}(5)^2 \longrightarrow Q \longrightarrow 2 \longrightarrow 1$$

is given by a homomorphism $\phi\colon 2 \to \mathrm{Out}(\mathrm{Alt}(5)^2)$, see [Bro94, Corollary IV.6.8]. Note that different equivalence classes of group extensions may give rise to isomorphic groups. The group $2 \wr 2$ contains 5 elements of order 2. Let $\phi\colon 2 \to \mathrm{Out}(\mathrm{Alt}(5)^2)$ and let Q_ϕ denote the resulting extension group. We analyse each possible extension in turn. We call ϕ trivial, symmetric, central, wreathing, or diagonal if the non-trivial element of 2 is mapped to an element with the corresponding property.

Suppose that ϕ is trivial. In this case, $Q_\phi \cong \mathrm{Alt}(5)^2 \times 2$. This clearly has a non-cyclic proper quotient, so we may rule it out.

Suppose that ϕ is symmetric. Clearly the resulting extension group is isomorphic to $\mathrm{Alt}(5) \times \mathrm{Sym}(5)$. This evidently has a non-cyclic proper quotient so we may rule it out.

Suppose that ϕ is central. We claim that either tautological Alt(5) subgroup is normal. For i=1,2 write $(\sigma_i, \tau_i; x_i) \in Q_\phi$ with $(\sigma_i, \tau_i) \in \mathrm{Alt}(5)^2$ and $x_i \in 2$ acts diagonally. The group multiplication is then given by a semidirect product

$$(\sigma_1, \tau_1; x_1)(\sigma_2, \tau_2; x_2) = (\sigma_1 x_1(\sigma_2), \tau_1 x_1(\tau_2); x_1 x_2)$$

and an inverse of $(\sigma, \tau; x)$ is given by $(x(\sigma^{-1}, x(\tau^{-1}), x)$. For $(\mu, 1; 0) \in \mathrm{Alt}(5) \times \{1\}$ we have

$$\begin{aligned}(\sigma, \tau; x)(\mu, 1; 0)(\sigma, \tau; x)^{-1} &= (\sigma x(\mu), \tau, x)(\sigma^{-1}, x(\tau^{-1}), x)\\ &= (\sigma x(\mu)x(\sigma^{-1}), \tau x(x(\tau^{-1})), xx)\\ &= (\sigma x(\mu)x(\sigma^{-1}), 1; 0)\end{aligned}$$

which is contained in $\mathrm{Alt}(5) \times \{1\}$. A similar argument also shows that $\{1\} \times \mathrm{Alt}(5)$ is normal. In either case, the quotient by a normal Alt(5) subgroup has shape Alt(5).2 and is therefore non-abelian.

The two remaining cases are when ϕ is wreathing or diagonal. Clearly in either case the normal closure of any Alt(5) subgroup is the subgroup $\mathrm{Alt}(5)^2$. Thus, one readily checks that every proper quotient of Q_ϕ is cyclic.

Let $Q_1 \cong \mathrm{Alt}(5) \wr 2$ and let Q_2 denote the group given by Q_ϕ where ϕ is diagonal. It remains to show that $Q_1 \cong Q_2$.

We have that Q_2 is a subgroup of $\mathrm{Aut}(\mathrm{Alt}(5)^2) \cong \mathrm{Sym}(5) \wr 2$. In particular we may view Q_2 as a subgroup of Sym(10). Indeed, we may generate Q_2 by the two copies of Alt(5), one on $\{1, \dots, 5\}$ and the other on $\{6, \dots, 9, 0\}$ and the element

$$x = (1\ 2)(6\ 7)(1\ 6)(2\ 7)(3\ 8)(4\ 9)(5\ 0) = (1\ 7)(2\ 6)(3\ 8)(4\ 9)(5\ 0).$$

But the latter element is $\mathrm{Sym}(5) \wr 2$ conjugate by $y = (1\ 2)$ to $z = (1\ 6)(2\ 7)(3\ 8)(4\ 9)(5\ 0)$. The element $(1\ 2)$ normalises $\mathrm{Alt}(5)^2$ so the group Q_2 is conjugate by y to $\langle \mathrm{Alt}(5)^2, z\rangle$. But this later group is exactly $\mathrm{Alt}(5) \wr 2 < \mathrm{Sym}(10)$ as required. ☐

Theorem 5.5. *The smallest non-cyclic quotient of $\mathcal{H}_4$ is $W(\mathcal{H}_4)/Z(\mathcal{H}_4)$, and it is unique up to automorphisms of the image.*

Proof. By Lemma 2.4, either Q is almost simple, or it is a subgroup of $\mathrm{Aut}(S^n)$ for some simple group S and $n > 1$. For all S and n except $\mathrm{Alt}(5)^2$, $\mathrm{Aut}(S^n) > |W(\mathcal{H}_4)|$, so Q is either almost simple or a subgroup of $\mathrm{Aut}(\mathrm{Alt}(5)^2)$. We first apply element order considerations to rule out almost simple groups of order at most $7200 = |W(\mathcal{H}_4)|$.

The simple groups Q with order at most 3600 are Alt(5), $\mathrm{PSL}_2(7)$, Alt(6), $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_2(11)$, $\mathrm{PSL}_2(13)$, $\mathrm{PSL}_2(17)$, Alt(7), $\mathrm{PSL}_2(19)$. One easily verifies via the ATLAS that $\mathrm{Aut}(Q)$ does not contain an element of order 15 for any of these. The simple groups Q with order between 3600 and 7200 are $\mathrm{PSL}_2(16)$, $\mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$, and $\mathrm{PSL}_2(23)$. In each of these cases, after consulting the ATLAS, Q contains has no element of order 10. Note we need not check $\mathrm{Aut}(Q)$ for these since $2|Q| > 7200$.

We are now left with groups of the shape $\mathrm{Alt}(5)^2.2$. By Lemma 5.4, there is exactly one group of this shape which does not admit a proper non-cyclic quotient, namely $\mathrm{Alt}(5) \wr 2$, so necessarily this group must be isomorphic $W = W(\mathcal{H}_4)/Z(\mathcal{H}_4)$ as required. One easily checks the quotient is unique up to automorphisms of the image using MAGMA, see Appendix B.A. □

6. The Artin group $\mathcal{E}_6$

In this section, we will compute the smallest non-abelian quotient of $\mathcal{E}_6$. Throughout, we will use Notation 2.1.

6.A. **The groups.** Our proof will use the classification of finite simple groups. We recall the finite simple groups smaller than $W(\mathcal{E}_6)$ in the table below.

Family	Group	Order	Out(G)
Alternating	Alt(5)	60	2
	Alt(6)	360	2^2
	Alt(7)	2,520	2
	Alt(8)	20,160	2
$\mathtt{A}_1$	$\mathrm{PSL}_2(q)$ $q \in \{7, 8, 11, 13, 17, 19, 16, 23, 25, 27, 29, 31, 32, 37, 41, 43\}$	$\frac{1}{d}q(q^2-1)$ where $d = \gcd(2, q-1)$	$d \times k$ where $d = \gcd(2, q-1)$ and $q = p^k$ for p a prime
$\mathtt{A}_2$	$\mathrm{PSL}_3(3)$	5,616	2
	$\mathrm{PSL}_3(4)$	20,160	$2 \times \mathrm{Sym}(3)$
${}^2\mathtt{A}_2$	$\mathrm{SU}_3(3)$	6,048	2
$\mathtt{C}_2$	$\mathrm{PSp}_4(3) \cong \mathrm{SU}_4(2)$	25,920	2
Exceptional	${}^2\mathtt{B}_2(8)$	29,120	3
Sporadic	M_{11}	7,920	1

Table 2. Simple groups with order at most 51,840.

Lemma 6.1. *A smallest non-abelian quotient of $\mathcal{E}_6$ has socle isomorphic to one of the following groups*

(1) $\mathrm{Alt}(n)$ *for* $n \in \{5, 6, 7, 8\}$*;*
(2) $\mathrm{PSL}_2(q)$ *for* $q \in \{7, 8, 11, 13, 17, 19, 16, 23, 25, 27, 29, 31, 37, 41, 43\}$*;*
(3) $\mathrm{PSL}_3(3)$*,* $\mathrm{PSL}(3, 4)$*;*
(4) $\mathrm{PSU}_3(3)$*;*
(5) $\mathrm{PSp}_4(3)$*;*
(6) ${}^2\mathtt{B}_2(8)$*;*
(7) M_{11}*;*
(8) $\mathrm{Alt}(5)^2$*;*
(9) $\mathrm{PSL}_2(7)^2$*.*

Proof. Recall that $\mathcal{E}_6^{ab}$ is cyclic. The subgroup $[\mathcal{E}_6, \mathcal{E}_6]$ is perfect by Theorem 4.3, so Lemma 2.4 applies.
The associated Coxeter group $W(\mathcal{E}_6)$ is isomorphic to $\mathrm{SO}_5(3)$ which has order $51,840$. The non-cyclic simple groups with order less than $51,840$ are listed in Table 2. □

6.B. **Excluding alternating groups.**

Lemma 6.2. *Let $k \leqslant 2$ and $n \leqslant 8$. If $\alpha\colon \mathcal{E}_6 \to \mathrm{Aut}(\mathrm{Alt}(n)^k) \wr 2$ is a homomorphism, then α has cyclic image.*

Proof. The group $\mathrm{Out}(\mathrm{Alt}(n))$ is a 2-group for each n; it is isomorphic to 2 for $n \neq 6$ and 2^2 for $n = 6$. Since $\mathrm{Alt}(n)$ for $n \leqslant 8$ does not contain an element of order 9, neither does $\mathrm{Aut}(\mathrm{Alt}(n))$ nor $\mathrm{Aut}(\mathrm{Alt}(n)^2) \wr 2 \cong (\mathrm{Aut}(\mathrm{Alt}(n)) \wr 2) \wr 2$. We conclude by Lemma 4.16. □

6.C. **Excluding groups in the family $\mathtt{A}_n$.** Here we give a uniform way of inductively excluding quotients isomorphic to subgroups of finite linear groups.

Theorem 6.3. [DFLO25, Theorem 2.2.8] *Let $x \in \mathrm{GL}_n(q)$ have minimal polynomial $f_1(t)^{e_1} \dots f_h(t)^{e_h}$ with $f_i \in \mathbb{F}_q[t]$ distinct and irreducible of degree d_i. Let the Jordan form of x be*

$$\bigoplus_{i=1}^{h} \left(\bigoplus_{l_{i,1}} B_{\lambda_{i,1}} \oplus \cdots \oplus \bigoplus_{l_{i,k_i}} B_{\lambda_{i,k_i}} \right)$$

where $\lambda_{i,1} < \cdots < \lambda_{i,k_i} = e_i$ for each i. Then $C_G(x) \cong U \rtimes R$ where

$$R \cong \prod_{i=1}^{h} \prod_{j=1}^{k_i} \mathrm{GL}_{l_{i,j}}(q^{d_i})$$

and $U = q^m$ where

$$m = \sum_{i=1}^{h} d_i \left(2 \sum_{a<b} \lambda_{ia} \right).$$

Corollary 6.4. *Let $(G_{m-2}, G_m) \in \{(\mathcal{A}_4, \mathcal{E}_6), (\mathcal{A}_5, \mathcal{E}_7), (\mathcal{E}_6, \mathcal{E}_8), (\mathcal{A}_n, \mathcal{A}_{n+2})\ n \geqslant 4\}$. If G_m admits a non-cyclic homomorphism to $\mathrm{GL}_n(q)$, then G_{m-2} admits a non-cyclic homomorphism to $\mathrm{GL}_{n-1}(q^d)$ for some $d \leqslant n$.*

Proof. Note that the centraliser of s_m contains a group isomorphic to G_{m-2}. Suppose $\alpha \colon G_m \to \mathrm{GL}_n(q)$ has non-cyclic image. Let the Jordan form of $\alpha(s_m)$ be

$$\bigoplus_{i=1}^{h} \left(\bigoplus_{l_{i,1}} B_{\lambda_{i,1}} \oplus \cdots \oplus \bigoplus_{l_{i,k_i}} B_{\lambda_{i,k_i}} \right)$$

where $\lambda_{i,1} < \cdots < \lambda_{i,k_i} = e_i$ for each i. Now, by Theorem 6.3 we have that $\alpha(G_{m-2})$ is contained in a subgroup of shape $U \rtimes R$, where R is a product of groups isomorphic to $\prod_{i=1}^{h} \prod_{j=1}^{k_i} \mathrm{GL}_{l_{i,j}}(q^{d_i})$. Moreover, $\alpha(G_{m-2})$ cannot be cyclic by Lemma 4.14. In particular, there must be some $\pi_{i,j} \colon U \rtimes R \twoheadrightarrow \mathrm{GL}_{l_{i,j}}(q^{d_i})$ such that $\pi(\alpha(G_{m-2}))$ is non-cyclic. Now, if some $l_{i,j} = n$, then $h = 1$, $i = j = 1$, and $\lambda_{1,1} = e_1 = d_1 = n$. From this we conclude that $\alpha(s_m)$ must be central. Since s_m satisfies an odd length Artin relation with s_{m-1}, this implies the image of α is cyclic by Lemma 4.11. We conclude that $l_{i,j} < n$ as required. □

Corollary 6.5. *Let $(G_m, G_{m+2}) \in \{(\mathcal{A}_4, \mathcal{A}_6), (\mathcal{A}_5, \mathcal{A}_7), (\mathcal{E}_6, \mathcal{E}_8), (\mathcal{A}_n, \mathcal{A}_{n+2})$ for $n \geqslant 4\}$. If G_{m+2} admits a non-cyclic homomorphism to $\mathrm{PGL}_n(q)$, then G_m admits a non-cyclic homomorphism to $\mathrm{PGL}_{n-1}(q^d)$ for some $d \leqslant n$.*

Proof. This is an easy consequence of Corollary 6.4. □

Remark 6.6. One could in principle formulate similar results in two ways. Firstly, with the homomorphism domain being a braid group. Secondly, with target other finite classical groups using the results of [DFLO25].

Lemma 6.7. *If $\alpha \colon \mathcal{E}_6 \to \mathrm{PGL}_2(q)$ is a homomorphism, then α has cyclic image.*

Proof. By Corollary 6.5, it suffices to check that every homomorphism $\mathcal{A}_4 \to \mathrm{PGL}_1(q^d)$ for every $d \geqslant 1$ has cyclic image. The result then follows from the fact that $\mathrm{PGL}_1(q^d)$ is solvable and Lemma 4.4. □

Remark 6.8. Note that $\mathrm{Aut}(\mathrm{PSL}_2(q)) \cong \mathrm{PGL}_2(q) : \mathrm{Aut}(\mathbb{F}_q)$. It thus follows from the previous lemma that the only groups in the family $\mathtt{A}_1$ we need to rule out are $\mathrm{Aut}(\mathrm{PSL}_2(q))$ when $\mathbb{F}_q$ admits a non-trivial automorphism. This occurs exactly when q is one of $8, 16, 27$, or 32.

Lemma 6.9. *If $\alpha \colon \mathcal{E}_6 \to \mathrm{Aut}(\mathrm{PSL}_2(q))$ with $q = 8, 16, 27, 32$ is a homomorphism, then α has cyclic image.*

Proof. Let $G = \mathrm{Aut}(\mathrm{PSL}_2(q))$ and $Q = \mathrm{PSL}_2(q)$. We can resolve three cases with element order considerations. If $q = 8$, then G does not contain an element of order 12 [CCN+85, pg. 6]. If $q = 16$, then G does not admit an element of order 9 [CCN+85, pg. 12]. If $q = 32$, then G does not admit an element of order 9 or 12 [CCN+85, pg. 29]. In any of the these cases, we conclude by Lemma 4.16.

The case $q = 27$ is more involved. In this case, $G = \mathrm{PSL}_2(27).6$. We claim that every homomorphism $\alpha\colon \mathcal{E}_6 \to Q.k$ for $k = 2, 3$ is cyclic. Indeed, the image of $\mathcal{A}_4$ must be a proper subgroup of $Q.k$ by Lemma 4.15. By [CCN+85, Page 18], the maximal subgroups of $Q.k$ are soluble groups of shapes $3^3 : 26$, D_{56}, D_{52}, $\mathrm{Sym}(4)$, $3^3 : 13 : 3$, $28 : 6$, $26 : 6$, $\mathrm{Sym}(4) \times 3$, and the non-soluble group Q.

The image of $\mathcal{A}_4$ cannot be one of the soluble groups because this would force its image to be cyclic by Lemma 4.4, and the image of α would then be cyclic by Lemma 4.14. Thus, the image must be Q. Observe, however, that $\alpha(\mathcal{A}_4)$ centralises $\alpha(s_1)$, and Q does not centralise any non-trivial elements of $Q.k$. It follows that Q cannot equal $\alpha(\mathcal{A}_4)$. Since every maximal subgroup of Q is solvable, we conclude that α has cyclic image.

We now repeat the same argument for G. The maximal subgroups of G are the soluble groups of shape $3^3 : 26 : 3$, $28 : 6$, $26 : 6$, $\mathrm{Sym}(4) \times 3$ as well as $Q.2$ and $Q.3$. By the same reasoning as before, the image of $\mathcal{A}_4$ cannot be $Q.2$ or $Q.3$ or Q. Since every other possible image is soluble, we conclude that α has cyclic image. □

Lemma 6.10. *Let $Q = \mathrm{PSL}_3(3)$. If $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(Q)$ is a homomorphism, then α has cyclic image.*

Proof. We first note that α cannot have image contained in Q because Q does not contain an element of order 12 [CCN+85, Page 13]. We now consider the image of $\mathcal{A}_4$ which centralises s_1. Since $G = \mathrm{Aut}(\mathrm{PSL}_3(3))$ has trivial centre and $\alpha(s_1)$ is non-trivial, we must have that $\alpha(\mathcal{A}_4)$ is a proper subgroup of G. Thus, $\alpha(\mathcal{A}_4)$ is contained in a maximal subgroup. This maximal subgroup cannot be Q because Q does not centralise any non-trivial elements. The remaining maximal subgroups of G are soluble, so we conclude by Lemma 4.4. □

6.D. Excluding other groups.

Lemma 6.11. *Let $Q = \mathrm{PSL}_3(4)$, $\mathrm{SU}_3(3)$, ${}^2\mathtt{B}_2(8)$, or M_{11}, and let $\alpha\colon \mathcal{E}_6/Z(\mathcal{E}_6) \to \mathrm{Aut}(Q)$ be a homomorphism. The image $\alpha(\mathcal{E}_6/Z(\mathcal{E}_6))$ is cyclic.*

Proof. Using the Atlas, one can verify that the groups $\mathrm{Aut}(\mathrm{SU}_3(3)) \cong G_2(2)$ and $\mathrm{Aut}({}^2\mathtt{B}_2(8)) \cong {}^2\mathtt{B}_2(8).3$ have no elements of order 9 [CCN+85, pg. 14, pg. 28], and $\mathrm{Aut}(M_{11}) \cong M_{11}$ has no elements of order 12 [CCN+85, pg. 18]. The statement then follows from Lemma 4.16. □

Lemma 6.12. *If $\alpha\colon \mathcal{E}_6 \to \mathrm{PSp}_4(3) : 2$ is a homomorphism, then it has cyclic image.*

Proof. Recall that we must have $\mathrm{PSp}_4(3) \leqslant \alpha(\mathcal{E}_6)$ by Lemma 2.4. Thus it suffices to show that there is no non-cyclic homomorphism $\alpha\colon \mathcal{E}_6 \to \mathrm{PSp}_4(3)$. The group $\mathrm{PSp}_4(3)$ does not have any elements of order 8 [CCN+85, pg. 27]. Thus we conclude by Lemma 4.16. □

Lemma 6.13. *If $\alpha\colon \mathcal{E}_6 \to \mathrm{Aut}(\mathrm{PSL}_2(7)^2) \wr 2$ is a homomorphism, then is has cyclic image.*

Proof. By [CCN+85, pg.3], the group $\mathrm{Aut}(\mathrm{PSL}_2(7)) \cong \mathrm{PSL}_2(7).2$ has no elements of order 9. It follows that $\mathrm{Aut}(\mathrm{PSL}_2(7)^2) \cong (\mathrm{PSL}_2(7).2) \wr 2$ has no elements of order 9. Therefore, $\mathrm{Aut}(\mathrm{PSL}_2(7)^2) \wr 2$ also has no elements of order 9. Thus we conclude by Lemma 4.16. □

6.E. Completing the proof.

Theorem 6.14. *The smallest non-abelian quotient of $\mathcal{E}_6$ is isomorphic to $\mathrm{PSU}_4(2) : 2 \cong W(\mathcal{E}_6)$. This quotient is unique up to an automorphism of the image.*

Remark 6.15. Note that $\mathrm{PSU}_4(2) : 2 \cong \mathrm{PGO}_6^-(2) \cong \mathrm{PSp}_4(3) : 2$.

Proof of Theorem 6.14. Let P be the smallest non-abelian quotient of $\mathcal{E}_6$. By Lemma 6.1, it suffices to consider subgroups of $\mathrm{Aut}(Q)$ where Q is one of the 9 classes of groups listed in the statement of the lemma. Lemma 6.2 excludes the possibility that $G < \mathrm{Aut}(\mathrm{Alt}(n)^2) \wr 2$, which encompasses classes (1) and (5) in the list. Lemmas 6.7 and 6.9 shows that G cannot be a subgroup of $\mathrm{Aut}(\mathrm{PSL}_2(q))$, eliminating class (2). Lemmas 6.11, 6.10, and 6.13 eliminate classes (2), (3), (4) (6), (7), and (9). Thus P must be a subgroup of $\mathrm{Aut}(\mathrm{PSU}_4(2)) \cong \mathrm{PSU}_4(2) : 2$. By Lemma 6.12, the image must be all of $\mathrm{PSU}_4(2) : 2$.

We now prove uniqueness. Let $G = \mathrm{PSU}_4(2) : 2$ and $Q = \mathrm{PSU}_4(2)$. It suffices to show that in any homomorphism $\alpha\colon \mathcal{E}_6 \to G$ with non-cyclic image, the image of s_1 has order 2. If $\alpha(s_1)$ has order 2, then α must factor through the Coxeter quotient map.

Note that such a quotient induces a non-trivial homomorphism of $\mathcal{A}_4$. Because $\mathcal{A}_4$ centralises s_1, the image of $\mathcal{A}_4$ must be a proper subgroup of G by Lemma 4.15. This subgroup cannot be Q because Q

does not centralise any non-trivial elements of G, so it must be contained in a maximal proper subgroup of G or Q.

By [CCN+85, Page 26], the possible candidates are $2^4 : \mathrm{Sym}(5)$, $\mathrm{Sym}(6) \times 2$, $3_+^{1+2} : 2\,\mathrm{Sym}(4)$, $3^3 : (\mathrm{Sym}(4) \times 2)$, and $(2 \cdot \mathrm{Alt}(4)^2.2).2$. Lemmas 4.4 and Lemma 4.14 allow us to eliminate any solvable groups from this list, leaving only $2^4 : \mathrm{Sym}(5)$ and $\mathrm{Sym}(6) \times 2$.

By Theorem 3.1, every homomorphism $\mathcal{E}_4 \to \mathrm{Sym}(5)$ is the composite of the Coxeter quotient map and an an automorphism of $\mathrm{Sym}(5)$. In particular, if the maximal proper subgroup is $2^4 : \mathrm{Sym}(5)$, then the image of s_1 must have order 2. If $\mathcal{A}_4$ instead had image contained in $\mathrm{Sym}(6) \times 2$, then by Lemma 3.3, either the image of $\mathcal{A}_4$ is cyclic or the image of $\mathcal{A}_4$ is $\mathrm{Sym}(5)$ and the homomorpism factors through $W(\mathcal{A}_4)$. It follows that either the entire quotient is cyclic by Lemma 4.14 or $\alpha(s_1)$ has order 2 as required. □

7. The Artin group $\mathcal{E}_7$

In this section we will compute the smallest non-abelian quotient of $\mathcal{E}_7$. Throughout we will use Notation 2.1.

7.A. The groups.

Lemma 7.1. *It suffices to exclude subgroups of the following groups:*

(1) $\mathrm{Aut}(Q)$ *for* Q *in Table 3;*
(2) $\mathrm{Aut}(Q) \wr 2$ *for* Q *one of* $\mathrm{PSL}_2(7)^2$, $\mathrm{PSL}_2(8)^2$, $\mathrm{PSL}_2(11)^2$, *and* $\mathrm{PSL}_2(13)^2$*;*
(3) $\mathrm{Aut}(\mathrm{Alt}(5)^3) \wr 6$*; and*
(4) $\mathrm{Aut}(\mathrm{Alt}(6)^2) \wr 2$*.*

Proof. We stress that these conditions are sufficient but in the case where the socle is a direct product they are overkill. By Lemma 2.4, it suffices to consider groups of the shape $S^n.k$ for some simple group S and where $|S^n| \leqslant |W(\mathcal{E}_7)/Z(W(\mathcal{E}_7))|$ and k divides $|\mathrm{Out}(S^n)|$. All simple groups S with $|S| \leqslant |W(\mathcal{E}_7)/Z(W(\mathcal{E}_7))|$ are listed in Table 3. The table also lists their orders, and the remainder of the statement follows from the universal embedding theorem [KK51]. □

7.B. Excluding the family $\mathtt{A}_1$.

To exclude groups in the family $\mathtt{A}_1$, we will work with the commutator subgroup $\mathcal{E}_7'$ of $\mathcal{E}_7$. As $\mathcal{E}_7'$ is a perfect group, it will suffice to rule out the simple groups $\mathrm{PSL}_2(q)$ rather than the groups of shape $\mathrm{PSL}_2(q)^n.k$.

Lemma 7.2. *Let* $Q \cong \mathrm{PSL}_2(q)$ *where* $q = p^k$ *with* p *a prime. If* $\alpha \colon \mathcal{E}_7' \to Q$ *is a homomorphism, then* α *has trivial image.*

Proof. Let X be the commutative totally symmetric set in Lemma 4.9. We aim to understand $H = \mathrm{Stab}_Q(\alpha(X))$. Since H is the stabiliser of a commutative totally symmetric set of size 3, we see that H admits a surjection to $\mathrm{Sym}(3)$. We now proceed as in [CK20, Lemma 4]. By [Kin05, §2.1] [1], subgroups of $\mathrm{PSL}_2(q)$ are as follows:

(1) elementary abelian p-groups;
(2) cyclic groups of order ℓ where $\ell|(q \pm 1)/d$ where $d = \gcd(q-1, 2$;
(3) dihedral groups of order 2ℓ with ℓ as in (2);
(4) $\mathrm{Alt}(4)$ if $p > 2$ or if $p = 2$ and $k \equiv 0 \pmod 2$;
(5) $\mathrm{Sym}(4)$ if $q^2 - 1 \equiv 0 \pmod{16}$;
(6) $\mathrm{Alt}(5)$ if $p = 5$ or $q^2 - 1 \pmod 5$;
(7) $p^m : t$ where $t|(p^m - 1)$ and $(q-1)/d$, note that in this case either the subgroup is dihedral as in case (3), or it is contained in the stabiliser of a point $\mathbf{P}^1_q$;
(8) $\mathrm{PSL}_2(p^{k_0})$ where $k_0|k$;
(9) $\mathrm{PGL}_2(p^{k_0})$ where $2k_0|k$.

Our goal now is to rule out the above cases for H. Since cases 1, 2, 4, 6, 8, and 9 do not admit $\mathrm{Sym}(3)$ as an epimorphic image, we can already rule them out. By [KLP21, Theorem 3.3], Dihedral groups do not contain a totally symmetric set of size 3, so we may rule out case 3.

[1] This reference is secondary; the canonical reference is [Dic58], but the first proofs may be found in [Wim00, HM03])

Family	Group	Order	Out(G)
Alternating	Alt(5)	60	2
	Alt(6)	360	2^2
	Alt(7)	2,520	2
	Alt(8) $\cong$ $\mathrm{PSL}_4(2)$	20,160	2
	Alt(9)	181,440	2
$\mathtt{A}_1$	$\mathrm{PSL}_2(q)$ $q \in \{7, 8, 11, 13, 17, 19, 16, 23, 25, 27, 29, 31, 32, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 64, 81, 83, 89, 97, 101, 103, 107, 109, 113, 121, 125, 127, 131, 137, 139\}$	$\frac{1}{\gcd(2,q-1)}q(q^2-1)$	$d \times k$ where $d = \gcd(2, q-1)$ and $q = p^k$ for p a prime
$\mathtt{A}_2$	$\mathrm{PSL}_3(3)$	5,616	2
	$\mathrm{PSL}_3(4)$	20,160	$2 \times \mathrm{Sym}(3)$
	$\mathrm{PSL}_3(5)$	372,000	2
${}^2\mathtt{A}_2$	$\mathrm{SU}_3(3)$	6,048	2
	$\mathrm{SU}_3(4)$	62,400	4
	$\mathrm{PSU}_3(5)$	126,000	Sym(3)
$\mathtt{C}_2$	$\mathrm{PSp}_4(3) \cong \mathrm{SU}_4(2)$	25,920	2
	$\mathrm{Sp}_4(4)$	979,200	4
Exceptional	${}^2\mathtt{B}_2(8)$	29,120	3
Sporadic	M_{11}	7,920	1
	M_{12}	95,040	2
	M_{22}	443,520	2
	J_1	175,560	1
	J_2	604,800	2

Table 3. Simple groups with order less than 1,451,520.

We now rule out case 5 where $H = \mathrm{Sym}(4)$. In this case the totally symmetric of size three is exactly the kernel of the projection $\mathrm{Sym}(4) \twoheadrightarrow \mathrm{Sym}(3)$ and so is contained in a Klein 4 group K. Write c_i for the image of $s_1 s_i^{-1}$ so $K = \{\mathrm{id}, c_2, c_5, c_7\}$. We have that $c_i c_j = c_\ell$ for any distinct $i, j, \ell \in \{2, 5, 7\}$. Now,

$$c_4c_7c_4c_5 = c_4(c_7c_5)c_4 = c_4c_2c_4 = c_2c_4c_2 = (c_7c_5)c_4(c_7c_5) = c_7c_4c_7c_5^2 = c_4c_7c_4c_5^2.$$

Hence $c_5 = 1$, which would imply the homomorphism α is trivial.

The remaining case is $H \cong p^m : t$ with $k \leqslant m$ and $t|q-1$, which was not treated by Caplinger–Kordek. In this case, H is a subgroup of M, the stabiliser of a point in $\mathbf{P}^1_q$. In particular, H is a subgroup of a maximal subgroup M isomorphic to a subgroup of the affine general linear group $\mathrm{AGL}_1(q) \cong \mathbb{F}_q \rtimes (\mathbb{F}_q^\times)$.

The subgroup H normalises $\langle X \rangle$, and elements of X commute, so X must be contained in the maximal non-trivial elementary abelian normal subgroup of M isomorphic to $\mathbb{F}_q$. This implies that the homomorphism of $H \to \mathrm{Sym}(X)$ induced by conjugation on X must factor through a cyclic group of order $q-1$, contradicting the assumption that X is totally symmetric of size 3.

It follows from the above analysis that α cannot be injective on X. By Lemma 2.5, the image of α must be cyclic. Since $\mathcal{E}_7'$ is perfect, the image of α is in fact trivial. □

Corollary 7.3. *Let $Q \cong \mathrm{PSL}_2(q)^\ell.k$ for some $k, \ell \geqslant 1$. If $\alpha\colon \mathcal{E}_7 \to Q$ is a homomorphism, then α has cyclic image.*

Proof. The homomorphism α induces a map $\mathcal{E}_7' \to \mathrm{PSL}_2(q)$. Lemma 7.2 then implies that the image $\alpha(\mathcal{E}_7')$ is trivial. In particular, α factors through $\mathcal{E}_7^{\mathrm{ab}} \cong \mathbb{Z}$, concluding the proof. □

7.C. 3^2s and 7s.

Lemma 7.4. *Let be Q be a group in Lemma 7.1 such that $Q \ncong \mathrm{Alt}(9)$ or a group in the family $\mathtt{A}_1$ or $\mathtt{A}_1 \times \mathtt{A}_1$. If $\alpha\colon \mathcal{E}_7 \to \mathrm{Aut}(Q)$ is a homomorphism, then α has cyclic image.*

Proof. We first deal with groups of type (1) in Lemma 7.1. Let Q be one of Alt(5), Alt(6), $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(5)$, $\mathrm{SU}(3, 4)$, $\mathrm{PSp}_4(3)$, $\mathrm{Sp}_4(4)$, M_{11}, or M_{12}. Then $|Q|$ is not divisible by 7. Since $|\,\mathrm{Out}(Q)| =$

$2, 3, 2, 2, 4, 2, 4, 1, 2$ respectively, we see that $|\operatorname{Aut}(Q)|$ is not divisible by 7. Consequently, $\operatorname{Aut}(Q)$ does not contain any elements of order 7. Lemma 4.17 then implies that the image of α is cyclic.

The group ${}^2\mathtt{B}_2(8)$ does not have order divisible by 3, so it cannot have elements of order 9. In the remaining cases, we will examine the Sylow 3-subgroups, which can be found in [CCN+85]. Since $\operatorname{Out}({}^2\mathtt{B}_2(8)) \cong 3$, $\operatorname{Aut}({}^2\mathtt{B}_2(8))$ has its 3-Sylow subgroups isomorphic to 3. A 3-Sylow subgroup of $\operatorname{Aut}(J_1)$ is isomorphic to 3. A 3-Sylow subgroup of $\operatorname{Aut}(Q)$ for Q one of $\operatorname{Alt}(7)$ or $\operatorname{Alt}(8)$ is isomorphic to 3^2. A 3-Sylow subgroup of $\operatorname{Aut}(Q)$ for Q one of $\mathrm{PSL}_3(4)$, $\mathrm{SU}_3(3)$, M_{22}, J_2 is isomorphic to $\mathrm{He}_3(3)$. A 3-Sylow subgroup of $\mathrm{PSU}_3(5)$ is isomorphic to 3^2. The automorphism group of $\mathrm{PSU}_3(5)$ contains ${}_3(5)$ as an index 2 subgroup and a 3-Sylow subgroup of $\mathrm{PSU}_3(5)$ is isomorphic to $\mathrm{He}_3(3)$. Hence, a 3-Sylow subgroup of $\operatorname{Aut}(\mathrm{PSU}_3(5))$ is isomorphic to $\mathrm{He}_3(3)$.

Since neither 3, 3^2, nor $\mathrm{He}_3(3)$ has elements of order 9, Lemma 4.17 implies that α has cyclic image. This proves the result for all groups in the hypothesis of type (1).

We now deal with types (3) and (4). The groups $\operatorname{Aut}(\operatorname{Alt}(5)) \wr 6$ and $\operatorname{Aut}(\operatorname{Alt}(6)^2) \cong (\operatorname{Sym}(6).2) \wr 2$ do not have an element of order 7. We again conclude by Lemma 4.17. □

7.D. **The alternating group of degree 9.**

Lemma 7.5. *If $\alpha\colon \mathcal{E}_7 \to \operatorname{Sym}(9) = \operatorname{Aut}(\operatorname{Alt}(9))$ is a homomorphism, then α has cyclic image.*

Proof. First, observe that since $\mathcal{A}_5$ is contained in the centraliser of s_1, the image of s_1 must have a centraliser with size at least 720. Otherwise, the image of $\mathcal{A}_5$ is cyclic by Theorem 3.1, and the image of α is cyclic by Lemma 4.14. Furthermore, the image of s_1 must have order larger than 2. If its image has order 2, then the homomorphism would factor through $W(\mathcal{E}_7)$, and this is impossible because no non-abelian subgroup of $\operatorname{Sym}(9)$ is a quotient of $W(\mathcal{E}_7)$.

The only conjugacy class of $\operatorname{Alt}(9)$ of elements of order greater than 2 with centraliser of size bigger than 720 is the conjugacy class of 3-cycles of shape $(1\ 2\ 3)$. This has centraliser isomorphic to $3 \times \operatorname{Sym}(6)$. Suppose for contradiction that $\alpha(s_1) = (a\ b\ c)$ and α has non-cyclic image. Then $\alpha(\mathcal{A}_{\{2,4,5,6,7\}})$ is exactly equal to $\operatorname{Sym}([9]\backslash\{a, b, c\})$. By Theorem 3.1, this quotient is unique up to an automorphism of $\operatorname{Sym}(6)$ and is induced by imposing the relation s_2^2. Since s_1 is conjugate to s_2, the element s_1 must satisfy $s_1^3 = 1$ and $s_1^2 = 1$, implying it is trivial. Hence, α must have cyclic image. □

7.E. **Completing the proof.**

Theorem 7.6. *The smallest non-cyclic quotient of $\mathcal{E}_7$ is isomorphic to $\mathrm{Sp}_6(2) \cong W(\mathcal{E}_7)/Z(W(\mathcal{E}_7))$. Any such quotient factors though the quotient to $W(\mathcal{E}_7)$. In particular, the quotient is unique up to an automorphism of the image.*

Remark 7.7. Note that $\mathrm{Sp}_6(2) \cong \mathrm{PSp}_6(2) \cong \mathrm{GO}_7(2) \cong \mathrm{PSO}_7(2)$.

Proof. Let Q be a finite non-cyclic quotient of $\mathcal{E}_7$, and suppose $|Q| \leqslant |\mathrm{Sp}_6(2)| = 1451520$. By Lemma 7.1, Q is a subgroup of $\operatorname{Aut}(G)$ where G is one of the groups in Table 3, or Q is a subgroup of $\operatorname{Aut}(G^2) \wr 2$ where G is one of $\mathrm{PSL}_2(7)^2$, $\mathrm{PSL}_2(8)^2$, $\mathrm{PSL}_2(11)^2$, $\mathrm{PSL}_2(13)^2$, $\mathrm{PSL}(5)^2$, $\operatorname{Alt}(5)^2$ or $\operatorname{Alt}(6)^2$, or Q is a subgroup of $\operatorname{Aut}(\operatorname{Alt}(5))^3 \wr 6$.

By Corollary 7.3, the image of $\mathcal{E}_7$ under any homomorphism to $\operatorname{Aut}(\mathrm{PSL}_2(q)^k)$ for $k \geqslant 1$ is cyclic. Lemma 7.4 rules out all the remaining possibilities except $G \cong \operatorname{Alt}(9)$, and the case $G \cong \operatorname{Alt}(9)$ is ruled out by Lemma 7.5. Noting that $W(\mathcal{E}_7)$ is a quotient of $\mathcal{E}_7$ and that $\mathrm{Sp}_6(2)$ is non-abelian completes the proof that $\mathrm{Sp}_6(2)$ is the smallest possible non-cyclic quotient.

We now prove that the homomorphism is unique up to automorphisms of the image. As in the case of $\mathcal{E}_6$, we show that any homomorphism $\alpha\colon \mathcal{E}_7 \to \mathrm{Sp}_6(2)$ sending a generator to an element of order other than 2 must be cyclic. We first claim that $\alpha(s_1)$ must be contained in the conjugacy class 3_A in the ATLAS [CCN+85, Page 47]. Indeed, this follows from the Orbit-Stabiliser Theorem: the centraliser of $\alpha(s_1)$ contains $\alpha(\mathcal{A}_5)$. This must be noncylic by Lemma 4.14, so it must have order at least 720 by Theorem 3.1, and 3_A is the only class with a centraliser of order at least 720.

By the information on [CCN+85, Page 46], the normaliser of the subgroup generated by an element in the class 3_A is isomorphic to $\operatorname{Sym}(3) \times \operatorname{Sym}(6)$. It follows that $\alpha(\mathcal{A}_5)$ must surject onto $\operatorname{Sym}(6)$. But by Theorem 3.1, any such homomorphism factors through $W(\mathcal{A}_5)$. In particular, if s_1 has order other than 2, then α has cyclic image. □

8. The Artin group $\mathcal{E}_8$

In this section we will compute the smallest non-abelian quotient of $\mathcal{E}_8$. Throughout we will use Notation 2.1.

8.A. **The groups.** Table 4 recalls the finite simple groups of order at least $1,451,520$ but below $348,364,800$ which are not in the family $\mathtt{A}_1$.

Family	Group	Order	Out(G)	7s	9s
Alternating	Alt(10)	1,814,400	2	y	y
	Alt(11)	19,958,400	2	y	y
	Alt(12)	239,500,800	2	y	y
$\mathtt{A}_2$	$\mathrm{PSL}_3(7)$	1,876,896	Sym(3)	y	n
	$\mathrm{PSL}_3(8)$	16,482,816	6	y	y
	$\mathrm{PSL}_3(9)$	42,456,960	2^2	y	y
	$\mathrm{PSL}_3(11)$	212,427,600	2	y	n
	$\mathrm{PSL}_3(13)$	270,178,272	Sym(3)	y	n
$^2\mathtt{A}_2$	$\mathrm{PSU}_3(8)$	5,515,776	$3 \times \mathrm{Sym}(3)$	y	y
	$\mathrm{SU}_3(7)$	5,663,616	2	y	n
	$\mathrm{SU}_3(9)$	42,573,600	4	n	y
	$\mathrm{PSU}_3(11)$	70,915,680	Sym(3)	n	n
$\mathtt{C}_2$	$\mathrm{PSp}_4(5)$	4,680,000	2	n	n
	$\mathrm{PSp}_4(7) \cong \mathrm{P}\Omega_5(7)$	138,297,600	2	y	n
$\mathtt{C}_3$	$\mathrm{PSp}_6(2)$	1451520	1	y	y
$\mathtt{A}_3$	$\mathrm{PSL}_4(3)$	6,065,280	2^2	n	n
$^2\mathtt{A}_3$	$\mathrm{PSU}_4(3) \cong O_6^-(3)$	3,265,920	D_8	y	y
$\mathtt{A}_4$	$\mathrm{PSL}_5(2)$	9,999,360	2	y	n
$^2\mathtt{A}_4$	$\mathrm{PSU}_5(2)$	13,685,760	2	n	y
$\mathtt{D}_4$	$\mathrm{P}\Omega_4^+(2)$	174,182,400	6	n	n
$^2\mathtt{D}_4$	$\mathrm{P}\Omega_4^-(2)$	197,406,720	2	y	y
Exceptional	$^2\mathtt{B}_2(32)$	32,537,600	5	n	n
	$^2\mathtt{F}_4'(2)$	17,971,200	2	n	n
	$\mathtt{G}_2(3)$	4,245,696	2	y	y
	$^3\mathtt{D}_4(2)$	211,341,312	3	y	y
	$\mathtt{G}_2(4)$	251,596,800	2	n	y
Sporadic	M_{23}	10,200,960	1	y	n
	HS	44,352,000	2	y	n
	J_3	50,232,960	2	n	y
	M_{24}	244,823,040	2	y	y

Table 4. Simple groups of order at least 1,451,520 and less than 348,364,800 not in the family $\mathtt{A}_1$.

Lemma 8.1. *It suffices to exclude subgroups of the following groups:*

(1) $\mathrm{Aut}(Q)$ *for any group* Q *appearing in Table 3;*
(2) $\mathrm{Aut}(Q)$ *for any group* Q *appearing in Table 4;*
(3) $\mathrm{Aut}(\mathrm{Alt}(5)^5) \wr k$ *for* k *the order of an element of* $\mathrm{Sym}(5) \wr \mathrm{Sym}(5)$*;*
(4) $\mathrm{Aut}(\mathrm{Alt}(6)^3) \wr k$ *for* k *the order of an element of* $\mathrm{Sym}(6).2 \wr \mathrm{Sym}(3)$*;*
(5) $\mathrm{Aut}(Q^2) \wr |\mathrm{Out}(Q)|$ *for* Q *one of* M_11, $\mathrm{PSL}_3(3)^2$, $\mathrm{PSU}_3(3)^2$*;*
(6) $\mathrm{PSL}_2(q)^n.k$ *for any* $n, k \leqslant 12$ *and any* q*.*

Proof. We stress that these conditions are sufficient but in the case where the socle is either $\mathrm{PSL}_2(q)$ or a direct product, they are stronger than necessary. By Lemma 2.4, it suffices to consider groups of the shape $S^n.k$ for some simple group S and where $|S^n| \leqslant |W(\mathcal{E}_8)/Z(W(\mathcal{E}_8))|$ and k divides $|\mathrm{Out}(S^n)|$. All simple groups S with $|S| \leqslant |W(\mathcal{E}_8)/Z(W(\mathcal{E}_8))|$ are listed in Table 3 and Table 4. The tables also lists their orders, and the remainder of the statement follows from the universal embedding theorem [KK51]. □

Remark 8.2. The Coxeter group $W(\mathcal{E}_8)$ modulo its centre has a simple subgroup of index 2. As such, to show that a homomorphism $\alpha : \mathcal{E}_8 \to Q$ to a particular group Q is cyclic, it will often suffice to show that it must factor through the Coxeter group.

8.B. **3^2s and 7s.**

Lemma 8.3. *A non-cyclic quotient of $\mathcal{E}_8$ contains elements of order 7 and 9.*

Proof. Let $\alpha\colon \mathcal{E}_8 \to Q$ be a finite quotient with Q non-cyclic. The homomorphism α induces a homomorphism $\beta\colon \mathcal{E}_7 \to Q$. The image $P = \mathrm{im}(\beta)$ is a finite quotient of $\mathcal{E}_7$, and it is non-cyclic by Lemma 4.14. By Lemma 4.17 we see that P contains elements of order 7 and 9. Whence, the lemma. □

Lemma 8.4. *Let Q be one of* $\mathrm{PSL}_3(7)$, $\mathrm{PSL}_3(11)$, $\mathrm{PSL}_3(13)$, $\mathrm{SU}_3(7)$, $\mathrm{SU}_3(9)$, $\mathrm{PSU}_3(11)$, $\mathrm{PSp}_4(5)$, $\mathrm{PSp}_4(7)$, $\mathrm{PSL}_4(3)$, $\mathrm{GL}_5(2)$, $\mathrm{PSU}_5(2)$, ${}^2\mathtt{B}_2(32)$, ${}^2\mathtt{F}_4(2)$, $\mathtt{G}_2(4)$, M_{23}, HS, *or* J_3. *If $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(Q)$ is a homomorphism, then α has cyclic image.*

Proof. Table 4 lists whether each group contains elements of order 7 and 9. With one exception, this information for Q and $\mathrm{Aut}(Q)$ can be found in [CCN+85]. If the group does not contain an element of order 7 and an element of order 9, we conclude by Lemma 8.3. The element order information for $Q \cong \mathrm{PSL}_3(13)$ is not in the ATLAS, so we verify there are no elements of order 9 in $\mathrm{Aut}(Q)$ via MAGMA; see Appendix B.B. □

8.C. **Alternating groups.**

Lemma 8.5. *Any homomorphism $\mathcal{E}_8 \to \mathrm{Sym}(12)$ has cyclic image.*

Proof. Let $\alpha\colon \mathcal{E}_8 \to \mathrm{Sym}(12)$ be a homomorphism We aim to constrain the possible images of the generator s_8. The element $\alpha(s_8)$ cannot have order 2 because this would imply that α factors through the quotient to $W(\mathcal{E}_8)$, and $\mathrm{Sym}(12)$ is not a quotient of $W(\mathcal{E}_8)$; see Remark 8.2.

The group $\mathrm{Sym}(12)$ has 77 conjugacy classes. Of these, 70 contain non-trivial elements with order distinct from 2. The third smallest conjugacy class of these 70 is the class of 5-cycles $(a\ b\ c\ d\ e)$ consisting of $\binom{12}{5} \times 4! = 19008$ elements. By the Orbit-Stabiliser Theorem, the third largest centraliser of a conjugacy class of non-trivial elements of $\mathrm{Sym}(12)$ with order distinct from 2 is $25,200$. This is smaller than the smallest non-abelian quotient of $\mathcal{E}_6$. It follows that if α had non-cyclic image, then $\alpha(s_8)$ must be contained in the one of the two smallest conjugacy classes of non-trivial elements of order not equal to 2.

The second smallest conjugacy class of the 70 classes is the class of 4-cycles $(a\ b\ c\ d)$ consisting of $\binom{12}{4} \times 3! = 2970$ elements. The centraliser of such an element has order $161,280$ and is isomorphic to $4 \times \mathrm{Sym}(8)$. In particular, we obtain a homomorphism $\mathcal{E}_6 \to \mathrm{Sym}(8)$. Since $|\mathrm{Sym}(8)| = 8! = 40,320$ is smaller than $51,840$, $\alpha(\mathcal{E}_6)$ must have cyclic image by Theorem 6.14. Lemma 4.14 then implies that $\alpha(\mathcal{E}_8)$ itself is cyclic.

The smallest conjugacy class of the 70 classes is the class of 3-cycles $(a\ b\ c)$ consisting of $\binom{12}{4} \times 2! = 440$ elements. The centraliser of such an element has order 1,088,640 and is isomorphic to $3 \times \mathrm{Sym}(9)$. In particular, we obtain a homomorphism $\beta\colon \mathcal{E}_6 \to \mathrm{Sym}(9)$ such that every Artin generator is mapped to a cycle of shape 3^1.

The centraliser of $\beta(s_1)$ must contain $\beta(\mathcal{A}_4)$ where $\mathcal{A}_4$ is generated by s_2, s_4, s_5, s_6. The centraliser of a cycle of shape 3^1 on $\mathrm{Sym}(9)$ is isomorphic to $3 \times \mathrm{Sym}(6)$, so we obtain an induced homomorphism $\theta\colon \mathcal{A}_4 \to \mathrm{Sym}(6)$. By Lemma 3.3, this either factors through Coxeter group $W(\mathcal{A}_4)$, implying that $\beta(s_2)$ has order 2 and contradicting the fact it has order 3, or the homomorphism β has cyclic image. We conclude that α has cyclic image as required. □

8.D. **Classical groups.** We begin by observing the folloiwing special case of Lemma 4.15.

Lemma 8.6. *Let Q be a non-trivial group with trivial centre, and let $\alpha\colon \mathcal{E}_8 \to Q$ be a homomorphism. The image $\alpha(\mathcal{E}_6)$ of the standard parabolic subgroup of type $\mathcal{E}_6$ must be a proper subgroup of Q.*

Proof. The standard parabolic subgroup of type $\mathcal{E}_6$ is contained in $\Gamma - \mathrm{St}(s_8)$. The result then follows from Lemma 4.15. □

Lemma 8.7. *If $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(\mathrm{PSL}_3(8))$ is a homomorphism, then α has cyclic image.*

Proof. Let $Q = \mathrm{PSL}_3(8)$ and recall that $\mathrm{Out}(Q) = 6$ from Table 4. In particular, $G = \mathrm{Aut}(\mathrm{PSL}_3(8))$ is a group of shape $\mathrm{PSL}_3(8).6$. Let $\alpha\colon \mathcal{E}_8 \to G$ be a homomorphism. The image of $\mathcal{E}_6$ under α must be a proper subgroup of G by Lemma 8.6.

By Corollary 6.5 if the image of α is non-cyclic and contained in Q, then $\mathcal{E}_6$ admits a non-cyclic homomorphism to $\mathrm{PGL}_2(8^n)$ for some $n \in \{1, 2, 3\}$. We claim we may take $n = 1$. One could prove this by examining the minimal polynomial of an element of $\mathrm{GL}_3(q)$; instead, we observe from [CCN+85, p. 73] that Q has no subgroups isomorphic to $\mathrm{PSL}_2(64)$ or $\mathrm{PSL}_2(512)$. By Lemma 6.7, every homomorphism $\mathcal{E}_6 \to \mathrm{PSL}_2(8)$ has cyclic image. Thus if α is non-cyclic, $\alpha(\mathcal{E}_6)$ must not be contained in Q.

Consulting [CCN+85, p. 73], we see that a proper maximal subgroup of G is isomorphic to one of the following

(1) $2^{3+6} : 7^2 : 6$, $73 : 18$, $7^2 : (6 \times \mathrm{Sym}(3))$,
(2) $(D_{14} \times \mathrm{PSL}_2(8)) : 3$,
(3) $\mathrm{PSL}_2(7) : 2 \times 3$,
(4) $Q.2$ and $Q.3$.

The maximal subgroups of $Q.2$ and $Q.3$ are isomorphic to subgroups of the maximal subgroups of $Q.6$ with extra addition of the subgroup Q. The subgroups Q, $Q.2$, and $Q.3$ do not centralise any non-trivial elements, so the image of $\mathcal{E}_6$ cannot be equal to one of these. Thus, $\alpha(\mathcal{E}_6)$ must be contained in a group of type (1), (2), or (3). The groups of type (1) are solvable, so any homomorphism $\mathcal{E}_6 \to M$ with M of type (1) has cyclic image by Lemma 4.4. A group of type (2) has order at most 42336, and a group of type (3) has order 1008. Both of these are smaller than the smallest non-cyclic quotient of $\mathcal{E}_6$, so we conclude by Theorem 6.14 and Lemma 4.14. □

Lemma 8.8. *If $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(\mathrm{PSL}_3(9))$ is a homomorphism, then α has cyclic image.*

Proof. Let $Q = \mathrm{PSL}_3(9)$ and recall that $\mathrm{Out}(Q) = 2^2$. Let $G = \mathrm{Aut}(\mathrm{PSL}_3(9))$, and note that because $\mathcal{E}_8^{ab}$ is cyclic, any homomorphism $\mathcal{E}_8 \to G$ must be contained in an index 2 subgroup of the shape $Q.2$. By [CCN+85, p. 78], any non-solvable maximal subgroup of an extension $Q.2$ admits an epimorphism with solvable kernel onto one of the following groups:

(1) Q;
(2) $\mathrm{Aut}(\mathrm{PSU}_3(3))$;
(3) $\mathrm{Aut}(\mathrm{PSL}_3(3))$; or
(4) $\mathrm{Aut}(\mathrm{Alt}(6))$.

Note that $\mathrm{Alt}(6) \cong \mathrm{PSL}_2(9)$. Lemmas 6.5 and 6.2 imply that that α cannot have a non-cyclic image contained in a group of type (1). If α has non-cyclic image, then $\alpha(\mathcal{E}_6)$, which is a proper subgroup of $\alpha(\mathcal{E}_8)$ by Lemma 8.6, must be contained in a group of type (1)-(4). Now, $\alpha(\mathcal{E}_6)$ cannot be equal to a group of type (1) since it does not centralise any non-trivial elements of G. If $\alpha(\mathcal{E}_6)$ was a proper subgroup of a group of type (1), then $\alpha(\mathcal{E}_6)$ would admit a quotient into a subgroup of a group of type (2), (3), or (4). Thus, we need to rule out the existence of homomorphisms $\mathcal{E}_6 \to M$ with non-cyclic image where M is one types (2), (3), or (4). This is handled by Lemma 6.11, Lemma 6.10, and Lemma 6.2 respectively. □

Lemma 8.9. *If $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(\mathrm{PSU}_3(8))$ is a homomorphism, then α has cyclic image.*

Proof. Let $Q = \mathrm{PSU}_3(8)$, and recall that $\mathrm{Out}(Q) = 3 \times \mathrm{Sym}(3)$. Note that any homomorphism $\mathcal{E}_8 \to \mathrm{Aut}(\mathrm{PSU}_3(8))$ must have image contained in a subgroup of shape Q.k where $k = 2, 3, 6$.

We first treat the case that the image of α is contained in Q. Observe that $\alpha(\mathcal{E}_6)$ must be a proper subgroup of Q by Lemma 8.6. Thus $\alpha(\mathcal{E}_6)$ must be contained in some proper maximal subgroup. The only non-soluble maximal proper subgroup of Q is $3 \times \mathrm{PSL}_2(8)$, and the order of $3 \times \mathrm{PSL}_2(8)$ is $1,512$. This is smaller than the smallest non-cyclic quotient of $\mathcal{E}_6$. Theorem 6.14 and Lemma 4.14 then imply $\alpha(\mathcal{E}_8)$ is cyclic.

Now, suppose α has image contained in a group of shape $Q.2$ or $Q.3$. A non-soluble maximal subgroup of such a group admits a homomorphism to either Q or to $\mathrm{Aut}(\mathrm{PSL}_2(8))$. The subgroup $\alpha(\mathcal{E}_6)$ cannot be Q by the same argument as in the previous paragraph, so $\alpha(\mathcal{E}_6)$ must admit a homomorphism to $\mathrm{Aut}(\mathrm{PSL}_2(8))$. The order of $\mathrm{Aut}(\mathrm{PSL}_2(8))$ is 1512, so any homomorphism from $\mathcal{E}_6$ to $\mathrm{Aut}(\mathrm{PSL}_2(8))$ is cyclic by Theorem 6.14. Lemma 4.14 then implies that $\alpha(\mathcal{E}_8)$ is cyclic.

Finally, suppose α has image contained in a group of shape $Q.6$. The only possible images for $\alpha(\mathcal{E}_6)$ which have not already been addressed are a group of shape $Q.2$ or $Q.3$. Subgroups of this form do not centralize any non-trivial elements, concluding the proof. □

Lemma 8.10. *If $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(\mathrm{PSU}_4(3))$ is a homomorphism, then α has cyclic image.*

Proof. Let $Q = \mathrm{PSU}_4(3)$, and recall $\mathrm{Out}(Q) = Dh_4$ which has order 8. If α factors through the canonical Coxeter quotient, it must be cyclic; see Remark 8.2. Otherwise, the image of an Artin generator has order larger than 2. By [CCN+85, Page 54], the largest centraliser of a class of elements of order greater than 2 in Q has size 5832. This is smaller than the smallest non-cyclic quotient of $\mathcal{E}_6$, so Lemma 4.14 implies that $\alpha(\mathcal{E}_8)$ is cyclic if it lies in Q.

We now show that any homomorphism $\alpha\colon \mathcal{E}_8 \to Q.2$ has cyclic image. According to the ATLAS [CCN+85, Page 52], there are exactly three non-isomorphic almost simple groups of shape Q.2. We study the possible images of $\mathcal{E}_6$ in a group of shape $Q.2$. By Lemma 8.6, $\alpha(\mathcal{E}_6)$ must be contained in a proper maximal subgroup of $Q.2$. Furthermore, this subgroup cannot be Q because Q does not centralize any non-trivial element.

The remaining possibilities are a soluble group or one of $3^4 : \mathrm{Alt}(6)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$, $2^4 : \mathrm{Alt}(6)$, $\mathrm{Alt}(7)$, $\mathrm{Alt}(6).2$, $\mathrm{PSU}_4(2) : 2$, $\mathrm{PSL}_3(4) : 2$, $\mathrm{PSU}_3(3) \times 2$, $\mathrm{Alt}(6).2^2$, $3^4 : \mathrm{Sym}(6)$, $\mathrm{PSU}_4(2) \times 2$, $2^4 : \mathrm{Sym}(6)$, $2^5 : \mathrm{Alt}(6)$, $\mathrm{Sym}(7)$, $3^4 : (\mathrm{Alt}(6).2)$, $\mathrm{PSU}_3(3)$, and $(2^4 \colon \mathrm{Alt}(6)).2)$. Note that by [JLPW95, Pg. 300], the group $2^4 : \mathrm{Sym}(6)$ is incorrectly listed in [CCN+85] and should be replaced by a group of shape $(2^4 \colon \mathrm{Alt}(6)).2$ as we have done. It follows that $\mathcal{E}_6$ admits a homomorphism to $\mathrm{Aut}(L)$ for L one of $\mathrm{Alt}(6)$, $\mathrm{Alt}(7)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$, or $\mathrm{PSU}_4(2)$. The only case where this is not immediate is $M = (2^4 \colon \mathrm{Alt}(6)).2$, where one observes first that the abelian factor of $\mathrm{Soc}(M)$ is exactly 2^4 and so one may pass to the quotient $\mathrm{Alt}(6).2$. The first two possibilities for $\mathrm{Aut}(L)$ are ruled out by Lemma 6.2, and the second two are ruled out by Lemma 6.11.

The group $\mathrm{PSU}_4(2) : 2$ is a quotient of $\mathcal{E}_6$, but by Theorem 6.14, every quotient map $\mathcal{E}_6 \to \mathrm{PSU}_4(2) : 2$ differs from the standard one by an automorphism of the image. Since all Artin generators are conjugate and thus have the same order in any quotient, this would imply the image of $\mathcal{E}_8$ in $Q.2$ factors through $W(\mathcal{E}_8)$. This is only possible if $\alpha(\mathcal{E}_8)$ is cyclic.

A nearly identical argument handles the case of $Q.4$. Since any other possible extension is by a soluble but not cyclic group, we conclude that every homomorphism $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(\mathrm{PSU}_4(3))$ has cyclic image. ☐

Lemma 8.11. *If $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(\mathrm{PSp}_6(2))$ is a homomorphism, then α has cyclic image.*

Proof. Let $Q = \mathrm{PSp}_6(2)$ and recall $\mathrm{Out}(Q) = 1$. By [CCN+85, pg. 47], the largest centraliser of a conjugacy class of non-trivial elements of order larger than 2 in $\mathrm{PSp}_6(2)$ has order 648. This is smaller than the smallest non-cyclic quotient of $\mathcal{E}_6$, so $\alpha(\mathcal{E}_6)$ is cyclic by Theorem 6.14, and $\alpha(\mathcal{E}_8)$ is consequently cyclic by Lemma 4.14. If the image of s_8 has order 2, we instead conclude by Remark 8.2. ☐

Lemma 8.12. *If $\alpha\colon \mathcal{E}_8 \to \mathrm{P}\Omega_4^+(2)$ is a homomorphism, then α has cyclic image.*

Proof. Let $Q = \mathrm{P}\Omega_4^+(2)$. By [CCN+85, Page 86], every conjugacy class of non-trivial elements of order greater than 2 in Q has centraliser smaller than the smallest non-cyclic quotient of $\mathcal{E}_6$ except for three classes of order 3 elements: classes 3_A, 3_B, and 3_C in the ATLAS. Each of these three classes of order 3 elements has centraliser of order $77,760$, and the ATLAS prints their normalisers as $3 \times \mathrm{PSU}_4(2)) : 2$ of order $155,520$. It follows that the centraliser must be index 2 and isomorphic to $3 \times \mathrm{PSU}_4(2)$. The map α cannot restrict to a non-cyclic map $\mathcal{E}_6 \to \mathrm{PSU}_4(2)$ by Lemma 6.11, and any homomorphism from $\mathcal{E}_6$ to an abelian group is cyclic. The entire homomorphism α then has cyclic image by either Lemma 4.14 or Remark 8.2. ☐

Lemma 8.13. *If $\alpha\colon \mathcal{E}_8 \to \mathrm{P}\Omega_4^-(2)$ is a homomorphism, then α has cyclic image.*

Proof. Let $Q = \mathrm{P}\Omega_4^-(2)$ and let $\alpha\colon \mathcal{E}_8 \to Q$ be a homomorphism. By [CCN+85, Page 88] and the Orbit-Stabiliser Theorem, the only conjugacy class of elements with order larger than 2 and with centraliser larger than the smallest non-cyclic quotient of $\mathcal{E}_6$ is the class 3_A, whose centraliser has order 60480. The ATLAS lists the normaliser of a cyclic subgroup generated by an element in the class 3_A as $(3 \times \mathrm{Alt}(8)) : 2$ of order 120,960. It follows the centraliser of such an element has index 2 and is isomorphic to $3 \times \mathrm{Alt}(8)$. By Lemma 6.2 every homomorphism $\mathcal{E}_6 \to \mathrm{Alt}(8)$ has cyclic image, so α must have cyclic image by Lemma 4.14 or Remark 8.2. ☐

8.E. **Exceptional and sporadic groups.**

Lemma 8.14. *If $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(\mathtt{G}_2(3))$ is a homomorphism, then α has cyclic image.*

Proof. Let $Q = G_2(3)$ and recall $G = \mathrm{Aut}(Q) = \mathtt{G}_2(3).2$. If the order of $\alpha(s_8)$ is 2, then we conclude by Remark 8.2. Otherwise, by [CCN+85, Page 61], the smallest conjugacy class of elements of order larger than 2 in Q is of size 728. By the Orbit-Stabiliser theorem, the centraliser of an element in this conjugacy class must have order 5832 in Q. This is smaller than the smallest non-cyclic quotient of $\mathcal{E}_6$, so if $\alpha(\mathcal{E}_8)$ lies in Q, it is cyclic by Theorem 6.14 and Lemma 4.14.

Now, consider a homomorphism $\alpha\colon \mathcal{E}_8 \to G$. The subgroup $\alpha(\mathcal{E}_6)$ is proper by Lemma 8.6, so the image of $\mathcal{E}_6$ must be contained in either Q or a subgroup a group of shape $\mathrm{PSU}_3(3):2$, $\mathrm{PSL}_3(3):2$, $\mathrm{PSL}_2(8):3\times 2$, $2^3\cdot\mathrm{PSL}_3(2):2$, $\mathrm{PSL}_2(13):2$, or a soluble group. In each case, we obtain a homomorphism from $\mathcal{E}_6$ to one of $\mathrm{Aut}(L)$ where $L \in \{\mathrm{PSU}_3(3), \mathrm{PSL}_3(2), \mathrm{PSL}_2(8), \mathrm{PSL}_2(13)\}$. Homomorphisms to the latter three automorphism groups have cyclic image by Lemmas 6.7, 6.9, and 6.11. The entire homomorphism α must then have cyclic image by Lemma 4.14. □

Lemma 8.15. *If $\alpha\colon \mathcal{E}_8 \to {}^3\mathtt{D}_4(2)$ is a homomorphism, then α has cyclic image.*

Proof. Let $G = {}^3\mathtt{D}_4(2)$ and $\alpha\colon \mathcal{E}_8 \to G$ be a homomorphism. By [CCN+85, Page 90], the largest centraliser of an element of order larger than 2 has order 3,584. This is smaller than the smallest non-cyclic quotient of $\mathcal{E}_6$, so we conclude that α must have cyclic image by Theorem 6.14 and Lemma 4.14 or Remark 8.2. □

Lemma 8.16. *If $\alpha\colon \mathcal{E}_8 \to M_{24}$ is a homomorphism, then α has cyclic image.*

Proof. Let $G = M_{24}$ and $\alpha\colon \mathcal{E}_8 \to G$ be a homomorphism. By [CCN+85, Page 96], the largest centraliser of an element of order larger than 2 has order 1,080. This is smaller than the smallest non-cyclic quotient of $\mathcal{E}_6$, so α must have cyclic image by Theorem 6.14 and Lemma 4.14 or Remark 8.2. □

8.F. **Products.**

Lemma 8.17. *Let Q be one of*

(1) $\mathrm{Aut}(\mathrm{Alt}(5)^5) \wr k$ *for k the order of an element of* $\mathrm{Sym}(5)\wr\mathrm{Sym}(5)$*;*
(2) $\mathrm{Aut}(\mathrm{Alt}(6)^3) \wr k$ *for k the order of an element of* $\mathrm{Sym}(6).2\wr\mathrm{Sym}(3)$*; or*
(3) $\mathrm{Aut}(Q^2) \wr |\mathrm{Out}(Q)|$ *for Q one of* M_{11}^2, $\mathrm{PSL}_3(3)^2$, *or* $\mathrm{PSU}_3(3)^2$*;*

If $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(Q)$ is a homomorphism, then α has cyclic image.

Proof. We first observe the structure of $\mathrm{Aut}(Q)$ in several of these cases: $\mathrm{Aut}(\mathrm{Alt}(5)^5) \cong \mathrm{Sym}(5)\wr\mathrm{Sym}(5)$ $\mathrm{Aut}(\mathrm{Alt}(6)^3) \cong (\mathrm{Sym}(6).2)\wr\mathrm{Sym}(3)$, $\mathrm{Aut}(M_{11}^2) \cong M_{11}\wr 2$, and $\mathrm{Aut}(\mathrm{PSL}_3(3)^2) \cong \mathrm{PGL}_3(3)\wr 2$. None of these groups have order divisible by 7, so they do not have any elements of order 7. Any positive integer k whose order divides both $\mathrm{Out}(Q^\ell)$ and $|Q^\ell|$ is not divisible by 7. Thus, the wreath product $\mathrm{Aut}(Q^\ell)\wr k$ does not contain any elements of order 7. The Universal Embedding Theorem [KK51] implies every possible group of shape $Q^\ell.k$ is a subgroup of $\mathrm{Aut}(Q^\ell)\wr k$ and so cannot have an element of order 7. Hence by Lemma 8.3, every homomorphism from $\mathcal{E}_8$ to one of these groups has cyclic image.

Now, $|\mathrm{Out}(\mathrm{Alt}(7)^2)| = 8$. Thus, any possible group of shape $\mathrm{Alt}(7)^2.\ell$ must have ℓ a power of 2 dividing 8. The group $\mathrm{Aut}(\mathrm{Alt}(7))\wr 8 \cong (\mathrm{Sym}(7)\wr 2)\wr 8$ does not have an element of order 9. Indeed, a 3-Sylow subgroup is isomorphic to the direct product of sixteen copies 3^2, i.e., it is isomorphic to 3^{32}. This is easily seen from the fact that a Sylow 3-subgroup of $\mathrm{Sym}(7)$ is isomorphic to 3^2. In particular, $\mathrm{Aut}(\mathrm{Alt}(7))\wr 8$ has no elements of order 9. By Lemma 8.3 every homomorphism from $\mathcal{E}_8$ to $\mathrm{Aut}(\mathrm{Alt}(7))\wr 8$ must be cyclic.

The group $\mathrm{Aut}(\mathrm{PSU}_3(3)^2 \cong \mathrm{PGU}_3(3)\wr 2$ does not have an element of order 9. Indeed, a 3-Sylow subgroup is isomorphic to the direct product of 2 copies of $\mathrm{He}_3(3)$, the isomorphism class of the 3-Sylow subgroup of $\mathrm{PSU}_3(3)$, since $|\mathrm{Out}(\mathrm{PSU}_3(3)^2)| = 8$. We conclude as in the last paragraph that every homomorphism $\mathcal{E}_8$ to $\mathrm{Aut}(\mathrm{PSU}_3(3)^2)\wr 8$ must be cyclic. □

Lemma 8.18. *Let $Q = \mathrm{PSL}_2(q)^n$. If $n \leqslant 12$, then any homomorphism $\alpha\colon \mathcal{E}_8 \to \mathrm{Aut}(Q)$ has cyclic image. If n is arbitrary, then any homomorphism $\alpha\colon \mathcal{E}_8 \to Q.n$ has cyclic image.*

Proof. First, we note that any homomorphism $\alpha\colon \mathcal{E}_8 \to \mathrm{PSL}_2(q)$ induces a homomorphism $\mathcal{E}_7' \to \mathrm{PSL}_2(q)$. By Lemma 7.2, the image of $\mathcal{E}_7'$ is trivial. This implies that $\alpha(\mathcal{E}_7)$ factors through the abelianization of $\mathcal{E}_7$, and is thus cyclic. By Lemma 4.14, this implies that $\alpha(\mathcal{E}_8)$ is also cyclic.

Next, we deal with the group $\mathrm{Aut}(\mathrm{PSL}_2(q)^n)) \cong \mathrm{Aut}(\mathrm{PSL}_2(q))\wr\mathrm{Sym}(n)$ with $n \leqslant 12$. By Lemma 8.5, the projection of the image of such a homomorphism to $\mathrm{Sym}(n)$ must be cyclic. Let C denote this cyclic group. The image $\alpha(\mathcal{E}_7') < \alpha(\mathcal{E}_8')$ must be contained in the commutator subgroup of $\mathrm{Aut}(\mathrm{PSL}_2(q))\wr C$, which is a product of copies of $\mathrm{PGL}_2(q)$. Since $\mathcal{E}_7'$ is perfect, it in fact must be contained in a product of copies of $\mathrm{PSL}_2(q)$. Projection onto any factor induces a homomorphism $\mathcal{E}_7' \to \mathrm{PSL}_2(q)$, and every such homomorphism has trivial image by Lemma 7.2. Then $\alpha(\mathcal{E}_7)$ must be cyclic, so Lemma 4.14 implies $\alpha(\mathcal{E}_8)$ is cyclic. □

8.G. **Completing the proof.**

Theorem 8.19. *The smallest non-abelian quotient of $\mathcal{E}_8$ is isomorphic to $W(\mathcal{E}_8)/Z(W(\mathcal{E}_8))$. It is unique up to automorphisms of the image.*

Proof. We now exclude all groups listed in Lemma 8.1. Notice that any homomorphism $\alpha : \mathcal{E}_8 \to \mathrm{Aut}(S)$ induces a homomorphism $\mathcal{E}_7 \to \mathrm{Aut}(S^n)$ by restricting to the standard parabolic subgroup of type $\mathcal{E}_7$. Corollary 7.3 and Lemma 7.4 showed that any homomorphism from $\mathcal{E}_7$ to the automorphism group of any of the groups in Lemma 7.1 has cyclic image, so the image of $\mathcal{E}_8$ under any such homomorphism must also be cyclic by Lemma 4.14. This handles case (1).

Note that all of the groups in Table 4 are large enough that $|S^2| > |W(\mathcal{E}_8)|$, so in these cases, it suffices to consider only $\mathrm{Aut}(S)$. Lemma 8.4 rules out the possibility that the smallest non-abelian quotient Q of $\mathcal{E}_8$ is a subgroup of the automorphism group of any of the groups in Table 4 except possibly $\mathrm{Alt}(10)$, $\mathrm{Alt}(11)$, $\mathrm{PSL}_3(8)$, $\mathrm{PSL}_3(9)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSp}_6(2)$, $\mathrm{PSU}_4(3) \cong O_6^-(3)$, and $\mathtt{G}_2(3)$). Lemma 8.5 eliminates the possibilities that Q is a subgroup of either $\mathrm{Sym}(12) = \mathrm{Aut}(\mathrm{Alt}(12))$, $\mathrm{Aut}(\mathrm{Alt}(11))$ or $\mathrm{Aut}(\mathrm{Alt}(10))$, the latter two of which are subgroups of the former. Lemmas 8.7, 8.8, 8.9, 8.11, 8.10, and 8.14 eliminate the possibilities that Q is a subgroup of $\mathrm{Aut}(\mathrm{PSL}_3(8))$, $\mathrm{Aut}(\mathrm{PSL}_3(9))$, $\mathrm{Aut}(\mathrm{PSU}_3(8))$, $\mathrm{Aut}(\mathrm{PSp}_6(2))$, $\mathrm{Aut}(\mathrm{PSU}_4(3))$, or $\mathrm{Aut}(\mathtt{G}_2(3))$ respectively. This handles case (2).

The remaining cases correspond to the socle of the smallest non-abelian quotient Q of $\mathcal{E}_8$ being a direct product S^n. These are listed in Lemma 8.17 except when $S \cong \mathrm{PSL}_2(q)$ which is given in Lemma 8.18. The lemmas show any the image of $\mathcal{E}_8$ under any homomorphism to one of these cases is cyclic. This completes the proof that $\overline{W} = W(\mathcal{E}_8)/Z(W(\mathcal{E}_8))$ is the smallest non-abelian quotient of $\mathcal{E}_8$.

We now prove that every epimorphism $\mathcal{E}_8 \to \overline{W}$ factors through $W = W(\mathcal{E}_8)$. In particular, the quotient is unique up to automorphisms of the image. We have that $\overline{W} \cong \mathrm{P}\Omega_4^+(2).2$ and as usual we will consider possible element orders of the image s_8 and the images of $\mathcal{E}_6$ in it centraliser. The only conjugacy classes with centraliser at least as large as $W(\mathcal{E}_6)$ are those listed as 3_A, 3_B, and 3_C in the ATLAS [CCN+85, pg 86]. In particular, if the epimorphism does not factor through the canonical Coxeter quotient (up to automorphisms), then the images $\overline{s}_1, \dots \overline{s}_8$ of $s_1, \dots, s_8$ must have order 3 and be contained in one of 3_A, 3_B, or 3_C. The ATLAS lists these conjugacy classes as having normaliser isomorphic to $\mathrm{Sym}(3) \times \mathrm{SU}_4(2) : 2$. One deduces that the centraliser $C_{\overline{W}}(\overline{s}_8)$ must be of the shape $3 \times \mathrm{SU}(4) : 2$ with $C_{\overline{W}}(\overline{s}_8)/\langle \overline{s}_8 \rangle \cong \mathrm{SU}_4(2) \cong W(\mathcal{E}_6)$. But by Theorem 6.14, every homomorphism with non-cyclic image $\beta \colon \mathcal{E}_6 \to W(\mathcal{E}_6)$ is equal to the canonical Coxeter homomorphism up to an automorphism. But this forces $\overline{s}_1$ to have order 2, contradicting that it had order 3. □

9. Spherical Artin groups

By combining previous theorems of this paper together with the work of Kolay, we have proven the first of our main theorems.

Theorem A. *Let G be an irreducible spherical Artin group. The following conclusions hold:*

(1) *If G is isomorphic to $\mathcal{A}_{n-1}$, $\mathcal{B}_n$, or $\mathcal{D}_n$, for $n \geqslant 5$, then the smallest non-abelian quotient of G is* $\mathrm{Sym}(n)$ *and it is unique up to automorphisms of the image.*
(2) *If G is isomorphic to one of $\mathcal{A}_2$ $\mathcal{A}_3$, $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{D}_4$, $\mathcal{F}_4$, or $\mathcal{I}_2(m)$ with $4|m$, then the smallest non-abelian quotient of G is* $\mathrm{Sym}(3)$.
(3) *If G is isomorphic to $\mathcal{I}_2(m)$ with $4 \nmid m$ then the smallest non-abelian quotient of G is D_p, where p is the smallest odd prime divisor of m, and it is unique up to automorphisms of the image.*
(4) *If G is isomorphic to $\mathcal{H}_3$, then the smallest non-abelian quotient of G is* $\mathrm{Alt}(5)$. *Up to automorphisms of the image there are two such homomorphisms.*
(5) *If G is isomorphic to $\mathcal{H}_4$, $\mathcal{E}_6$, $\mathcal{E}_7$, or $\mathcal{E}_8$, then the smallest non-abelian quotient of G is isomorphic to $W(G)/Z(W(G))$ and it is unique up to automorphisms of the image.*

Proof. Combine Theorems 3.1, 3.9, 3.12, 3.13, 5.1, 5.3, 5.5, 6.14, 7.6, and 8.19. □

10. Affine Artin groups

Theorem B. *Let G be an irreducible affine Artin group. The following conclusions hold:*

(1) If G is isomorphic to $\widetilde{\mathcal{A}}_{n-1}$, $\widetilde{\mathcal{B}}_n$, $\widetilde{\mathcal{C}}_n$ or $\widetilde{\mathcal{D}}_n$, for $n \geqslant 5$, then the smallest non-abelian quotient of G is $\mathrm{Sym}(n)$.
(2) If G is isomorphic to one of $\widetilde{\mathcal{A}}_1$, $\widetilde{\mathcal{A}}_2$ $\widetilde{\mathcal{A}}_3$, $\widetilde{\mathcal{B}}_2$, $\widetilde{\mathcal{B}}_3$, $\widetilde{\mathcal{B}}_4$, $\widetilde{\mathcal{D}}_4$, $\widetilde{\mathcal{F}}_4$, or $\widetilde{\mathcal{G}}_2$, then the smallest non-abelian quotient of G is $\mathrm{Sym}(3)$.
(3) If G is isomorphic to $\widetilde{\mathcal{E}}_n$ for $n = 6, 7, 8$ then the smallest non-abelian quotient of G is isomorphic to $W(\mathcal{E}_n)/Z(W(\mathcal{E}_n))$.

Proof. The group $\widetilde{\mathcal{A}}_1$ is a rank 2 free group, so its smallest non-abelian quotient is $\mathrm{Sym}(3)$. Let $G = \widetilde{\mathcal{X}}_n$ be an irreducible affine Artin group such that $G \neq \widetilde{\mathcal{A}}_1$. Now, $W(\widetilde{\mathcal{X}}_n) \cong B_{\mathcal{X}} \rtimes W(\mathcal{X}_n)$ where $B_{\mathcal{X}}$ is free abelian and acts by translations on $\mathbb{E}^n$, see [Hum92, §4.2 Proposition]. Thus, we have an epimorphism

$$\widetilde{\mathcal{X}}_n \to W(\widetilde{\mathcal{X}}_n) \to W(\mathcal{X}_n).$$

This proves (2) noting $\mathcal{G}_2 \cong \mathcal{I}_2(6)$.

We now prove (1). We first prove the result for $G = \widetilde{\mathcal{A}}_{n-1}$ with $n \geqslant 5$. In particular, by Theorem A the image of any $\mathcal{A}_{n-1}$ in a finite non-abelian quotient of G must either be cyclic or have size at least $n!$ with equality if the image is isomorphic to $\mathrm{Sym}(n)$. By Lemma 4.11, the Artin generators of G form a collapsing set. If the image of any such $\mathcal{A}_{n-1}$ is cyclic, then the image of G must be cyclic as well by Lemma 4.14. Hence the smallest non-abelian quotient is $\mathrm{Sym}(n)$ as required.

We next prove the result for $G = \widetilde{\mathcal{B}}_n$ with $n \geqslant 5$. Note that there are two parabolic $\mathcal{A}_{n-1}$ subgroups of G that together generate a parabolic $\mathcal{D}_n$ subgroup. By Theorem A, in any quotient, these either have cyclic image or size at least $n!$, with equality if the image is isomorphic to $\mathrm{Sym}(n)$. We now proceed as in Theorem 3.12. If the image of either parabolic $\mathcal{A}_{n-1}$ is cyclic, then the quotient collapses the entire $\mathcal{D}_n$ to a cyclic group and the quotient must factor through $\langle \overline{s}_1, \overline{s}_2 \mid [\overline{s}_1, \overline{s}_2]_4,\ [\overline{s}_1, \overline{s}_2] \rangle \cong \mathbb{Z}^2$ where $\overline{s_1}$ generates the image of the parabolic $\mathcal{D}_n$ and $\overline{s_2}$ generates the image of the one remaining generator. In particular, the quotient of G must be abelian. Hence the smallest non-abelian quotient is $\mathrm{Sym}(n)$ as required.

The proofs of $\widetilde{\mathcal{C}}_n$ and $\widetilde{\mathcal{D}}_n$ with $n \geqslant 5$ are similar.

To prove (3) note that $\mathcal{E}_n$ is naturally a subgroup of $\widetilde{\mathcal{E}}_n$ for $n = 6, 7, 8$. By Lemma 4.11, the Artin generators of $G = \widetilde{\mathcal{E}}_n$ form a collapsing set. In particular, by Theorem A, the image of $\mathcal{E}_n$ in a finite non-abelian quotient of G must have size at least $|W(\mathcal{E}_n)/Z(W(\mathcal{E}_n))|$ with equality if the image is isomorphic to $W(\mathcal{E}_n)/Z(W(\mathcal{E}_n))$. Hence the theorem. □

11. Profinite rigidity

In this section we will prove Theorem C. The arguments are of a distinctly different flavour to much of the rest of the paper and combine Theorem A with some standard techniques from profinite rigidity. We the refer the reader to Reid's excellent survey [Rei15] which introduces all of the notions we will use below. We denote by $\widehat{G}$ the profinite completion of a group G. Note that by [DFPR82], if G is finitely generated this contains the same information as the set of isomorphism classes of finite quotient groups of G.

Lemma 11.1. *The following hold:*
(1) [Mar12, Proposition 1] *Let G be one of $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{D}_n$, and $\mathcal{I}_2(m)$. Then G is good the sense of Serre. Namely, for every finite discrete $\widehat{G}$-module M, the natural map $H^k(\widehat{G}; M) \to H^k(G; M)$ is an isomorphism for all $k \geqslant 0$.*
(2) Let G and H be irreducible spherical Artin groups of type A, D, or $I_2(2m+1)$. If $\widehat{G} \cong \widehat{H}$, then $G \cong H$.
(3) (a) For $n \geqslant 1$ we have $\mathrm{cd}(\widehat{\mathcal{A}}_n) = n$,
(b) for $n \geqslant 3$ we have $\mathrm{cd}(\widehat{\mathcal{B}}_n) = n$,
(c) for $n \geqslant 4$ we have $\mathrm{cd}(\widehat{\mathcal{D}}_n) = n$,
(d) $\mathrm{cd}(\widehat{\mathcal{I}_2(m)}) = 2$,
(e) $\mathrm{cd}(\widehat{\mathcal{F}}_4) \geqslant 3$.
(4) Neither $\mathcal{B}_3$ nor $\mathcal{B}_4$ admits $W(\mathcal{F}_4)$ as a quotient.

Proof. (1) is as cited. (2) follows from applying (1) to the discussion after Question 3 in [Mar12] which in turn depends on [CM14] (note this reference is quite secondary for cohomology of the Artin groups we are interested so we also refer the reader to the classics [Arn69, Del72, Bri73, Gor78, Vaj78]).

We now prove (3). Begin by letting G be one of the groups in cases (a)-(d). In case (d), we let $n = 2$. Note that the goodness proved in (1) implies that $\mathrm{cd}(\widehat{G}) \leqslant n$, where $\mathrm{cd}(\widehat{G})$ refers to the cohomological dimension of $\widehat{G}$; see [Rei15, Proposition 7.4]. Thus we need to exhibit some non-trivial cohomology in degree n for some finite discrete $\widehat{G}$-module.

The kernel of the map $G \to W(G)$, denoted PG, is the fundamental group of an aspherical complexified hyperplane arrangement X which has non-vanishing cohomology in degree n [Del72, Bri73]. Moreover, the nth integral cohomology of X is non-zero [Del72, Bri73]. The Universal Coefficient Theorem implies $H^n(X; \mathbb{F}_p) \neq 0$ for some prime p. By asphericity of X, goodness of G proved in (1), and several applications of Shapiro's Lemma [Bro94] we obtain isomorphisms

$$H^n(X; \mathbb{F}_p) \cong H^n(PG; \mathbb{F}_p) \cong H^n(G; \mathbb{F}_p[W(G)]) \cong H^n(\widehat{G}; \mathbb{F}_p[W(G)]).$$

Hence, $\mathrm{cd}(\widehat{G}) = n$ as required.

We now treat case (e). Since cohomological dimension is monotonic on subgroups, it suffices to exhibit a subgroup of $\widehat{\mathcal{F}}_4$ which has cohomological dimension at least 3. There is a retract

$$\alpha\colon \mathcal{F}_4 \twoheadrightarrow A = \langle s_1, s_2, s_4 \rangle \cong \mathcal{A}_2 \times \mathbb{Z}$$

given by $s_3 \mapsto s_4$ and fixing the other generators. It follows from [Min21, Lemma 2.2] that A has the full induced profinite topology, that is every finite index subgroup of A is separable in $\mathcal{F}_4$. In particular, by [Rei15, Lemma 4.6] we have $\widehat{A} \rightarrowtail \widehat{\mathcal{F}}_4$. That is, $\widehat{A}$ is a subgroup of $\widehat{G}$. Moreover by (3)(a) and the Künneth formula we have $\mathrm{cd}(\widehat{A}) = 3$, as required.

The proof of (4) is a simple computation in MAGMA, see Appendix B.C. □

Theorem C. *Let G and H be irreducible spherical Artin groups. If $\widehat{G} \cong \widehat{H}$, then $G \cong H$.*

Proof. First suppose that $b_1(G) = 1$, where $b_1(G) = \dim_{\mathbb{Q}} H^1(G; \mathbb{Q})$ refers to the first Betti number of G. By [Rei15, Proposition 3.2], $b_1(H) = 1$. In particular, H is not $\mathcal{I}_2(2m)$, $\mathcal{F}_4$, or $\mathcal{B}_n$ since these groups satisfy $b_1 > 1$. The result follows for the exceptional Artin groups $\mathcal{H}_3$, $\mathcal{H}_4$, $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{E}_8$ by Theorem A. The remaining cases with $b_1 = 1$ are all of type A, D, and I_2 and so the result follows from Lemma 11.1.

Next, suppose $b_1(G) = 2$. By by [Rei15, Proposition 3.2], $b_1(H) = 2$. Thus, H is either $\mathcal{I}_2(2m)$, $\mathcal{F}_4$, or $\mathcal{B}_n$. For $n \geqslant 5$ the group $\mathcal{B}_n$ is distinguished by Theorem A. The computations of cohomological dimension in Lemma 11.1(3) distinguish types $\mathcal{B}_3$, $\mathcal{B}_4$, and $\mathcal{F}_4$ from $\mathcal{I}_2(2m)$. We distinguish $\mathcal{B}_3$, and $\mathcal{B}_4$ from $\mathcal{F}_4$ by Lemma 11.1(4).

It remains to distinguish $\mathcal{I}_2(2m)$ from $\mathcal{I}_2(n)$ when $m \neq n$. The idea to use cup products in the sequel is inspired by [HIP+25, Proof of Theorem 4.12]. By work of Landi [Lan00, Table II], we have

$$H^*(\mathcal{I}_2(2m); \mathbb{Z}) = \mathbb{Z}[x, y, z]/(x^2, y^2, xy = mz)$$

where x, y have degree 1 and z has degree 2, so

$$H^n(\mathcal{I}_2(2m)); \mathbb{Z} \cong \begin{cases} \mathbb{Z} & n = 0, 2; \\ \mathbb{Z} & n = 1; \\ 0 & \text{otherwise.} \end{cases}$$

By the Universal Coefficient Theorem we obtain

$$H^n(\mathcal{I}_2(2m); \mathbb{Z}/k\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & n = 0, 2; \\ (\mathbb{Z}/k\mathbb{Z})^2 & n = 1; \\ 0 & \text{otherwise.} \end{cases}$$

A careful reading of §5.1, Table I and Table II of [Lan00] and substituting the $\mathbb{Z}$ coefficients for $\mathbb{Z}/k$ coefficients, yields that

$$H^*(\mathcal{I}_2(2m); \mathbb{Z}/k\mathbb{Z})) = (\mathbb{Z}/k\mathbb{Z})[x, y, z]/(x^2, y^2, xy = mz).$$

Note that [Lan00, Example 1] is particularly instructive. It follows that m equals the smallest integer $k \geqslant 2$ such that $H^*(\mathcal{I}_2(2m); \mathbb{Z}/k\mathbb{Z}))$ has no non-trivial cup products. Since cup products are preserved by goodness see [HN26, §2.C.1] (or [KW16, Proposition 6]), the same is true for $H^*(\widehat{\mathcal{I}}_2(2m); \mathbb{Z}/k\mathbb{Z}))$. Whence, the theorem. □

Appendix A. Quotient computations

This section contains the computations of Section 4.C. Our computations are always accomplished by applying one of a small handful of possible rearrangements of a positive word w in the Artin group and by cancelling adjacent pairs of letters which consist of an Artin generator and its inverse. While none of these rearrangements are themselves difficult, it is somewhat tedious to work out which rearrangements should be applied in which order to obtain the desired equality. Rather than present lengthy line-by-line computations, we develop the following notation to record which sequence of rearrangements we used. We recommend that a reader interested in following through these steps does write them out line-by-line. We note that the sequences are not necessarily unique; we simply provide one effective option.

A.A. **Notation.**

Throughout, let S denote the collection of Artin generators, and let w denote a positive word in this generating set. We will often equate the positive word $w = s_1s_3s_5s_2s_1$, for example, with the string of integers 13521. When necessary, we will equate the inverse of the positive word 13521 with the string $1^{-1}2^{-1}5^{-1}3^{-1}1^{-1}$. We will primarily use this to improve readability when a word is used as a subscript.

We will be almost exclusively concerned with simplifying equations of the form $ws_iw^{-1} = vs_jv^{-1}$ or $ws_iw^{-1} = v^{-1}s_jv$. Because w is a positive word, it will always be clear which generator should be the distinguished s_j; it is the unique generator which is adjacent to both a positive letter and an inverse. There are a few methods of rearranging one or both sides of the equation which will be used repeatedly; we classify them below.

Throughout, we will adopt the convention that any subword of the form $s_is_i^{-1}$ should be immediately deleted. We do not mark rearrangements of this kind. We are also using the Artin generator labellings of Figure 1 throughout.

The majority of our methods are *sided operations.* These are rearrangements that are applied to a particular side of the equation, so either to ws_iw^{-1} or to vs_jv^{-1}, but not both. The correct side of the equation will be denoted via an *RHS* or *LHS* preceding the relevant step except in the case where we have defined a variable ϵ and are attempting to reduce an expression of the form $\epsilon s_i\epsilon^{-1} := ws_iw^{-1}$. In this setting, it is clear that sided operations should be applied to the right hand side.

- **$M_{ij,\square}$:** Suppose that w in the above equation contains a unique subword which is an alternating product $s_is_j...$ of length m_{ij}. To denote that one should apply the Artin relation of length m_{ij} to this subword and to the corresponding subword of w^{-1}, we write M_{ij}.

  If there are multiple such subwords, we write $M_{ij,n}$ to indicate that one should apply the Artin relation to the alternating product subword beginning at the n^{th} appearance, when read from left to right, of the letter s_i in w, but this is rarely necessary. To avoid confusion between the index of the Artin generator and the position, we write the position n in Roman numerals when it is needed, e.g., $M_{12,IV}$ means that one should apply the Artin relation m_{12} beginning at the 4th appearance of the letter s_1 in w and working left to right, and at the 4th appearance of s_1^{-1} in w^{-1} working right to left.

  Example A.1. Define $\epsilon = s_1s_2s_1$ in the Artin group $\mathcal{A}_3$. To simplify $\epsilon s_2\epsilon^{-1} := (s_1s_2s_1)s_2(s_1^{-1}s_2^{-1}s_1^{-1})$, we can apply M_{12}. We do not need a position indicator, because there is only one alternating product of s_1 and s_2 of the appropriate length in w and w^{-1}. Applying M_{12} and cancelling the resulting adjacent s_2 and s_2^{-1} yields the shorter expression $\epsilon s_2\epsilon^{-1} = (s_2s_1)s_2(s_1^{-1}s_2^{-1})$.

- **$\tilde{M}_{ij}$:** Recall that Artin relations yield a conjugacy relationship in an obvious way: if two generators s_i and s_j satisfy a braid relation, then $(s_is_j)s_i(s_j^{-1}s_i^{-1}) = s_js_is_js_j^{-1}s_i^{-1} = s_j$. If they satisfy an Artin relation of a different length, then the same holds for s_is_j replaced with the alternating product $s_is_j...$ of length $m_{ij} - 1$. This follows from applying the Artin relation to the initial subword and then cancelling inverses. We denote the replacement $s_is_j...s_is_j^{-1}s_i^{-1}... = s_j$ with $\tilde{M}_{ij}$. Notice that this is specifically a conjugacy relation, so it always occurs on the rightmost subword of w and the leftmost subword of w^{-1}, and no position indicator is needed.

  Example A.2. Define $\epsilon = s_3s_1s_2$ in the Artin group $\mathcal{A}_4$. To reduce the expression $\epsilon s_1\epsilon^{-1} := s_3s_1s_2s_1s_2^{-1}s_1^{-1}s_3^{-1}$, we apply $\tilde{M}_{12}$. This replaces the interior subword $s_1s_2s_1s_2^{-1}s_1^{-1}$ with s_2, yielding $\epsilon s_1\epsilon^{-1} = s_3s_2s_3^{-1}$.

- $S_{i,R/L,\square}$**:** Often, we will want to move an Artin generator in w as far to the left or right as possible via commuting relations. While this could be written as several M operations, it is shorter to define a third operation: $S_{i,n,R/L}$ denotes that one should shift the nth appearance, from left to right, of Artin generator s_i in w as far to the right or left as possible in ws_iw^{-1} via commuting relations with the adjacent letters. The precise opposite should occur with w^{-1} so that the expressions for w and w^{-1} are still mirrored. If there is a unique instance of s_i in w, we omit the position indicator n and simply write $S_{i,R}$ or $S_{i,L}$. As before, the position indicator n, when it is needed, is given in Roman numerals to avoid confusion with the index i.

  If we are considering an expression of the form $w^{-1}s_iw$, we will write $S_{i^{-1},R}$ to denote that one should shift the unique instance of s_i^{-1} in w^{-1} as far to the right as possible and the unique instance of s_i in w as far to the left as possible.

  Example A.3. Define $\epsilon = s_4s_1^7$ in $\mathcal{A}_4$. To simplify the expression $\epsilon s_2\epsilon^{-1}$, we apply $S_{4,R}$ to the equation $\epsilon s_2\epsilon^{-1} := s_4s_1s_2s_1^{-1}s_4^{-1}$ to obtain $\epsilon s_2\epsilon^{-1} := s_1s_4s_2s_4^{-1}s_1^{-1}$. If we were instead interested in $\epsilon^{-1}s_2\epsilon$, we could instead apply $S_{4^{-1},L}$ to $\epsilon^{-1}s_2\epsilon := s_1^{-1}s_4^{-1}s_2s_4s_1$ to obtain $\epsilon s_2\epsilon^{-1} := s_4^{-1}s_1^{-1}s_2s_1s_4$.

- X_v**:** Let $w = uv$ where v is the longest suffix, i.e., subword read from right to left, of w which commutes with s_j. We write X_v to denote the replacement $ws_jw^{-1} = us_ju^{-1}$. As with the shift operation S and with M, the operation X_v could be accomplished via successive applications of $\tilde{M}_{ij}$ for each letter in v, but X is significantly shorter. As with $\tilde{M}_{ij}$, no position indicator is needed, since this always occurs in the middle of the expression.

  Example A.4. Define $\epsilon = s_2s_5s_6$ in $\mathcal{A}_6$. To simplify $\epsilon s_1\epsilon^{-1} := s_2s_5s_6s_1s_6^{-1}s_5^{-1}s_2^{-1}$, we apply X_{56} to denote the replacement $s_5s_6s_1s_6^{-1}s_5^{-1} = s_1$, yielding $\epsilon s_1\epsilon^{-1} := s_2s_1s_2^{-1}$.

We will also occasionally modify both sides of an equation of the form $ws_iw^{-1} = vs_jv^{-1}$ at the same time. These are *unsided* operations.

- C_v denotes conjugating both sides of the equation by v. Here we say that conjugating a by v is replacing a with vav^{-1}.

  Example A.5. Given the equation $s_1 = s_4s_1s_4^{-1}$ in $\mathcal{A}_5$, the operation $C_{4^{-1}}$ denotes replacing $s_1 = s_4s_1s_4^{-1}$ with $s_4^{-1}s_1s_4 = s_1$.

- R_v denotes right-multiplying both sides of the equation by v. L_v denotes left-multiplying both sides of the equation by v.

In all of these, v is typically represented by its associated string of integers.

We give an example which combines several steps, and then the remainder of our computations are provided only via the shorthand notation.

Example A.6. In the Artin group $\mathcal{E}_6$, one can show that $(s_4s_2s_3s_1s_5s_6)s_5(s_6^{-1}s_5^{-1}s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1}) = s_6$ via the following sequence.

$$^{LHS}\tilde{M}_{56} \to X_{4231}$$

The first step, $\tilde{M}_{56}$, is applied to the left-hand side, and replaces $(s_4s_2s_3s_1s_5s_6)s_5(s_6^{-1}s_5^{-1}s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1})$ with $(s_4s_2s_3s_1)s_6(s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1})$.

The Artin generators s_1, s_2, s_3, and s_4 all commute with s_6, so the entire subword $(s_4s_2s_3s_1)$ commutes with s_6. The second step, X_{4231}, denotes replacing $(s_4s_2s_3s_1)s_6(s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1})$ with s_6.

Example A.7. In the Artin group $\mathcal{E}_6$, one reduces the expression

$$\epsilon s_1\epsilon^{-1} := (s_4s_2s_3s_1s_5s_6)(s_4s_2s_3)s_1(s_3^{-1}s_2^{-1}s_4^{-1})(s_6^{-1}s_5^{-1}s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1})$$

as follows:

$$\begin{aligned}
\epsilon s_3\epsilon^{-1} &= (s_4s_2s_3s_1s_5s_6s_4s_2s_3)s_1(s_3^{-1}s_2^{-1}s_4^{-1}s_6^{-1}s_5^{-1}s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1})\\
&= (s_4s_2s_3s_1s_5s_4s_2s_3s_6)s_1(s_6^{-1}s_3^{-1}s_2^{-1}s_4^{-1}s_5^{-1}s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1})\\
&= (s_4s_2s_3s_1s_5s_4s_2s_3)s_1(s_3^{-1}s_2^{-1}s_4^{-1}s_5^{-1}s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1})\\
&= (s_4s_2s_3s_5s_4s_2s_1s_3)s_1(s_3^{-1}s_1^{-1}s_2^{-1}s_4^{-1}s_5^{-1}s_3^{-1}s_2^{-1}s_4^{-1})\\
&= (s_4s_2s_3s_5s_4s_2)s_3(s_2^{-1}s_4^{-1}s_5^{-1}s_3^{-1}s_2^{-1}s_4^{-1})\\
&= (s_4s_2s_3s_5s_4)s_3(s_4^{-1}s_5^{-1}s_3^{-1}s_2^{-1}s_4^{-1})\\
&= (s_4s_2s_5s_3s_4)s_3(s_4^{-1}s_3^{-1}s_5^{-1}s_2^{-1}s_4^{-1})\\
&= (s_4s_2s_5)s_4(s_5^{-1}s_2^{-1}s_4^{-1})
\end{aligned}$$

In the shorthand, this is $\epsilon s_3\epsilon^{-1} := (s_4s_2s_3s_1s_5s_6s_4s_2s_3)s_1(s_3^{-1}s_2^{-1}s_4^{-1}s_6^{-1}s_5^{-1}s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1})$

$$S_{6,R} \to X_6 \to S_{1,R} \to \tilde{M}_{13} \to X_2 \to S_{3,R} \to \tilde{M}_{34}$$

$\epsilon s_3\epsilon^{-1} = (s_4s_2s_5)s_4(s_5^{-1}s_2^{-1}s_4^{-1})$.

We now proceed to the computations used in the paper.

A.B. **The group $\mathcal{E}_6$.** Recall the following definitions of elements and their orders in $\mathcal{E}_6/Z(\mathcal{E}_6)$, as given in [Sor21].

- $\epsilon_{12} = s_4s_2s_3s_1s_5s_6$ has order 12.
- $\epsilon_9 = s_4s_2s_5s_4s_3s_1s_5s_6$ has order 9.
- $\epsilon_8 = s_4s_3s_1s_5s_4s_2s_3s_6s_5$ has order 8.

A.B.1. *The element* ϵ_{12}. We begin with ϵ_{12}. In each row, it should be understood that when $|i| \neq 1$, the beginning expression for $\epsilon_{12}^i \cdot s_5 \cdot \epsilon_{12}^{-i}$ is the conjugate of the reduced expression for $\epsilon_{12}^{i-1} \cdot s_5 \cdot \epsilon_{12}^{-i+1}$ rather than the conjugate of s_5 by i copies of the expression for ϵ_{12} given above.

To see that $\mathcal{E}_6\langle\langle\epsilon_{12}^6\rangle\rangle$ is cyclic, we compute the following.

Starting expression	Ending expression	Path
$\epsilon_{12}\cdot s_3\cdot\epsilon_{12}^{-1}$	s_1	$X_{56} \to \tilde{M}_{13} \to X_{42}$
$\epsilon_{12}^2\cdot s_3\cdot\epsilon_{12}^{-2}$	$(s_4s_2s_3)s_1(s_3^{-1}s_2^{-1}s_4^{-1})$	X_{156}
$\epsilon_{12}^3\cdot s_3\cdot\epsilon_{12}^{-3}$	$(s_4s_2s_5)s_4(s_5^{-1}s_2^{-1}s_4^{-1})$	$S_{1,R} \to \tilde{M}_{13} \to X_2 \to S_{3,R} \to \tilde{M}_{34}$
$\epsilon_{12}^{-1}\cdot s_3\cdot\epsilon_{12}$	$(s_6^{-1}s_5^{-1}s_2^{-1})s_4(s_2s_5s_6)$	$S_{3^{-1},R} \to \tilde{M}_{34} \to S_{1^{-1},R} \to X_{1^{-1}}$
$\epsilon_{12}^{-2}\cdot s_3\cdot\epsilon_{12}^{2}$	$(s_5^{-1}s_1^{-1}s_3^{-1})s_4(s_3s_1s_5)$	$S_{6,II,R} \to M_{65,I} \to S_{5^{-1},II,R} \to \tilde{M}_{24}$ $\to X_{5^{-1}} \to \tilde{M}_{24} \to S_{6^{-1},R} \to X_{6^{-1}}$
$\epsilon_{12}^{-3}\cdot s_3\cdot\epsilon_{12}^{3}$	$(s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1}s_1^{-1})s_3(s_1s_4s_2s_3s_1)$	$S_{5^{-1},I,R} \to M_{54,I} \to S_{4^{-1},II,R} \to \tilde{M}_{34}$ $\to S_{5^{-1},R} \to X_{5^{-1}} \to S_{6^{-1},R} \to X_{6^{-1}}$

In any quotient where ϵ_{12}^6 is trivial, we must have

$$(s_4s_2s_5)s_4(s_5^{-1}s_2^{-1}s_4^{-1}) = (s_1^{-1}s_3^{-1}s_2^{-1}s_4^{-1}s_1^{-1})s_3(s_1s_4s_2s_3s_1)$$

This imposes the relation $s_5 = s_3$ via the following sequence of reductions.

$$\begin{aligned}
C_{2^{-1}4^{-1}} \to^{RHS} S_{1^{-1},I,L} \to M_{24} \to M_{34} \to \tilde{M}_{13} \to X_{4^{-1}} \to S_{1^{-1},R} \to \tilde{M}_{13} \to X_{2^{-1}}\\
\to C_4 \to^{LHS} \tilde{M}_{45}
\end{aligned}$$

The quotient $\mathcal{E}_6/\langle\langle\epsilon_{12}^6\rangle\rangle$ must then be cyclic by Lemma 4.11.

◆

A.B.2. *The element* ϵ_9. Using the commuting relations between Artin generators, ϵ_9 can be rearranged to the form $\epsilon_9 = s_4s_2s_5s_4s_5s_6s_3s_1$. We compute the following using this form of ϵ_9. In each row, it should be understood that when $|i| \neq 1$, the beginning expression for $\epsilon_9^i \cdot s_2 \cdot \epsilon_9^{-i}$ is the conjugate of the reduced expression for $\epsilon_9^{i-1} \cdot s_2 \cdot \epsilon_9^{-i+1}$ rather than the conjugate of s_2 by i copies of the expression for ϵ_9 given above.

Starting expression	Ending expression	Path
$\epsilon_9 s_2 \epsilon_9^{-1}$	s_5	$X_{5631} \to S_{2,R} \to \tilde{M}_{24} \to \tilde{M}_{45}$
$\epsilon_9^2 \cdot s_2 \cdot \epsilon_9^{-2}$	$(s_4s_5)s_6(s_5^{-1}s_4^{-1})$	$X_{31} \to \tilde{M}_{56} \to S_{2,R} \to X_{24}$
$\epsilon_9^{-1} \cdot s_2 \cdot \epsilon_9$	$s_6^{-1}s_5s_6$	$\tilde{M}_{24} \to \tilde{M}_{45} \to X_{5^{-1}} \to S_{6^{-1},L} \to X_{1^{-1}3^{-1}}$

In a quotient where ϵ_9^3 is trivial, we must have $(s_4s_5)s_6(s_5^{-1}s_4^{-1}) = s_6^{-1}s_5s_6$. This imposes the relation $s_4s_5 = s_5s_4$ via the following sequence of reductions.

$$C_6 \to^{LHS} S_{4,L} \to \tilde{M}_{56} \to R_4$$

The relation $s_4s_5 = s_5s_4$ implies that $\mathcal{E}_6/\langle\langle \epsilon_9^3 \rangle\rangle$ is cyclic by Lemma 2.8 and Lemma 4.11. ◆

A.B.3. *The element* ϵ_8. For ϵ_8, we compute the following. In each row, it should be understood that when $|i| \neq 1$, the beginning expression for $\epsilon_8^i \cdot s_6 \cdot \epsilon_8^{-i}$ is the conjugate of the reduced expression for $\epsilon_8^{i-1} \cdot s_6 \cdot \epsilon_8^{-i+1}$ rather than the conjugate of s_6 by i copies of the expression for ϵ_8 given at the beginning of the section.

Starting expression	Ending expression	Path
$\epsilon_8 s_6 \epsilon_8^{-1}$	s_3	$\tilde{M}_{56} \to X_{23} \to \tilde{M}_{45} \to X_1 \to \tilde{M}_{34}$
$\epsilon_8^2 \cdot s_6 \cdot \epsilon_8^{-2}$	$(s_4s_3s_1s_5s_4)s_3(s_4^{-1}s_5^{-1}s_1^{-1}s_3^{-1}s_4^{-1})$	X_{2365}
$\epsilon_8^{-1} \cdot s_6 \cdot \epsilon_8$	$(s_3^{-1}s_2^{-1})s_4(s_2s_3)$	$X_{1^{-1}3^{-1}4^{-1}} \to S_{6^{-1},R} \to \tilde{M}_{56} \to S_{5^{-1},R} \to \tilde{M}_{45}$
$\epsilon_8^{-2} \cdot s_6 \cdot \epsilon_8^2$	$(s_5^{-1}s_6^{-1}s_3^{-1}s_2^{-1}s_4^{-1}s_5^{-1}s_4^{-1})s_2(s_4s_5s_4s_2s_3s_6s_5)$	$M_{34} \to \tilde{M}_{24} \to S_{1^{-1},R} \to X_{1^{-1}3^{-1}}$

In a quotient where ϵ_8^4 is trivial, we must have the following relation.

$$(s_4s_3s_1s_5s_4)s_3(s_4^{-1}s_5^{-1}s_1^{-1}s_3^{-1}s_4^{-1}) = (s_5^{-1}s_6^{-1}s_3^{-1}s_2^{-1}s_4^{-1}s_5^{-1}s_4^{-1})s_2(s_4s_5s_4s_2s_3s_6s_5).$$

This implies $s_6 = s_2$ via the following sequence of reductions.

$$\begin{gathered} {}^{RHS}S_{3^{-1},L} \to C_3 \to^{LHS} M_{34} \to S_{4,II,R} \to M_{45} \to X_5 \to S_{3,R} \to \tilde{M}_{34} \to S_{4,R} \to \tilde{M}_{45} \to X_1 \\ \to^{RHS} C_5 \to S_{2^{-1},L} \to C_{62} \to^{LHS} X_2 \to^{RHS} M_{45} \to C_4 \to^{LHS} S_{4,R} \to \\ X_4 \to C_5 \to^{LHS} \tilde{M}_{56} \to C_4 \to^{LHS} X_4 \end{gathered}$$

We conclude that $\mathcal{E}_6/\langle\langle \epsilon_8^4 \rangle\rangle$ is cyclic by Lemma 4.11. Consequently, $\mathcal{E}_6/\langle\langle \epsilon_8^2 \rangle\rangle$ and $\mathcal{E}_6/\langle\langle \epsilon_8 \rangle\rangle$ are also cyclic. ◆

A.C. **The group $\mathcal{E}_7$.** Recall the following definitions of elements and their orders in $\mathcal{E}_7/Z(\mathcal{E}_7)$, as given in [Sor21].

- $\epsilon_7 = s_4s_2s_7s_6s_5s_4s_2s_3s_1$ has order 7.
- $\epsilon_9 = s_4s_2s_3s_1s_5s_6s_7$ has order 9.

A.C.1. *The element* ϵ_7. The expression $\epsilon_7 s_3 \epsilon_7^{-1}$ can be reduced to s_1 via the following sequence of operations.

$$\tilde{M}_{13} \to X_{4276542}$$

We conclude that the quotient $\mathcal{E}_7/\langle\langle \epsilon_7 \rangle\rangle$ is cyclic by Lemma 4.11. ◆

A.C.2. *The element* ϵ_9. Applying a rearrangement of commuting terms, $\epsilon_9 = s_4s_5s_6s_7s_2s_3s_1$. Using this expression, we compute the following. In each row, it should be understood that when $|i| \neq 1$, the beginning expression for $\epsilon_9^i \cdot s_6 \cdot \epsilon_9^{-i}$ is the conjugate of the reduced expression for $\epsilon_9^{i-1} \cdot s_6 \cdot \epsilon_9^{-i+1}$ rather than the conjugate of s_6 by i copies of the expression for ϵ_9 given above.

Starting expression	Ending expression	Path
$\epsilon_9 \cdot s_6 \cdot \epsilon_9^{-1}$	s_7	$X_{231} \to \tilde{M}_{67} \to X_{45}$
$\epsilon_9^2 \cdot s_6 \cdot \epsilon_9^{-2}$	$(s_4s_5s_6)s_7(s_6^{-1}s_5^{-1}s_4^{-1})$	X_{7231}
$\epsilon_9^{-1} \cdot s_6 \cdot \epsilon_9$	s_5	$X_4 \to \tilde{M}_{56} \to X_{1327}$

In any quotient where ϵ_9^3 is trivial, we have the following relation.

$$s_4s_5s_6s_7s_6^{-1}s_5^{-1}s_4^{-1} = s_5$$

This imposes the relation $s_7 = s_4$ via the following sequence of reductions.

$$C_{5^{-1}4^{-1}} \to^{RHS} \tilde{M}_{45} \to C_{6^{-1}} \to^{RHS} X_{6^{-1}}$$

As in the previous cases, we can conclude that $\mathcal{E}_7/\langle\langle \epsilon_9^3 \rangle\rangle$ is cyclic by Lemma 4.11 ◆

A.D. **The group $\mathcal{H}_4$.** Recall the following definitions of elements and their orders in $\mathcal{H}_4/Z(\mathcal{H}_4)$, as given in [Sor21].

- $\epsilon_{10} = s_1s_2s_1s_2s_3s_4$ has order 10.
- $\epsilon_{15} = s_1s_2s_3s_4$ has order 15.

Before we proceed to the computations of the quotients, we highlight that because s_1 and s_2 satisfy an Artin relation of length 5 rather than length 3, it is no longer true that $(s_1s_2)s_1(s_2^{-1}s_1^{-1}) = s_1$. In $\mathcal{H}_4$, we instead have the relation $(s_1s_2s_1s_2)s_1(s_2^{-1}s_1^{-1}s_2^{-1}s_1^{-1}) = s_1$. We also highlight that, because we are working in a graph with no valence 3 vertices, the set of indices which commute with one another is different than in the $\mathcal{E}_n$ cases; see Figure 1.

A.D.1. *The element* ϵ_{10}. We begin with ϵ_{10}. We compute the following. In each row, it should be understood that when $|i| \neq 1$, the beginning expression for $\epsilon_{10}^i \cdot s_1 \cdot \epsilon_{10}^{-i}$ is the conjugate of the reduced expression for $\epsilon_{10}^{i-1} \cdot s_1 \cdot \epsilon_{10}^{-i+1}$ rather than the conjugate of s_1 by i copies of the expression for ϵ_{10} given above.

Starting expression	Ending expression	Path
$\epsilon_{10}s_1\epsilon_{10}^{-1}$	s_2	$X_{34} \to \tilde{M}_{12}$
$\epsilon_{10}^2s_1\epsilon_{10}^{-2}$	$(s_1s_2)s_3(s_2^{-1}s_1^{-1})$	$X_4 \to \tilde{M}_{23} \to X_1$
$\epsilon_{10}^3s_1\epsilon_{10}^{-3}$	$(s_2s_1s_2s_3)s_4(s_3^{-1}s_2^{-1}s_1^{-1}s_2^{-1})$	$S_{1,III,L} \to M_{12} \to M_{23} \to$ $\tilde{M}_{34} \to S_{3,L} \to X_{12}$
$\epsilon_{10}^4s_1\epsilon_{10}^{-4}$	$(s_1s_2s_1s_3s_2s_1)s_2(s_1^{-1}s_2^{-1}s_3^{-1}s_1^{-1}s_2^{-1}s_1^{-1})$	$S_{4,R} \to \tilde{M}_{34} \to M_{23} \to S_{3,II,R} \to \tilde{M}_{23}$
$\epsilon_{10}^{-1}s_1\epsilon_{10}$	$(s_4^{-1}s_3^{-1}s_2^{-1}s_1^{-1}s_2^{-1})s_1(s_2s_1s_2s_3s_4)$	X_1

In any quotient where ϵ_{10}^2 is trivial, the following relation is satisfied.

$$s_1 = (s_1s_2)s_3(s_2^{-1}s_1^{-1})$$

Since s_1 commutes with s_4, the image of the right hand side must also commute with the image of s_4 in the quotient, yielding the following.

$$s_4 = (s_1s_2)s_3(s_2^{-1}s_1^{-1})s_4(s_1s_2)s_3^{-1}(s_2^{-1}s_1^{-1})$$

This imposes the relation $s_4s_3 = s_3s_4$ via the following sequence of reductions.

$$C_{2^{-1}1^{-1}} \to^{LHS} X_{2^{-1}1^{-1}} \to^{RHS} X_{2^{-1}1^{-1}} \to R_3$$

As in the previous cases, we conclude that the quotients $\mathcal{H}_4/\langle\langle \epsilon_{10} \rangle\rangle$ and $\mathcal{H}_4/\langle\langle \epsilon_{10}^2 \rangle\rangle$ are cyclic by Lemmas 2.8 and 4.11. ◆

In any quotient where ϵ_{10}^5 is trivial, the following relation is satisfied.

$$(s_1s_2s_1s_3s_2s_1)s_2(s_1^{-1}s_2^{-1}s_3^{-1}s_1^{-1}s_2^{-1}s_1^{-1}) = (s_4^{-1}s_3^{-1}s_2^{-1}s_1^{-1}s_2^{-1})s_1(s_2s_1s_2s_3s_4)$$

This imposes the relation $s_2 = s_4$ via the following sequence of reductions.

$$C_1^{-1} \to^{RHS} S_{1^{-1},I,R} \to \tilde{M}_{12} \to C_{2^{-1}} \to^{RHS} S_{2,R} \to \tilde{M}_{23} \to C_{3^{-1}1^{-1}} \to^{RHS} S_{1^{-1},R} \to X_{1^{-1}} \\ \to \tilde{M}_{34} \to C_{1^{-1}2^{-1}} \to^{RHS} X_{1^{-1}2^{-1}}$$

We conclude that $\mathcal{H}_4/\langle\langle\epsilon_{10}^5\rangle\rangle$ is cyclic by Lemma 4.11. ◆

A.D.2. *The element* ϵ_{15}. For ϵ_{15}, we compute the following. In each row, it should be understood that when $|i| \neq 1$, the beginning expression for $\epsilon_{15}^i \cdot s_2 \cdot \epsilon_{15}^{-i}$ is the conjugate of the reduced expression for $\epsilon_{15}^{i-1} \cdot s_2 \cdot \epsilon_{15}^{-i+1}$ rather than the conjugate of s_2 by i copies of the expression for ϵ_{15} given above.

Starting expression	Ending expression	Path
$\epsilon_{15} \cdot s_2 \cdot \epsilon_{15}^{-1}$	s_3	$X_4 \to \tilde{M}_{23} \to X_1$
$\epsilon_{15}^2 \cdot s_2 \cdot \epsilon_{15}^{-2}$	s_4	$\tilde{M}_{34} \to X_{12}$
$\epsilon_{15}^3 \cdot s_2 \cdot \epsilon_{15}^{-3}$	$(s_1s_2s_3)s_4(s_3^{-1}s_2^{-1}s_1^{-1})$	X_4

In any quotient where ϵ_{15}^3, the following relation is satisfied.

$$s_2 = (s_1s_2s_3)s_4(s_3^{-1}s_2^{-1}s_1^{-1})$$

This imposes the relation $s_3 = (s_2s_1)s_2(s_1^{-1}s_2^{-1})$ via the following sequence of reductions.

$$C_4 \to^{LHS} X_4 \to^{RHS} S_{4,R} \to \tilde{M}_{34} \to C_{23} \to^{LHS} \tilde{M}_{23} \to^{RHS} S_{3,R} \to \tilde{M}_{23}$$

The right hand side commutes with the image of s_4, so the image of s_3 commutes with the image of s_4. Since s_3 and s_4 also satisfy a braid relation, they are identified in the quotient by 2.8. The quotient is then cyclic by Lemma 4.11.

In any quotient where ϵ_{15}^5 is trivial, we must have $\epsilon_{15}^3 s_3 \epsilon_{15}^{-3} = \epsilon_{15}^{-2} s_3 \epsilon_{15}^2$. Applying our previous calculation of $\epsilon_{15}^3 s_3 \epsilon_{15}^{-3}$ and the unreduced form of $\epsilon_{15}^{-2} s_3 \epsilon_{15}^2$, the following relation is satisfied in the quotient.

$$(s_1s_2s_3)s_4(s_3^{-1}s_2^{-1}s_1^{-1}) = (s_4^{-1}s_3^{-1}s_2^{-1}s_1^{-1}s_4^{-1}s_3^{-1}s_2^{-1}s_1^{-1})s_3(s_1s_2s_3s_4s_1s_2s_3s_4)$$

This imposes the relation $s_1 = s_2$ via the following sequence of reductions.

$$C_4 \to^{LHS} S_{4,R}\tilde{M}_{34} \to C_3 \to^{LHS} S_{3,R} \to \tilde{M}_{23} \to C_{412} \to^{LHS} S_{4,R} \to X_4 \to^{RHS} X_1 \to \tilde{23} \to C_2 \to^{LHS} \tilde{M}_{12}$$

Lemma 4.11 then shows that $\mathcal{H}_4/\langle\langle\epsilon_{15}^5\rangle\rangle$ must be cyclic. ◆

Appendix B. MAGMA code

B.A. Code for Theorem 5.5.

```
H4:=Group<a,b,c,d | a*b*a*b*a=b*a*b*a*b, b*c*b=c*b*c, c*d*c=d*c*d,
                    a*c=c*a, a*d=d*a, b*d=d*b>;
WH4:=PermutationGroup(quo<H4 | H4.1^2>);
Q:=WH4/centre(WH4);

Homomorphisms(H4, Q : Surjective:=true);
```

The code returns two homomorphisms both sending the generators to elements of order 2. Hence the quotient factors through the Coxeter group as required.

B.B. Code for Lemma 8.3.

```
G:=PermutationGroup(AutomorphismGroup(Group("PSL(3,13)")));
CCL:=ConjugacyClasses(G);
for x in CCL do x[1]; end for;
```

This code prints the orders of elements in $\mathrm{Aut}(\mathrm{PSL}_3(13))$. The output shows there are no elements of order 9.

B.C. **Code for Lemma 11.1.**

```
F4:=Group<a,b,c,d | a*b*a=b*a*b,b*c*b*c=c*b*c*b, c*d*c=d*c*d,
                    a*c=c*a, a*d=d*a, b*d=d*b, a^2, c^2>;
WF4:=PermutationGroup(F4);
B3:=Group<a,b,c | a*b*a*b=b*a*b*a, a*c=c*a, b*c*b=c*b*c >;
B4:=Group<a,b,c,d | a*b*a*b=b*a*b*a, a*c=c*a, a*d=d*a,
                    b*c*b=c*b*c, b*d=d*b, c*d*c=d*c*d >;

Homomorphisms(B3, WF4 : Surjective:=true, Limit:=1);
Homomorphisms(B4, WF4 : Surjective:=true, Limit:=1);
```

The code returns the empty set in both cases.

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