LECTURE NOTES ON ℓ^2-HOMOLOGY

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Abstract. These lecture notes are for a minicourse I taught at GTA:Gran Bilbao VI. The lectures will cover the following topics:

(1) Review of group cohomology via chain complexes, example computations, finiteness properties, G-CW complexes, equivariant cohomology.
(2) Group von Neumann algebras, trace, dimension, ℓ^2-Betti numbers, basic properties and examples.
(3) Applications to mapping tori, approximation, and more.
(4) The algebra of affiliated operators, the Linnell ring, and the Atiyah Conjecture.
(5) ℓ^2-homology of right-angled Artin groups.

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Other resources I recommend are

- *H. Kammeyer*, Introduction to ℓ^2-invariants. Cham: Springer (2019; Zbl 1458.55001)
- *B. Eckmann*, Introduction to ℓ_2-methods in topology. Isr. J. Math. 117, 183–219 (2000; Zbl 0948.55006)
- *W. Lück*, L^2-invariants: Theory and applications to geometry and K-theory. Berlin: Springer (2002; Zbl 1009.55001)

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1. Review of group homology

1.A. Resolutions and finiteness properties.

Definition 1.1 (Projective resolution)**.** Let R be a ring (associative and unital), let G be a group, and let M be an RG-module. A *projective resolution* for M by RG-modules is an exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is a projective RG-module.

Definition 1.2 (Type FP_n)**.** If there exists a projective resolution of M over RG such that P_i is a finitely generated RG-module, then we say that M is *type* FP_n. If the trivial RG-module R is type FP_n, then we say that G is *type* $\mathsf{FP}_n(R)$.

Exercise 1.3. Every RG module admits a free resolution.

Often we can build free resolutions of the trivial module using topology.

Definition 1.4 (Model of a $K(G,1)$ space)**.** We say X is a model for a $K(G,1)$ space, if X is a topological space such that

$$\pi_i X \cong \begin{cases} G & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that such a space is unique up to homotopy.

Remark 1.5. If G is a discrete group, then a model for a $K(G,1)$ is exactly a classifying space BG. The universal cover of a classifying space is often denoted EG. This latter space has the property of being a free contractible G-space and is unique up to G-homotopy.

Exercise 1.6. Every group G admits a model for a $K(G,1)$.

Example 1.7. Suppose that X is a CW-complex model for a $K(G,1)$. The universal cover $\widetilde{X}$ of X is a contractible free G-CW complex. Since G acts on $\widetilde{X}$, we obtain an action of G on $C_\bullet(\widetilde{X};R)$, the cellular chain complex of $\widetilde{X}$. Since the action on X is free, this turns the chain complex $C_\bullet(\widetilde{X};R)$ into a chain complex of free RG-modules. Now, the homology groups $H_n(\widetilde{X};R) = 0$ for $n \neq 0$ and equal R when $n = 0$. Thus, the sequence $C_\bullet(\widetilde{X};R) \twoheadrightarrow R$ is a free resolution of R over RG.

Definition 1.8. We say that G is *type* F_n if G admits a model for a $K(G,1)$ with finitely many k-cells for $k \leqslant n$.

Exercise 1.9. For a group G, the following are equivalent:

(1) G is finitely generated;
(2) G is type F_1;
(3) G is type $\mathsf{FP}_1(R)$ for any non-trivial ring R.

Exercise 1.10. A group G is finitely presented if and only if G is type F_2.

Theorem 1.11 (Bestvina–Brady)**.** *There exist groups of type* FP_2 *but not* F_2*.*

1.B. **Group homology.** Let $P_\bullet \twoheadrightarrow R$ be a projective resolution of the trivial RG-module R. The homology of G with coefficients in M, denoted, $H_*(G;M)$ is defined to be $H_*(P_\bullet \otimes_{RG} M)$, that is, the homology of the chain complex

$$\cdots \longrightarrow P_1 \otimes_{RG} M \longrightarrow P_0 \otimes_{RG} M \longrightarrow 0.$$

Definition 1.12 (Fox derivatives)**.** The *Fox partial derivatives* $\frac{\partial}{\partial x_i}$ are defined by the rules

- $\frac{\partial 1}{\partial x_i} = 0$, and
- $\frac{\partial x_i}{\partial x_i} = 1$.

We extend this to a product $u = y_1 \dots y_n$ where $y_i = x_k$ or $y_i = x_k^{-1}$ for some $k = k(i)$ by the formula

$$\frac{\partial u}{\partial x_i} = \sum_{s=1}^{n} y_1 \cdots y_{s-1} \frac{\partial y_s}{\partial x_i}.$$

Exercise 1.13. Show that

(1) $\frac{\partial x_i^{-1}}{\partial x_i} = -x_i^{-1}$.
(2) $\frac{\partial x_j^{\pm 1}}{\partial x_i} = 0, \quad i \neq j$.
(3) $\partial(tat^{-1}a^{-2})/\partial t = 1 - tat^{-1}$.
(4) $\partial(tat^{-1}a^{-2})/\partial a = t - tat^{-1}a^{-1} - tat^{-1}a^{-2}$.

Exercise 1.14. Let G be your favourite one relator group $\langle a, b \mid r \rangle$ such that r is not a proper power. Prove that the following chain complex

$$0 \to \mathbb{Z}G \xrightarrow{\partial_2} \mathbb{Z}G^2 \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} \mathbb{Z} \to 0,$$

where

$$\partial_2 = \left(\frac{\partial r}{\partial a}, \frac{\partial r}{\partial b} \right), \quad \partial_1 = \begin{pmatrix} a-1 \\ b-1 \end{pmatrix}, \quad \text{and} \quad \partial_0(g) = 1$$

is a free resolution.

Example 1.15 (Alexander modules)**.** Let $G = \mathrm{BS}(1,2)$, this is the soluble Baumslag–Solitar group with presentation

$$\langle a, t \mid tat^{-1} = a^2 \rangle.$$

The group G admits a homomorphism $\varphi\colon G \twoheadrightarrow \mathbb{Z}$ by $\varphi(a) = 0$ and $\varphi(t) = 1$. Let $\mathbb{Z}[t^{\pm 1}]$ denote the $\mathbb{Z}G$-module where the action is given by $g \cdot x = t^{\varphi(g)}x$.

A $K(G,1)$ space X is given by a rose with two circles with edge labels σ_a and σ_t, as well a single 2-cell σ_2 attached with the obvious attaching map. Thus, we obtain a length two free resolution of the trivial module $\mathbb{Z}$ by $\mathbb{Z}G$-modules when passing to the universal cover and looking at the cellular chain complex.

We shall compute $H_n(G;\mathbb{Z}[t^{\pm 1}])$. We have a chain complex

$$0 \to \mathbb{Z}G\langle \sigma_2 \rangle \otimes_{\mathbb{Z}G} \mathbb{Z}[t^{\pm 1}] \xrightarrow{\partial_2} \mathbb{Z}G\langle \sigma_a, \sigma_e \rangle \otimes_{\mathbb{Z}G} \mathbb{Z}[t^{\pm 1}] \xrightarrow{\partial_1} \mathbb{Z}G\langle \sigma_0 \rangle \otimes_{\mathbb{Z}G} \mathbb{Z}[t^{\pm 1}] \to 0.$$

Computing the tensor products we obtain

$$0 \to \mathbb{Z}[t^{\pm 1}]\langle \sigma_2 \rangle \xrightarrow{\partial_2} \mathbb{Z}[t^{\pm 1}]\langle \sigma_a, \sigma_t \rangle \xrightarrow{\partial_1} \mathbb{Z}[t^{\pm 1}]\langle \sigma_0 \rangle \to 0.$$

The differentials become

$$\begin{aligned}
\partial_1(\sigma_a) &= (\sigma_0 - a\cdot\sigma_0)\otimes 1\\
&= 0;\\
\partial_1(\sigma_t) &= (\sigma_0 - t\cdot\sigma_0)\otimes 1\\
&= (1-t)\sigma_0;\\
\partial_2(\sigma_2) &= \left(\frac{\partial r}{\partial a}, \frac{\partial r}{\partial t}\right)^{\mathrm{T}}(\sigma_2)\\
&= \big((t - tat^{-1}a^{-1} - tat^{-1}a^{-2})\otimes 1, (1-tat^{-1})\otimes 1\big)\\
&= (t-2, 0).
\end{aligned}$$

Computing homology we obtain

$$H_n(\mathrm{BS}(1,2);\mathbb{Z}[t^{\pm 1}] = \begin{cases} \mathbb{Z}[t^{\pm1}]/(1-t)\cong \mathbb{Z} & n=0\\ \mathbb{Z}[t^{\pm1}]/(2-t)\cong \mathbb{Z}[\frac{1}{2}] & n=1\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 1.16. Generalise the previous computation to $\mathrm{BS}(m,n)$.

1.C. **Equivariant homology.**

Definition 1.17 (G-CW complex)**.** A *G-CW complex* is a CW complex X equipped with a G action that permutes the cells of X such that if $g\in G$ fixes a cell $\sigma\in X$ setwise, then it fixes it pointwise. Equivalently X is a G-space equipped with a filtration

$$\varnothing = X_{-1}\subset X_1\subset\cdots\subset X_n\subset\dots\bigcup_{n\geqslant -1} X_n = X$$

such that X carries the colimit topology with respect to this filtration and such that X_n is obtained from X_{n-1} by a G-pushout

$$\begin{array}{ccc}
\coprod_{i\in I} G/H_i\times S^{n-1} & \longrightarrow & X_{n-1}\\
\big\downarrow & & \big\downarrow\\
\coprod_{i\in I} G/H_i\times D^{n} & \longrightarrow & X_{n}.
\end{array}$$

A G-CW complex is *finite* if it has finitely many G-orbits of cells, or equivalently finitely many equivariant cells, or equivalently, if G acts on X cocompactly, that is if X/G is compact.

Define indcued space.

Definition 1.18 (Naïve equivariant homology)**.** Let X be a G-CW-complex, let $C_\bullet(X)$ denote its cellular chain complex, and let M be a G-module. The *(naïve) G-equivariant homology* of X with coefficients in M, denoted $H_n^G(X;M)$, is the homology of the complex $C_\bullet(X)\otimes_{\mathbb{Z}G} M$.

Example 1.19. Let X be a finite CW complex. The space $\widetilde{X}$ is a finite free $\pi_1 X$-CW complex, where each cell of X lifts to a free orbit of cells in $\widetilde{X}$. Moreover, for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ we have $H_n^G(\widetilde{X};\mathbb{Z}) = H_n(X;\mathbb{Z})$.

2. Enter ℓ^2-homology

We will begin with a very analytic definition of ℓ^2-homology and slowly introduce more machinery from operator theory to give the theory an algebraic foundation.

2.A. **A first attempt.**
In this section we will give the naïve approach to ℓ^2-cohomology.

Definition 2.1. Let X be a CW-complex and let

$$C_\bullet(\widetilde{X}) \coloneqq C_0 \xleftarrow{\partial_0} C_1 \xrightarrow{\partial_1} C_2 \leftarrow \cdots$$

denote the cellular chain complex of the universal cover $\widetilde{X}$. Let $C_\bullet(\widetilde{X}) = \hom(C_\bullet(\widetilde{X}); \mathbb{R})$ denote the cellular cochain complex with differential d_n and let $C^{(2)}_\bullet(\widetilde{X})$ be the subcomplex consisting of square summable cochains Note that the differential d_n restricts to $C^\bullet_{(2)}$ and we denote the restriction by $d^{(2)}_n$. We define the *unreduced ℓ^2-cohomology of X* to be

$$H^n_{(2)}(X) = \ker d^{(2)}_n / \operatorname{im} d^{(2)}_{n+1}$$

and the *reduced ℓ^2-cohomology of X* to be

$$\overline{H}^{(2)}_n(X) = \ker d^{(2)}_n / \overline{\operatorname{im} d^{(2)}_{n+1}}$$

where $\overline{\operatorname{im} d^{(2)}_{n+1}}$ is the closure of $\operatorname{im} d^{(2)}_{n+1}$ in $\ker d^{(2)}_n$.

For a discrete group G we define $H^n_{(2)}(G) = H^n_{(2)}(EG)$.

Remark 2.2. The ℓ^2-cohomology groups $\overline{H}^{(2)}_n(-)$ are functorial with respect to bi-Lipschitz maps.

Theorem 2.3 (Pansu, Sauer)**.** *Suppose G and H are finitely generated groups. If G and H are quasi-isometric, then $\overline{H}^n_{(2)}(G) \cong \overline{H}^n_{(2)}(H)$.*

2.B. **A second attempt.**
In our next version we will switch to homology, this approach requires defining a new algebra.

Definition 2.4. Define ℓ^2G to be the set of square summable sequences on G. That is,

$$\ell^2 G = \left\{ \sum_{g\in G} c_g g \mid c_g \in \mathbb{C},\ \sum_{g\in G} c_g \overline{c_g} < \infty \right\}.$$

This is exactly the completion of $\mathbb{C}G$ with respect to the inner product

$$\langle \cdot, \cdot \rangle \colon \mathbb{C}G \times \mathbb{C}G \to \mathbb{C} \quad \text{by} \quad \left\langle \sum_{g\in G} a_g g, \sum_{g\in G} b_g G \right\rangle \mapsto \sum_{g\in G} a_g \cdot \overline{b}_g.$$

Note that the inner product extends from $\mathbb{C}G$ to ℓ^2G.

Lemma 2.5. *Let X be a free cocompact G-CW complex. Then,*

$$H^G_n(X; \ell^2 G) \cong H^n_{(2)}(X).$$

Example 2.6. Let $G = \mathbb{Z} = \langle t \rangle$ and $X = S^1$. We have that $Y = \widetilde{X} = \mathbb{R}$ with vertices indexed by $\mathbb{Z}$. The ℓ^2-chain complex of $\mathbb{R}$ is then given by

$$0 \longrightarrow C_1^{(2)}(Y) \xrightarrow{\partial_1^{(2)}} C_0^{(2)}(Y) \longrightarrow 0$$

Note that both $C_0^{(2)}(T)$ and $C_1^{(2)}(T)$ are isomorphic to $\ell^2(\mathbb{Z})$ with bases e and v respectively. We also have $\partial_1^{(2)}(e) = (1-t)v$ which is clearly injective. Hence, $H_1^{(2)}(\mathbb{Z}) = \ker(\partial_1^{(2)}) = 0$. Now, we have a right exact sequence

$$C_1^{(2)}(T) \xrightarrow{\partial_1^{(2)}} C_0^{(2)}(T) \longrightarrow \ell^2(\mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z} \longrightarrow 0$$

so $\overline{\mathrm{im}(\partial_1^{(2)})}^{\perp}$ is mapped injectively into the coinvariants $\ell^2(\mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}$. But, this consists of $\mathbb{Z}$-invariant elements and so is trivial. It follows that $\mathrm{im}(\partial_1^{(2)})$ is dense and $\overline{H}_0^{(2)}(\mathbb{Z}) = 0$.

We still want to show the zeroth unreduced homology group is non-zero. Observe that 1 is not in $\mathrm{im}(\partial_1^{(2)})$, because

$$1 = (1-t)\sum_{i \in \mathbb{Z}} a_i t^i$$

implies that the a_i are all equal for all $i < 0$ hence are all 0, but also that the a_i are all equal for $i \geqslant 0$, hence are all 0.

Adapting the arguments in the previous example we can deduce two facts.

Proposition 2.7. *The following hold:*

(1) If G is an infinite discrete group, then $\overline{H}_0^{(2)}(G) = 0$.
(2) If X is an aspherical n-manifold and $G = \pi_1(X)$, then $H_n^{(2)}(G) = 0$.

Exercise 2.8. Compute $\overline{H}_*^{(2)}(\mathbb{Z}^2)$.

Example 2.9. Let F_2 be a free group, let X be a wedge of 2-circles, and let $T = \widetilde{X}$ denote the 4-valent tree with edges labelled by the free group on a and b. We have the ℓ^2 chain complex

$$0 \longrightarrow C_1^{(2)}(T) \xrightarrow{\partial_1^{(2)}} C_0^{(2)}(T) \longrightarrow 0.$$

We claim that $\overline{H}_1^{(2)}(F_2) \neq 0$ for $n \geqslant 2$.

Let us construct an ℓ^2-1-chain as follows:

$$\begin{aligned} ce = \Bigg(& 1 + \frac{1}{2}\left(a + a^{-1} + b + b^{-1}\right) \\ & + \frac{1}{4}\left(a^2 + ab + ba + b^2 + a^{-2} + a^{-1}b^{-1} + b^{-1}a^{-1} + b^{-2}\right) + \dots \Bigg) e \end{aligned}$$

for some edge e in T. Note that the chain is square summable. Indeed,

$$1 + \cdot 4\frac{1}{2^2} + 8 \cdot \frac{1}{4^2} + 16 \cdot \frac{1}{8^2} + \dots = 1 + \sum_{n \geqslant 2} \frac{2^n}{2^{2n-2}} = 1 + 2 = 3.$$

Now, we compute the boundary of the chain

$$\begin{aligned}\partial_1^{(2)}(ce) &= (1-a)v + \frac{1}{2}\left((a-a^2)+(a^{-1}-1)+(b-ab)+b^{-1}-ab^{-1}\right)v+\dots\\ &= \left(1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\dots\right)v-\left(1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\dots\right)av+\dots\\ &= 0.\end{aligned}$$

Whence, it is an ℓ^2-1-cycle as required.

Exercise 2.10. Show that the non-trivial cycle constructed in the previous example has a boundary in $C_2^{(2)}(\mathbb{Z}\times F_n)$.

Exercise 2.11. Find a non-trivial cycle in $C_2^{(2)}(T\times T)$. Deduce $H_2^{(2)}(F_2\times F_2)$ is non-trivial.

2.C. **Group von Neumann algebras and trace.** An operator A on a Hilbert space H is *bounded* if there exists a constant C such that for all $v\in H$ we have $||Av||\leqslant C||v||$.

We define the *group von Neumann algebra* $\mathcal{N}G$ of G to be the algebra of G-equivariant bounded operators $\ell^2G\to\ell^2G$.

A *Hilbert G-module* is a complex Hilbert space V equipped with an isometric G-action such that there exists an isometric G-embedding $V\to(\ell^2G)^n$ for some n. A *morphism of Hilbert G-modules* $V\to W$ is a bounded $\mathbb{C}$-linear G-map.

The algebra $\mathcal{N}G$ comes equipped with a trace

$$\mathrm{tr}_G\colon \mathcal{N}G\to\mathbb{C}\quad\text{by}\quad a\mapsto\langle a(e),e\rangle_{\ell^2G}$$

where $e\in G$ is the identity.

Lemma 2.12. *The von Neumann trace satisfies the following properties:*

1. *For all $a,b\in\mathcal{N}G$ we have $\mathrm{tr}_G(ab)=\mathrm{tr}_G(ba)$.*
2. *For all $a\in\mathcal{N}G$ we have $\mathrm{tr}_G(aa^*)=0$ if and only if $a=0$*

Exercise 2.13. Let H be a finite subgroup of G. Show that $\ell^2(G/H)$ is a Hilbert G-module and that $\dim_G\ell^2(G/H)=|H|^{-1}$.

[Hint: use the projection $\frac{1}{|H|}\sum_{h\in H}h$.]

We can extend the trace to matrices over the von Neumann algebra as follows

$$\mathrm{tr}_G\colon \mathbf{M}_n(\mathcal{N}G)\to\mathbb{C}\quad\text{by}\quad (M_{i,j})\mapsto\sum_{i=1}^n\mathrm{tr}_G M_{i,i}.$$

The *von Neumann dimension* $\dim_G V$ of a Hilbert G-module V is defined as follows: let $i\colon V\to(\ell^2G)^n$ be the given embedding and let $\pi:(\ell^2G)^n\twoheadrightarrow i(V)$ denote the orthogonal projection. We set

$$\dim_G V=\mathrm{tr}_G(\pi)\in\mathbb{R}_{\geqslant 0}\cup\{\infty\}.$$

Theorem 2.14. *The above definition is independent of the embedding i.*

Proposition 2.15. *Let G be a countable group and let U,V,W be Hilbert G-modules.*

(1) $\dim_G \ell^2 G = 1$*;*
(2) $\dim_G V = 0$ *if and only if* $V = 0$*;*
(3) *if* $0 \to U \to V \to W \to 0$ *is exact then* $\dim_G V = \dim_G U + \dim_G W$*.*

2.D. **Lück's formulation.** So far all of the definitions have required a free action and we have to worry about closures. This can be extremely limiting. To remedy this, the category of Hilbert G-modules embeds into the category of $\mathcal{N}G$-modules. Moreover, Lück shows that this embedding can be refined to an equivalence of categories F between the category of Hilbert G modules and the category of finitely generated projective $\mathcal{N}G$-modules, such that $F(\ell^2 G) = \mathcal{N}G$. Lück shows for finitely generated modules that $\dim_G V = \dim_G F(V)$. For a non finitely generated module we take the supremum of the dimensions of finitely generated projective sub-modules.

From here on out for a G-CW complex X we define the ℓ^2-homology of X to be the homology groups $H_n^G(X;\mathcal{N}G)$. (Note there is no hypothesis on the stabilisers.)

The upshot of all this is we can use things like spectral sequences to compute the ℓ^2-homology.

2.E. **Betti numbers.** Let X be a G-CW-complex. We define the *nth* ℓ^2*-Betti number of* X to be
$$b_n^{(2)}(X) = \dim_G H_n^G(X;\mathcal{N}G).$$
For a group G we set
$$b_n^{(2)}(G) = b_n^{(2)}(EG).$$
That these are well defined group invariants follows from the next theorem.

Theorem 2.16 (Properties)**.** *Let* G, H *be groups.*

(1) *If* $f\colon X \to Y$ *is a* G*-homotopy equivalence, then*
$$b_p^{(2)}(X) = b_p^{(2)}(Y) \quad \text{for } p \geqslant 0.$$
(2) *Let* X *be a* G*-CW complex and* Y *be an* H*-CW complex. Then,* $X \times Y$ *is a* $G \times H$*-CW complex and*
$$b_n^{(2)}(X \times Y) = \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(Y).$$
(3) *Let* K *be a finite index subgroup of* G*. If* X *is a* G*-CW complex (and hence a* K*-CW complex by restriction), then*
$$b_p^{(2)}(X;K) = |G:K| \cdot b_p^{(2)}(X;G).$$
In particular,
$$b_p^{(2)}(K) = |G:K| \cdot b_p^{(2)}(G).$$
(4) *Suppose* $H \leqslant G$ *and let* X *be an* H*-CW complex. Then,*
$$b_p^{(2)}(G \times_H X; G) = b_p^{(2)}(X;H).$$
(5) $b_0^{(2)}(X;G) = |G|^{-1}$*, where* $|G|^{-1} = 0$ *if* G *is infinite.*
(6) $b_n^{(2)}(X) \leqslant c_n$*, where* c_n *is the number of equivariant* n*-cells in* X*.*

(7) If X is a finite free G-CW-complex, then

$$\chi(X/G) = \sum_{i\geqslant 0}(-1)^i b_i^{(2)}(X).$$

(8) If M is an n-manifold, then

$$b_i^{(2)}(\widetilde{M}) = b_{n-i}^{(2)}(\widetilde{M}).$$

(9) Let X be a finite CW-complex. Then, $b_1^{(2)}(\tilde{X}) = b_1^{(2)}(\pi_1 X)$.

(10) Let $X_1, \dots X_r$ be pointed CW complexes and let $X = \bigvee_{i=1}^r X_i$. Then,

$$b_1^{(2)}(\tilde{X}) - b_0^{(2)}(\tilde{X}) = r - 1 + \sum_{j=1}^r \left(b_1^{(2)}(\tilde{X}_j) - b_0^{(2)}(\tilde{X}_j)\right)$$

$$b_p^{(2)}(\tilde{X}) = \sum_{j=1}^r b_p^{(2)}(\tilde{X}_j) \text{ for } 2 \leqslant p.$$

Exercise 2.17. Compute the ℓ^2-Betti numbers of surface groups, free groups, and direct products of free groups.

Exercise 2.18. Let X be a closed (triangulated) 4-manifold with Euler characteristic c. Compute $b_p^{(2)}(\tilde{X})$, for all p, in terms of c and $b_1^{(2)}(\pi_1 X)$.

2.F. **Measure equivalence invariance.** Two groups are *measure equivalent* if they admit free, measure-preserving actions on a common standard probability space that share the same orbits (almost everywhere).

The key example of measure equivalent groups are lattices in the same locally compact groups.

Theorem 2.19 (Gaboriau)**.** *Let G and H be countable measure equivalent groups with measure coupling C. Then, $b_p^{(2)}(G) = C \cdot b_p^{(2)}(H)$.*

Exercise 2.20. Let Γ be a lattice in $\mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_2)$ where each $\mathcal{T}_i$ is an n_i-regular tree with $n_i \geqslant 3$. Show $b_p^{(2)}(\Gamma) = 0$ for $p \neq 2$.

Theorem 2.21 (Lück)**.** *Let G be an infinite amenable group. Then, $b_p^{(2)}(G) = 0$ for all $p \geqslant 0$.*

Proof. Every infinite amenable group is measure equivalent to $\mathbb{Z}$. The result follows from Gaboriau's Theorem. □

Remark 2.22. Lück's original proof instead shows: if G is amenable, then $\mathcal{N}G$ is dimension-flat over $\mathbb{C}G$, i.e. $\dim_G \mathrm{Tor}_p^{\mathbb{C}G}(\mathcal{N}G; M) = 0$ for $p \geqslant 1$ and every $\mathbb{C}G$-module M.

3. Mayer–Vietoris sequences

Recall the classical Mayer–Vietoris sequence for singular homology. Namely, given a space X which can be written as a union $X_1 \cup X_2$ with intersection $Z = X_1 \cap X_2$, there is a long exact sequence in singular homomology

$$\cdots \to H_{n+1}(X;R) \xrightarrow{\delta_n} H_n(Z;R) \xrightarrow{(i_n,j_n)} H_n(X_1;R) \oplus H_n(X_2;R) \xrightarrow{k_n - \ell_n} H_n(X;R) \to \cdots$$

where i, j, k, ℓ are the natural inclusions.

In group homology there is a very natural situation where this occurs. Namely, given a group G written as an amalgamated free product $A_1 *_C A_2$, we obtain a long exact sequence in group homology

$$\cdots \to H_{n+1}(G;M) \xrightarrow{\delta_n} H_n(C;M) \xrightarrow{(i_n,j_n)} H_n(A_1;M) \oplus H_n(A_2;M) \xrightarrow{k_n - \ell_n} H_n(G;R) \to \cdots$$

for any G-module M. Note that the coefficients for A_1, A_2, C are the restrictions of M to each group.

The main challenge in computations (especially with non-trivial coefficients) is understanding the maps involved.

Remark 3.1. The following observation is often helpful when computing L^2-homology. Suppose $A \leqslant G$. Then,

$$\dim_A H_n(A; \mathcal{N}A) = \dim_G H_n^G(G \times_A EA; \mathcal{N}G)$$

and

$$H_n^G(G \times_A EA; \mathcal{N}G) \cong H_n(A; \mathcal{N}G).$$

In particular, we have

$$\dim_A H_n(A; \mathcal{N}A) = \dim_G H_n(A; \mathcal{N}G).$$

Example 3.2 (Fernós–Valette, Chatterji–H.–Kropholler)**.** Let G be the fundamental group of a finite graph of groups such that each edge group satisfies $b_1^{(2)}(G_e) = 0$. Then,

$$b_1^{(2)}(G) = \frac{1}{|G|} + \sum_{v \in V} \left(b_1^{(2)}(G_v) - \frac{1}{|F_v|} \right) + \sum_{e \in E} \frac{1}{|F_e|}.$$

Consider the relevant portion of the Mayer–Vietoris sequence

$$\cdots \to H_2(G; \mathcal{N}G) \to \bigoplus_{e \in E} H_1(G_e; \mathcal{N}G) \to \bigoplus_{v \in V} H_1(G_v; \mathcal{N}G) \to H_1(G; \mathcal{N}G) \to \ldots$$

and using the remark and the hypothesis this becomes the exact sequence

$$0 \to \bigoplus_{v \in V} H_1(G_v; \mathcal{N}G) \to H_1(G; \mathcal{N}G) \to \bigoplus_{e \in E} H_0(G_e; \mathcal{N}G) \to \bigoplus_{v \in V} H_0(G_v; \mathcal{N}G) \to H_0(G : \mathcal{N}G) \to 0.$$

Computing von Neumann dimensions with respect to G and rearranging gives the equation

$$b_1^{(2)}(G) = \frac{1}{|G|} + \sum_{v \in V} \left(b_1^{(2)}(G_v) - \frac{1}{|F_v|} \right) + \sum_{e \in E} \frac{1}{|F_e|}$$

as required. ■

4. The Mapping Torus Theorem

Let $f\colon X \to X$ be a selfmap. It *mapping torus* T_f is the space

$$X \times [0,1]/\sim \quad \text{where} \quad (x,0) \sim (f(x),1).$$

There is a canonical map $p\colon T_f \to S^1$ by $(x,t) \mapsto e^{2i\pi t}$. If X is path connected, then p induces a canonical epimorphism $\pi_1 T_f \twoheadrightarrow \mathbb{Z}$.

Theorem 4.1 (Lück)**.** *Let $f\colon X \to X$ be a cellular self map of a finite connected CW complex. Let T_f denote the mapping torus with $G = \pi_1 T_f$. Then, $b_p^{(2)}(\widetilde{T}_f) = 0$ for all $p \geqslant 0$.*

Proof. Write $G = \pi_1 X \rtimes \mathbb{Z}$ with the $\mathbb{Z}$ factor corresponding to 'going-around-the-mapping-torus'. Let G_n denote the preimage of $n\mathbb{Z}$ under the projection $\psi\colon G \twoheadrightarrow \mathbb{Z}$. Note $|G : G_n| = n$.

We have

$$b_p^{(2)}(\widetilde{T}_f; G) = \frac{1}{n} b_p^{(2)}(\widetilde{T}_f; G_n). \tag{1}$$

There is a homotopy equivalence

$$h\colon T_{f^n} \to \widetilde{T}_f / G_n$$

where $f^n = f \circ \cdots \circ f$. The map h induces a G_n homotopy equivalence

$$\widetilde{T}_{f^n} \to T_f.$$

Thus,

$$b_p^{(2)}(\widetilde{\widetilde{T}_f}; G_n) = b_p^{(2)}(\widetilde{T}_{f^n}; G_n). \tag{2}$$

Let c_p denote the number of p-cells in X. We may endow T_{f^n} with a CW structure consisting of $c_p + c_{p-1}$ many p-cells [exercise]. Hence,

$$b_p^{(2)}(\widetilde{T}_{f^n}) \leqslant c_p + c_{p-1}. \tag{3}$$

Now, combining (1), (2), and (3) we obtain

$$b_p^{(2)}(\widetilde{T}_f) \leqslant \frac{1}{n}\left(c_p + c_{p-1}\right).$$

Since $c_p + c_{p-1}$ is independent of n, the claim follows from taking the limit as $n \to \infty$. □

4.A. Fibring theorems.

Theorem 4.2 (H.–Kielak)**.** *Let G be a group of type $\mathsf{FP}_n(\mathbb{Q})$. If $b_n^{(2)}(G) \neq 0$, then $\Sigma^n(G;\mathbb{Q}) = \varnothing$.*

Theorem 4.3 (Kielak, Fisher)**.** *Suppose G is a RFRS group of type $\mathsf{FP}_n(\mathbb{Q})$. Then, G is virtually $\mathsf{FP}_n(\mathbb{Q})$-fibred if and only if $b_i^{(2)}(G) = 0$ for $i \leqslant n$.*

Theorem 4.4 (Kielak–Linton, Fisher)**.** *Suppose G is a finitely generated RFRS group with $\mathrm{cd}_{\mathbb{Q}}(G) = 2$. If $b_2^{(2)}(G) = 0$, then G is virtually free-by-cyclic.*

4.B. **Normal subgroups.**

Theorem 4.5 (Gaboriau, Sánchez-Peralta)**.** *Let G be a countable group and let $N \lhd G$ have infinite index. If $b_p^{(2)}(N) = 0$ for $p \leqslant n-1$ and $b_n^{(2)}(N) < \infty$, then $b_n^{(2)}(G) = 0$.*

Corollary 4.6 (Gaboriau)**.** *Let G be a finitely generated group. If $b_1^{(2)}(G) \neq 0$, then every infinite index normal subgroup is infinitely generated.*

5. Lück's approximation theorem

Let G be a residually finite group. We say a chain $G = G_0 \geqslant G_1 \geqslant \cdots$ of finite index subgroups of G is a *residual chain* if each $G_i \trianglelefteq G$ and if $\bigcap_{i \geqslant 0} G_i = 1$.

Theorem 5.1 (Lück)**.** *Let X be a G-space with finite $(n+1)$-skeleton and let (G_i) be a residual chain. Then,*

$$b_n^{(2)}(X) = \lim_{i \to \infty} \frac{b_n(X/G_i)}{|G : G_i|}.$$

Note that even the statement that the right hand side is a limit and not a limit supremum is non-trivial.

State Lück–Osin.

Open Question 5.2. *Let X be a G-space with finite $(n+1)$-skeleton and let (G_i) be a residual chain. Is the quantity*

$$\limsup_{n \to \infty} \frac{b_n(X/G_i; \mathbb{F}_p)}{|G : G_i|}$$

a genuine limit? Is it independent of the residual chain? What does it converge to?

5.A. **Deficiency and rank gradient.**

Define the *deficiency* of G to be maximum $g(P) - r(P)$ where P runs over all finite presentations of G. Here $g(P)$ is the number of generators and $r(P)$ is the number of relations in P.

Exercise 5.3. Let $G = \langle S \mid R \rangle$ be a finitely presented group. Then, $\operatorname{def}(G) \leqslant 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G)$.

For a residually finite group G and residual chain of finite index normal subgroups (G_n), the *rank gradient* of G with respect to (G_n) is

$$\mathrm{RG}(G; (G_n)) = \lim_{n \to \infty} \frac{d(G_n) - 1}{|G : G_n|}$$

Exercise 5.4. Let G be a finitely presented residually finite group. Then,

$$b_1^{(2)}(G) \leqslant \mathrm{RG}(G; (G_n)).$$

Open Question 5.5. *Let G be a finitely presented residually finite group. Is $b_1^{(2)}(G) = \mathrm{RG}(G; (G_n))$?*

5.B. **Profinite invariance.**

Theorem 5.6. *Let G and H be finitely generated residually finite groups such that $\widehat{G} = \widehat{H}$.*

(1) [Bridson–Conder–Reid] *Then, $b_1^{(2)}(G) = b_1^{(2)}(H)$.*
(2) [Kammeyer–Kionke–Raimbault–Sauer] *$b_n^{(2)}(G)$ is not a profinite invariant for $n \geqslant 2$.*

6. Affiliated operators

6.A. **Ore localisation.** In this section we will describe an analogue of localisation for non-commutative rings.

Definition 6.1. Let R be a ring. An element $x \in R$ is a *zero-divisor* if $x \neq 0$, and $xy = 0$ or $yx = 0$ for some non-zero $y \in R$. A non-zero element that is not a zero-divisor will be called *regular.*

Definition 6.2 (Right Ore condition)**.** Let R be a ring and $S \subseteq R$ a multiplicatively closed subset consisting of regular elements. The pair (R, S) satisfies the *right Ore condition* if for every $r \in R$ and $s \in S$ there are elements $r' \in R$ and $s' \in S$ satisfying $rs' = sr'$.

Definition 6.3 (Right Ore localisation)**.** If (R, S) satisfies the right Ore condition we may define the *right Ore localisation*, denoted RS^{-1}, to be the following ring. Elements are represented by pairs $(r, s) \in R \times S$ up to the following equivalence relation: $(r, s) \sim (r', s')$ if and only if there exists $u, u' \in R$ such that the equations $ru = r'u'$ and $su = s'u'$ hold, and $su = s'u'$ belongs to S. The addition is given by

$$(r, s) + (r', s') = (rc + r'd, t), \text{ where } t = sc = s'd \in S,$$

and the multiplication is given by

$$(r, s)(r', s') = (rc, s't), \text{ where } sc = r't \text{ with } t \in S.$$

6.B. **The algebra of affiliated operators.** Let G be a group. An operator A on a Hilbert space H is *closed* if the graph of A is closed; is *densely defined* if its domain $\mathrm{dom}(f)$ is dense in H; is a *G-operator* if $\mathrm{dom}(f)$ is a linear G-invariant subspace and f satisfies $f(x) \cdot g = f(x \cdot g)$ for all $g \in G$.

Definition 6.4 (Affiliated operators)**.** We say that an operator

$$f\colon \mathrm{dom}(f) \to \ell^2 G$$

with $\mathrm{dom}(f) \subseteq \ell^2 G$ is *affiliated* (to $\mathcal{N}G$) if f is densely defined closed G-operator (recall that G acts on $\ell^2 G$ on the right). The set of all operators affiliated to $\mathcal{N}G$ forms the *algebra of affiliated operators $\mathcal{U}G$* of G.

Since an adjoint of a densely defined closed operator is densely defined and closed, every $x \in \mathcal{U}G$ has a well-defined adjoint $x^* \in \mathcal{U}G$.

Note that we have inclusions of $\mathbb{Q}G$-modules

$$\mathbb{Q}G \rightarrowtail \mathbb{C}G \rightarrowtail \mathcal{N}G \rightarrowtail \mathcal{U}G.$$

Theorem 6.5 (Roos)**.** *The set S of regular elements of $\mathcal{N}G$ forms a right Ore set. Moreover, $\mathcal{U}G$ is canonically isomorphic to $(\mathcal{N}G)S^{-1}$.*

Definition 6.6. For a finitely generated projective $\mathcal{U}G$-module Q define

$$\dim_{\mathcal{U}G} Q := \dim_G P$$

where P is any finitely generated $\mathcal{N}G$-module P such that $\mathcal{U}G \otimes_{\mathcal{N}G} P \cong_{\mathcal{U}G} Q$. For a general $\mathcal{U}G$-module Q we take the supremum of $\dim_{\mathcal{U}G}$-dimensions of the finitely generated projective submodules.

Since $\mathcal{U}G$ is flat over $\mathcal{N}G$ we obtain that

$$b_p^{(2)}(X;G) = \dim_{\mathcal{U}G} H_p^G(G;\mathcal{U}G).$$

Wolfgang Lück describes the passage of $\mathcal{N}G$ to $\mathcal{U}G$ as being like the passage from $\mathbb{Z}$ to $\mathbb{Q}$. One loses all of the torsion submodule information, but often computations are simpler.

6.C. **The Linnell ring.**

Definition 6.7 (Division and rational closure)**.** Let R be a ring and S a subring. We say that S is *division closed* if every element of S invertible over R is invertible over S. We say that S is *rationally closed* if every finite square matrix over S invertible over R is invertible over S.

Define the *division closure of S in R*, denoted by $\mathcal{D}(S \subset R)$, to be the smallest division-closed subring of R containing S. Define the *rational closure of S in R*, denoted by $\mathcal{R}(S \subset R)$, to be the smallest rationally closed subring of R containing S.

Definition 6.8. For a group G, the *Linnell ring* $\mathcal{D}_{\mathbb{Q}G}$ is defined ot be the ring $\mathcal{D}(\mathbb{Q}G \subset \mathcal{U}G)$.

6.D. **The Atiyah Conjecture.**

Conjecture 6.9 (The Atiyah Conjecture)**.** *For every countable torsionfree group G and every $A \in \mathbf{M}_n(\mathbb{Q}G)$, the kernel K of the operator $A\colon (\ell^2 G)^n \to (\ell^2 G)^n$ satisfies $\dim_G K \in \mathbb{Z}$.*

Theorem 6.10 (Linnell)**.** *For a torsionfree group G the following are equivalent:*

(1) the Atiyah Conjecture is true for G;
(2) $\mathcal{D}_{\mathbb{Q}G}$ is a skew field.

Theorem 6.11 (Jaikin Zapirain–López-Álvarez)**.** *Locally indicable groups satisfy the Atiyah Conjecture.*

6.E. **One relator groups.**

Theorem 6.12 (Dicks–Linnell)**.** *Let G be a non-trivial torsion-free one-relator group. Then, $b_p^{(2)}(G) = 0$ for all $p \neq 1$ and $b_1^{(2)}(G) = -\chi(G)$.*

Proof. For a one-relator group $G = \langle x_1, \ldots, x_s \mid r \rangle$ we have a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ coming from an aspherical presentation 2-complex, namely

$$0 \to \mathbb{Z}G.r \xrightarrow{J} \mathbb{Z}G.\{x_1, \ldots, x_s\} \xrightarrow{\partial_0} \mathbb{Z}G \to \mathbb{Z} \to 0, \tag{4}$$

where J is Jacobian of fox derivatives. Unaugmenting the resolution and tensoring with $\mathcal{U}G$ we obtain

$$0 \to \mathcal{U}G.r \xrightarrow{J} \mathcal{U}G.\{x_1, \ldots, x_s\} \xrightarrow{\partial_0} \mathcal{U}G \to 0. \tag{5}$$

Claim 6.13. *Let G be a left orderable group. Let $y \in \mathcal{U}G$ and $a \in \mathbb{C}G$ both be non-zero. Then, $y \cdot a \neq 0$.*

Lets see how the claim proves the theorem.

In (4) the map J is injective. So either $r = 0$ or there exists some x_i such that $\partial r/\partial x \neq 0$. The claim then implies that J is injective in (5). Hence, $H_p(G;\mathcal{U}G) = 0$ for all $p \geqslant 2$. Taking $\mathcal{U}G$-dimensions we obtain that

$$\dim_{\mathcal{U}G} \ker \partial_0 = s - 1$$
$$\dim_{\mathcal{U}G} \operatorname{im} J = 1.$$

In particular, $b_1^{(2)}(G) = s - 2 = -\chi(G)$. □

Proof of Claim 6.13. We first establish the fact that a is invertible in $\mathcal{U}G$. Observe that since locally indicable groups satisfy the Atiyah Conjecture we have

$$0 \neq \dim_{\mathcal{U}G} a \cdot \mathcal{U}G \geqslant 1 = 1.$$

Whence, a is invertible. To prove the claim we now suppose that $ya = 0$ and that $a \neq 0$. Then, $y^*yaa^* = 0$ with $y^*y \in \mathcal{U}G$ and $0 \neq aa^* \in \mathbb{C}G$. But, aa^* is invertible in $\mathcal{U}G$ since both a and a^* are. Hence, $y^*y = 0$ and so $y = 0$. ■

7. Right-angled Artin groups

Let $C_{\bullet,\bullet}$ be a double complex with horizontal differential d_h and vertical differential d_v. The *total complex* in degree n is given by $TC_n = \bigoplus_{i+j=n} C_{i,j}$. The *total differential* $d_t : TC_n \to TC_{n-1}$ is given by $d_t = d_h + (-1)^i d_v$. We have two filtrations, the *horizontal filtration*

$$F_p^h TC_n = \bigoplus_{i+j=n, i \leqslant p} C_{i,j}$$

and the *vertical filtration*

$$F_q^v TC_n = \bigoplus_{i+j=n, j \leqslant q} C_{i,j}.$$

Each filtration gives rise to a spectral sequence $E^*_{*,*}$ converging to the homology of the total complex. The spectral sequence consists of a series of *"pages"* E^n and in favourable circumstances we get a stabilisation

$$E^n = E^{n+1} = \cdots = E^\infty.$$

In theory each page can be computed from the previous one, but in practice this can be rather tricky.

Theorem 7.1 (Davis–Leary)**.** *Let L be a connected flag complex and let A_L be a right-angled Artin group. Then, $b_n^{(2)}(A_L) = \widetilde{b}_{n-1}(L)$.*

Proof. Let $\mathcal{L}$ denote the maximal simplicies in L. For $\sigma \in L$ let X_σ denote the subcomplex of the Salvetti complex X spanned by the vertices σ. Note that X_σ is a k-torus for some k. Let Y_σ be the union of the lifts of X_σ to the universal cover $\widetilde{X}$. For $\varnothing \neq S \subseteq \mathcal{L}$ define

$$Y_S = \bigcap_{\sigma \in S} Y_\sigma.$$

Note that each Y_S is a free A_L-CW complex (in fact it is an A_L-orbit of m-flats).

If $S \neq \varnothing$, then we have that $H_*^{A_L}(Y_S; \mathcal{N}A_L) \cong H_*(\mathbb{Z}^m; \mathcal{N}\mathbb{Z}^m)$ which, after applying Lück's equivalence of categories, vanishes.

If $S = \varnothing$, then $Y_S = \widetilde{X}$.

We define a double complex so we can run a spectral sequence argument. The double complex comes from filtering $\widetilde{X}$ by the Y_σ. Let

- $C_{\bullet,0} = C_\bullet(\widetilde{X})$;
- $C_\bullet, j = \bigoplus_{S \subseteq \mathcal{L},\ |S| = j} C_\bullet(Y_S)$ for $j > 0$;
- the boundary map of degree $(-1, 0)$ are the boundary maps in $C_\bullet(Y_S)$;
- the boundary map of degree $(0, -1)$ are given by matrices whose (S, T) entry is given by $\epsilon(S, T)$ times the map induced by the inclusion of $Y_S \to Y_T$, where $\epsilon(S, T) = (-1)^i$ if T is obtained from S by omitting the ith element of S (for some fixed ordering of $\mathcal{L}$).

Note that this double complex has trivial homology because the boundary map of degree $(0, -1)$ is exact. Since $C_{i,j}$ is free, the chain complex $C_{\bullet,i}$ is split exact.

Define a double complex $E^0_{i,j} := C_{i,j} \otimes_{A_L} \mathcal{N}A_L$ and let $E^*_{i,j}$ denote the spectral sequence of the double complex with differential d_0 induced by the boundary map of degree $(-1, 0)$.

$$\begin{tikzcd}
3 & C_{0,3}\otimes_A \mathcal{N}A & C_{1,3}\otimes_A \mathcal{N}A \arrow[l] & C_{2,3}\otimes_A \mathcal{N}A \arrow[l] & C_{3,3}\otimes_A \mathcal{N}A \arrow[l] \\
2 & C_{0,2}\otimes_A \mathcal{N}A & C_{1,2}\otimes_A \mathcal{N}A \arrow[l] & C_{2,2}\otimes_A \mathcal{N}A \arrow[l] & C_{3,2}\otimes_A \mathcal{N}A \arrow[l] \\
1 & C_{0,1}\otimes_A \mathcal{N}A & C_{1,1}\otimes_A \mathcal{N}A \arrow[l] & C_{2,1}\otimes_A \mathcal{N}A \arrow[l] & C_{3,1}\otimes_A \mathcal{N}A \arrow[l] \\
0 & C_0(\widetilde{X})\otimes_A \mathcal{N}A & C_1(\widetilde{X})\otimes_A \mathcal{N}A \arrow[l] & C_2(\widetilde{X})\otimes_A \mathcal{N}A \arrow[l] & C_3(\widetilde{X})\otimes_A \mathcal{N}A \arrow[l] \\
j/i & 0 & 1 & 2 & 3
\end{tikzcd}$$

The boundary map of degree $(0,-1)$ is exact and so the homology of the total complex TE_n vanishes. It follows that $E^\infty_{i,j} = 0$ for all i,j.

We have that the jth row is

$$\bigoplus_{S\subseteq\mathcal{L},\ |S|=j} C_0(Y_S)\otimes_A\mathcal{N}A \leftarrow \bigoplus_{S\subseteq\mathcal{L},\ |S|=j} C_1(Y_S)\otimes_A\mathcal{N}A \leftarrow \bigoplus_{S\subseteq\mathcal{L},\ |S|=j} C_2(Y_S)\otimes_A\mathcal{N}A \leftarrow \cdots$$

We now describe the E^1-page,

- $E^1_{i,0} = H_i(A;\mathcal{N}A)$ if $i>0$;
- $E^1_{i,j} = 0$ if both $i,j>0$;
- $E^1_{0,j} = \bigoplus_{k_j}\mathcal{N}A$;
- k_j is the number of j-element subsets of $\mathcal{L}$ such that the intersection of the corresponding simplices of L is empty.

$$\begin{tikzcd}
3 & \bigoplus_{k_3}\mathcal{N}A & 0 & & \\
2 & \bigoplus_{k_2}\mathcal{N}A & 0 & & \\
1 & \bigoplus_{k_1}\mathcal{N}A & 0 & 0 & 0 \\
0 & \bigoplus_{k_0}\mathcal{N}A & H_1(A;\mathcal{N}A) & H_2(A;\mathcal{N}A) \arrow[ull, "d^2"'] & H_3(A;\mathcal{N}A) \arrow[uulll, "d^3"'] \\
j/i & 0 & 1 & 2 & 3
\end{tikzcd}$$

Claim 7.2. $E^2_{0,j} = \widetilde{H}_j(L;\mathcal{N}A)$ *for* $j>0$.

Proof of claim. The chain complex $E^1_{0,\bullet}$ embeds as a subcomplex in an exact complex $C_\bullet$, where

$$C_j = \bigoplus_{S\subseteq\mathcal{L},\ |S|=j}\mathcal{L}A.$$

Let $Q_\bullet = C_\bullet/E^1_{0,\bullet}$. The short exact sequence of chain complexes

$$0\to E^1_{0,\bullet}\to C_\bullet\to Q_\bullet\to 0$$

gives a homology long exact sequence

$$\cdots \to H_n(E^1_{0,\bullet}) \to H_n(C_\bullet) \to H_n(Q_\bullet) \to H_{n-1}(E^1_{0,\bullet}) \to \cdots$$

and so $H_n(E^1_{0,\bullet}) \cong H_{n-1}(Q_\bullet)$.

Now, $Q_\bullet$ is isomorphic to the augmented chain complex for the nerve of the covering of L by the elements of $\mathcal{L}$ shifted in degree by one (with coefficients in $\mathcal{N}A$). That is, $H_{n-1}(Q_\bullet) = \widetilde{H}_n(L; \mathcal{N}A)$. ■

Thus, we have

$$\begin{tikzcd}
3 & H_3(L;\mathcal{N}A) & 0 & & \\
2 & H_2(L;\mathcal{N}A) & 0 & & \\
1 & H_1(L;\mathcal{N}A) & 0 & 0 & 0 \\
0 & H_0(A;\mathcal{N}A) & H_1(A;\mathcal{N}A) & H_2(A;\mathcal{N}A) \arrow[ull, "d^2"'] & H_3(A;\mathcal{N}A) \arrow[uulll, "d^3"'] \\
j/i & 0 & 1 & 2 & 3
\end{tikzcd}$$

and the d^i must be isomorphisms. □

Theorem 7.3 (Fisher–H.–Leary)**.** *Let R be a skew field, let A_L be a right-angled Artin group, and let $RA_L \to \mathcal{D}$ be an embedding where $\mathcal{D}$ is skew-field. Then, $b_n^{\mathcal{D}}(A_L) = \widetilde{b}_{n-1}(L; R)$.*

Theorem 7.4 (Avramidi–Okun–Schreve)**.** *Let A_L be a RAAG and let (G_i) be a residual chain. Then,*

$$\lim_{i\to\infty} \frac{b_n(G_i; \mathbb{F}_p)}{|G : G_i|} = \widetilde{b}_{n-1}(L; \mathbb{F}_p).$$

Theorem 7.5 (Fisher–H.–Leary)**.** *Let $\mathbb{F}$ be a skew field, let $\varphi \colon A_L \to \mathbb{Z}$ be an epimorphism and let BB_L^φ denote* $\ker\varphi$*. If BB_L^φ is of type $\mathsf{FP}_{n+1}(\mathbb{F})$ then*

$$b_m^{\mathcal{D}_{\mathbb{F}BB_L^\varphi}}(BB_L^\varphi) = b_m^{(2)}(BB_L^\varphi; \mathbb{F}) = \sum_{v \in L^{(0)}} |\varphi(v)| \cdot \widetilde{b}_{m-1}(\mathrm{Lk}(v); \mathbb{F}).$$

for all $m \leqslant n$.

8. Some other applications

8.A. **Acylindrical hyperbolicity.**

Theorem 8.1 (Osin)**.** *Let G be a finitely presented indicable group. If $b_1^{(2)}(G) \neq 0$, then G is acylindrically hyperbolic.*

8.B. **Simple algebras.**

Theorem 8.2 (Breuillard–Kalantar–Kennedy–Ozawa)**.** *If a group G has no non-trivial finite normal subgroup and some $b_k^{(2)}(G) \neq 0$, then $C_r^*(G)$ is a simple algebra.*

8.C. **Coherence.**

Theorem 8.3 (Jaikin-Zapirain–Linton)**.** *Let G be a locally indicable group of type FP_2 with $\mathrm{cd}(G) = 2$. If $b_2^{(2)}(G) = 0$, then G is coherent.*