Profinite and residual methods in geometric group theory

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Contents

Introduction

Todo 1: Write introduction

MOTIVATING QUESTION (Profinite rigidity). *Given a finitely generated group G and a class of groups $\mathcal{C}$. To what extent do the isomorphism classes of finite quotients of G determine G amongst groups in $\mathcal{C}$?*

CHAPTER I

Residual finiteness and the profinite topology

I.A. Residual finiteness

I.A.1. Definitions, examples, and basic properties. We will make repeated use of the following lemma of Marshall Hall Jr. [**Hal50**]

Lemma I.A.1.1. *A finitely generated group G has only finitely many subgroups of a given finite index n.*

Proof. Let H be a subgroup of G of index n and consider the action of G on the set of cosets G/H. Note that $|G/H| = n$. The action defines a homomorphism $\psi\colon G \to \mathrm{Sym}(G/H)$. Now, $g \in H$ if and only if $gH = H$. So $H = \mathrm{Stab}_G(H)$ and this is completely determined by the homomorphism ψ. Let X be a finite generating set for G. The homomorphism ψ is determined by its images $\psi(x)$ for $x \in X$. But $\mathrm{Sym}(n)$ is finite so there can only be finitely many possibilities for ψ and hence only finitely many possibilities for H. □

We now turn to the star of the section: residual properties.

Definition I.A.1.2 (Residually $\mathcal{P}$)**.** Let $\mathcal{P}$ be a property of a group (e.g. finite, nilpotent, soluble, amenable). A group G is *residually* $\mathcal{P}$ if for every non-identity element $g \in G$, there exists a homomorphism $\alpha\colon G \twoheadrightarrow Q$ such that Q has $\mathcal{P}$ and $\alpha(g) \neq 1_Q$.

Note that this is equivalent to the following: for every element $g \in G$ there exists a normal subgroup $N \lhd G$ such that G/N has $\mathcal{P}$ and $g \notin N$.

We will now hone in on residual finiteness.

Examples I.A.1.3. We record a number of groups that are easily seen to be residually finite.

(1) Finite groups are clearly residually finite.
(2) The group $\mathbb{Z}$ is residually finite.
Proof. for each $n \in \mathbb{Z}$ such that $n \neq 0$, there exists a quotient $\alpha_k\mathbb{Z} \twoheadrightarrow \mathbb{Z}/k$ with k relatively prime to n. In particular, $n \pmod{k} \not\equiv 0$ so $\alpha_k(n) \neq 0$. ◆
(3) Direct products of residually finite groups are residually finite.
Proof. To see this note that every element has non-trivial image when projected to one of the factors. Now, compose with a finite quotient there. ◆
(4) Free groups are residually finite.
Proof. Let X be the generators of a free group F. Consider a word $x_n \dots x_1$ in reduced form where x_i or x_i^{-1} is in X. We will build a homomorphism $F \twoheadrightarrow \mathrm{Sym}(n+1)$, the group of permutations of

$\Omega := \{1, \ldots, n+1\}$. Define a function $X \to \Omega$ by $f(y) = 1$ if y does not equal any x_i or x_i^{-1}. For the remaining elements we define f as follows:

Let A be the set of consisting of the i such that $x_i = y$ and B be the set of consisting of the j such that $x_j^{-1} = y$. Set $f(y)$ as any permutation σ that sends each $i \in A$ to $i + 1$, and for each $j \in B$, sends $j+1$ to j. This is well-defined since an element and its inverse cannot occur adjacently in the reduced form for a word.

The function f extends to a homomorphism $\phi\colon F \to \mathrm{Sym}(n+1)$ by the universal property of F. Moreover, x has non-trivial image since $\phi(x)(1) = n + 1$. As x was arbitrary we are done. ◆

(5) An infinite simple group is not residually finite.

Lemma I.A.1.4. *A group G is residually finite if and only if the intersection of all finite index normal subgroups of G is trivial.*

Proof. Suppose G is residually finite and let $g \in G$ be non-trivial. By the definition of residual finiteness there is a finite quotient $\alpha_g\colon Q \twoheadrightarrow G$ such that $\alpha_g(g) \neq 1_Q$. Now, $\ker\alpha$ is a finite index normal subgroup of G such that $g \notin \ker\alpha$. Since g was arbitrary it follows the intersection of all finite index normal subgroups of G equals $\{1\}$.

Conversely, suppose that the intersection of all finite index normal subgroups of G is trivial and let $g \in G$ be non-trivial. By hypothesis there exists a finite index normal subgroup $N \trianglelefteq G$ such that $g \notin N$. Thus, g has non-trivial image in the finite quotient G/N. Since g was arbitrary it follows that G is residually finite. □

For a general group G we will denote the intersection of all finite index normal subgroups by $G^{(\infty)}$. We call this subgroup the *finite residual* of G. By the previous lemma G is residually finite if and only if $G^{(\infty)}$ is the trivial group.

Exercise I.A.1.5. Explain how Lemma I.A.1.4 is equivalent to 'the intersection of all finite index subgroups of G is trivial'.

Exercise I.A.1.6. Let G be a residually finite group. If $H \leqslant G$, then H is residually finite.

Exercise I.A.1.7. Let I be a set and let G_i be a residually finite group for each $i \in I$. Then, $\prod G_i$ is residually finite.

The following style of argument will appear a number of times. The key idea dates back to work of McKinsey on symbolic logic [**McK43**], but we provide an adaptation for groups first noticed independently by Dyson [**H.64**] and Mostowski [**Mos66**]. As well as G. Higman and A. Turing (unpublished).

Theorem I.A.1.8 (McKinsey's Algorithm). *A finitely presented residually finite group G has solvable word problem.*

Proof. Note that finite groups are recursively enumerable. For each positive integer n we can generate all Cayley multiplication tables of size $n \times n$ and check whether both a given table represents a group and if the group is already on the list. Note that the existence of the table gives a solution to the word problem for any given fintie group.

Fix an enumeration of all finite groups $\{Q_i\}$ and presentation $\langle X \mid R\rangle$ of G. The set $\hom(G; Q_i)$ is finite since a homorphism is determined by its images on the generators and Q_i is finite. We need to check whether we can extend a map $f\colon X \to Q_i$ to a homomorphism $\phi\colon G \to Q_i$. This amounts to checking the relations $r \in R$ are satisfied. Fortunately, as R is finite a brute force approach will terminate. Hence, we may list all elements of $\hom(G, Q_i)$ in a finite amount of time. Fix an enumeration of all homomorphism from G to finite groups $\phi_1, \phi_2, \ldots$

Let g be a word in X. To solve the word problem we now run two machines:

The machine runs over all homomorphism ϕ_i and checks if $\phi_i(g) \neq 1$ (here we are using the solvability of the word problem in the codomain, a finite group). Since G is residually finite, if $g \neq_G 1$, this process will stop.

The second machine runs over all representatives w_j of 1 in G and checks whether $g \equiv w_i$. If $g =_G 1$ this process will stop.

Since exactly one of the machines must stop we can algorithmically decide if g is the trivial word. □

Remark I.A.1.9. There exists a finitely generated residually finite group with unsolvable word problem. See **[Mes74]**.

A *characteristic subgroup* $K \leqslant G$ is one that is invariant under all automorphisms of G. Note that if G is finitely generated, then then the intersection of all subgroups of index n is both finite index and characteristic in G.

Proposition I.A.1.10 (Baumslag)**.** **[Bau63]** *If G is a finitely generated residually finite group, then* $\mathrm{Aut}(G)$ *is residually finite.*

Proof. Let $a \in \mathrm{Aut}(G)$ be non-trivial. There exists a $g \in G$ such that $a(g) \neq g$. Let $h = a(g)g^{-1}$. Since $h \neq 1 \in G$, by Lemma I.A.1.4, there exists a finite index normal subgroup $N \trianglelefteq G$ such that $h \notin N$. Let K denote the intersection of all subgroups of index $|G : N|$ in G. Since K is characteristic, the sujection $\alpha\colon G \twoheadrightarrow G/K$ induces a homomorphism $\psi\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/K)$. Moreover, $\psi(a)$ is a non-trivial automorphism of G/K because $\pi(h) \neq 1 \in G/K$. Since a was arbitrary, we have verified that $\mathrm{Aut}(G)$ is residually finite. □

Corollary I.A.1.11. *Let N and Q be a residually finite groups. If N is finitely generated, then any semi-direct product $G = N \rtimes Q$ is residually finite.*

Proof. It is easy to see that elements of the form nq with $n \in N$, $q \in Q$ and $q \neq 1$ are non-trivial in a finite quotient. Indeed, the projection $\pi\colon G \twoheadrightarrow Q$ composed with a finite quotient α of Q where $\alpha(q) \neq 1$ suffices. It remains to find finite quotients for the elements $n \in N$ with $n \neq 1$. Since N is residually finite there is a finite quotient $\beta\colon N \twoheadrightarrow L$ such that $\beta(n) \neq 1_L$. Since N is finitely generated and L is finite there are only finitely many homomorphisms $N \to L$. The intersection of these homomorphisms is a finite index characteristic subgroup C of N and so preserved by Q. The subgroup CQ is normal and has finite index in G. Moreover, n has non-trivial image in the finite group G/CQ. □

There is also a set of dynamical criterion for residual finiteness due to Cairns, Davis, Elton, Kolganova, and Perversi.

Definition I.A.1.12. Suppose that a group G acts continuously on a Hausdorff topological space X. Then we say that the action of G on X is *chaotic* if both of the following conditions hold

(1) (*topological transitivity*) for every pair of non-empty open subsets U and V of X, there is an element $g \in G$ such that $g(U) \cap V \neq \varnothing$;
(2) (*finite orbits dense*) the set of points in X whose orbit under G is finite is a dense subset of X.

THEOREM I.A.1.13. [**CDE+95**, Theorem 1] *For a group G, the following are equivalent:*

(1) G is residually finite;
(2) there is a faithful action of G with finite orbits dense on some Hausdorff topological space X;
(3) there is a faithful action of G with all orbits finite on some Hausdorff topological space X;
(4) there is a faithful chaotic action of G on some Hausdorff topological space X.

PROOF. https://www.e-periodica.ch/cntmng?pid=ens-001:1995:41::68

Todo 2: add proof

□

I.A.2. Mal'cev's Theorem. In [**Mal65**] Mal'cev proved the following remarkable theorem:

THEOREM I.A.2.1 (Mal'cev's Theorem). *A finitely generated linear group is residually finite.*

To prove Mal'cev's Theorem we will need to recall some basic ring theory.

Let R be a ring. Here a ring is associative and contains a multiplicative unit 1. A ring without a multiplicative identity is a *rng*.

A *zero-divisor* is a non-zero element $z \in R$ such that there exists a non-zero element z' satisfying $zz' = 0$. A ring R is a *domain* if it has no zero-divisors.

We say a ring R is *left Noetherian* if R has the *ascending chain condition* on left ideals, that is, for every chain of left ideals $I_1 \subset I_2 \subset \dots$ has a largest element. Said differently, this means there exists an n such that $I_N = I_{N+1}$ for $N > n$.

Let S be a commutative ring and let R be an S-algebra. We say R is *finitely generated* if there exists a finite set of elements $x_1, \dots, x_n \in R$ such that every element of R can be expressed as a polynomial in the x_i with coefficients in S.

The key examples of a finitely generated $\mathbb{Z}$- or $\mathbb{F}_p$-algebra for us is as follows.

Example I.A.2.2. Let k be a field, let $G \leqslant \mathrm{GL}_n(k)$ be a finitely generated subgroup, and let X be a finite set of matrices (closed under inversion) generating G. The subring R of k generated as a k-algebra by the entries of the matrices in X and the multiplicative identity of k is a finitely generated algebra over $\mathbb{Z}$ if $\operatorname{char} k = 0$ and over $\mathbb{F}_p$ if $\operatorname{char} k = p$ a prime. Moreover, $G \leqslant \mathrm{GL}_n(R)$.

Lemma I.A.2.3. *Let S be a commutative Noetherian ring and let R be a finitely generated S-algebra. The following assertions hold:*

(1) R is Noetherian;
(2) if R is a field, then it is finite;
(3) the intersection of the maximal ideals of R is 0.

Proof of Theorem I.A.2.1. Let G be a finitely generated linear group over some field k. Thus, $G \leqslant \mathrm{GL}_n(k)$ for some $n \geqslant 0$. As in Example I.A.2.2 we find that $G \leqslant \mathrm{GL}_n(R)$ where R is a finitely generated $\mathbb{Z}$-algebra.

For an ideal $I \subseteq R$, let $\Gamma(I)$ to be the *Ith principle congruence subgroup* of $\mathrm{GL}_n(R)$ defined by

$$\Gamma(I) := \ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)),$$

where the homomorphism is defined by taking entries of $\mathrm{GL}_n(R)$ modulo I.

Now, since R is Noetherian (Lemma I.A.2.3(i)), if I is a maximal ideal then R/I is a field and by Lemma I.A.2.3(ii) it is finite. Hence, $\mathrm{GL}_n(R/I)$ is finite and so $\Gamma(I)$ is a finite index normal subgroup of $\mathrm{GL}_n(R/I)$. Now, the intersection

$$\bigcap_{I \subset R} \Gamma(I) \quad \text{where } I \text{ ranges over all maximal ideals}$$

is finite by Lemma I.A.2.3(iii). The theorem follows from Lemma I.A.1.4. ☐

The following example of Druţu and Sapir gives an example of a non-linear residually finite one-relator group.

Example I.A.2.4. [**DS05**] (Druţu–Sapir) The group $\langle a, t \mid t^2 a t^{-2} = a^2 \rangle$ is non-linear but is residually finite. Their proof depends in an essential way on [**Weh73**].

I.A.3. An aside: Hopficity.

Definition I.A.3.1. A group G is *Hopfian* if every surjection $G \twoheadrightarrow G$ is injective.

Another result of Mal'cev shows that for finitely generated groups the Hopf property follows from residual finiteness.

Proposition I.A.3.2. *Let G be a finitely generated group. If G is residually finite, then G is Hopfian.*

Proof. Suppose $G \cong G/N$ for some normal subgroup N, our aim is to show that N is trivial. As G is finitely generated, that the number of subgroups of index m in G is finite. Moreover, the number of subgroups of index n in G/N is equal to the number of subgroups of index n in G. The bijection is $H_i/N \leftrightarrow H_i$. But then it is clear that $N \leqslant H_i$ for every finite index subgroup of G. Thus, $N \leqslant G^{(\infty)} = \{1\}$ where the equality follows from the fact G is residually finite and Lemma I.A.1.4. ☐

Example I.A.3.3 (Baumslag–Solitar groups)**.** We define a family of one-relator groups, known as Baumslag–Solitar groups, by

$$\mathrm{BS}(m,n) := \langle a, t \mid ta^m t^{-1} = a^n \rangle.$$

We claim *if* $(|m|,|n|) = 1$*, then* $\mathrm{BS}(m,n)$ *is non-Hopfian.* Let p be a prime dividing m but not n. Define an homomorphism

$$\nu\colon \mathrm{BS}(m,n) \to \mathrm{BS}(m,n) \text{ by } \begin{cases} a \mapsto a^p, \\ t \mapsto t. \end{cases}$$

We first prove that ν is an epimorphism. We have $\nu(t) = t$ and

$$\nu([t, a^{m/p}]) = ta^m t^{-1} a^{-m} = a^n a^{-m} = a^{n-m},$$

now since $\nu(a) = a^p$ and $(p, n-m) = 1$, by Bézout's Identity there exist integers x, y such $a^x p a^{y(n-m)} = a$. It remains to check that $\ker \nu$ is non-empty. Note that the element $[t, a^{m/p}]^p a^{m/p}$ is non-trivial in $\mathrm{BS}(m,n)$, but

$$\nu([t, a^{m/p}]^p a^{m-n}) = (ta^m t^{-1} a^m)^p a^{(m-n)p} = (a^n a^{-m})^p a^{(m-n)p} = 1.$$

Hence, ν is not injective. ◆

In fact a complete study of the residual finiteness and Hopficity of Baumslag–Solitar groups has been completed. The Hopfian property was verfied by Baumslag and Solitar in [**BS62**], however, their claim of which $\mathrm{BS}(m,n)$ are residually finite was incorrect. A correct and complete argument was later given by Meskin [**Mes72**].

THEOREM I.A.3.4 (Baumslag–Solitar, Meskin). *The groups* $\mathrm{BS}(m,n)$ *are*

(1) Hopfian if and only if m *and* n *have the same prime factors;*
(2) residually finite if and only if $|m| = |n|$ *or at least or of* $|m|$ *or* $|n|$ *equals* 1.

PROOF. To be added, for now see [**Mes72**, Section 2].

Todo 3: Add proof

□

Definition I.A.3.5. A group G is *co-Hopfian* if every injection $G \rightarrowtail G$ is surjective. We say G is *finitely co-Hopfian* is every injection $G \rightarrowtail G$ whose image has finite index is surjective.

Exercise I.A.3.6. Let G be a finitely generated group. Suppose $X(G)$ is a group invariant taking values in $\mathbb{R}$ which for every finite index subgroup $H \leqslant G$ satisfies $X(H) = |G : H| X(G)$. If $X(G) \neq 0$, then G is finitely co-Hopfian.

Examples of invariants (when they are defined) satisfying the hypothesis of the exercise include: Euler characteristic $\chi(G)$, ℓ^2-Betti numbers $b_i^{(2)}(G)$, and ℓ^2-torsion $\rho^{(2)}(G)$.

I.B. Subgroup separability

I.B.1. The definition. We will now extend the notion of residual finiteness to subgroups.

Definition I.B.1.1 (Separable subgroup)**.** Let G be a group and let $H \leqslant G$. We say H is *separable* in G if for each $g \in G \backslash H$ there exists a finite index subgroup $K_g \leqslant G$ with $H \leqslant K_g$ and $g \notin K_g$.

It is easy to see a subgroup H is separable in G if and only if H is an intersection of finite index subgroups of G. In particular, G is residually finite if and only if the trivial subgroup is separable in G.

Lemma I.B.1.2. *Let G be a finitely generated group and let $H \leqslant G$. Then, H is separable in G if and only if for every $g \in G \backslash H$ there exists a finite quotient $\alpha\colon G \twoheadrightarrow Q$ with $\alpha(g) \notin \alpha(H)$.*

Proof. Let $g \in G \backslash H$. We have a homomorphism $\alpha\colon G \twoheadrightarrow Q$ with Q finite and $\alpha(g) \notin \alpha(H)$. Define $K := \alpha^{-1}(\alpha(H))$ and note that K contains H, but $g \notin K$. Thus, it suffices to show that K is finite index in G, but this is clear since it contains $\ker \alpha$ with has index $|Q|$ in G.

Conversely, suppose H is separable in G and let $g \in G \backslash H$. By hypothesis there exists a finite index subgroup $K_g \leqslant G$ containing H but not g. Let N denote the *core* of K_g, that is the intersection of all of the conjugates of a K_g in G. Since G is finitely generated, N is a finite index normal subgroup of G. Let α denote the quotient $G \twoheadrightarrow G/N$. The claim that $\alpha(g) \notin \alpha(H)$ will follows from $\alpha(H) \leqslant \alpha(K_g)$ once we show $\alpha(g) \notin \alpha(K_g)$. The latter statement follows from the finite index of N, the fourth (or lattice) isomorphism theorem, and the fact that gK_g and K_g are distinct cosets of K_g. □

Exercise I.B.1.3. Let H be a separable subgroup in a finitely generated group K. If K is a finite index subgroup of a group G, then H is separable in G.

Let $H \leqslant G$. We say H has *solvable membership problem* in G if there exists an algorithm which takes as input an element $g \in G$ and decides if $g \in H$.

Theorem I.B.1.4. *Let H be a finitely generated subgroup of a finitely presented group G. If H is separable in G, then H has solvable membership problem in G.*

Proof. Let $g \in G$ and fix a generating set $S = \{h_1, \dots, h_k\}$ of H. The algorithm consists of two processes run in parallel of which one will stop. The first process enumerates finite quotients of G and checks to see if the image of H contains the image of g. If the image of g is not contained in the image of H, then the algorithm stops since $g \notin H$. The second process works in a finitely generated free group F such that $F \twoheadrightarrow G$ is a finite presentation with relations R. We enumerate the words w_ℓ in the free monoid on $S \cup S^{-1}$ and the products p_k in F of conjugates of relators $r \in R$. We check to see if each $g^{-1}w_\ell$ is freely equal to p_k, it is then the algorithm stops since $g \in H$. □

The following result, due to Mihaĭlova [**Mih58, Mih66**], shows that $F_2 \times F_2$ does not in fact have solvable subgroup membership problem.

THEOREM I.B.1.5 (Mihaĭlova). *The group $F_2 \times F_2$ has unsolvable subgroup membership problem.*

PROOF. Note that F_2^2 contains F_n^2 for every $n \geqslant 1$ so we may work in F_n^2 for some n. Let Q be a finitely presented group with unsolvable word problem and let S generate Q. Let $|S| = n$ and let $\pi\colon F_n^2 \twoheadrightarrow Q^2$ be the natural projection. Let $D = \{(g,g)\} < Q^2$ be the diagonal subgroup and define $\Delta = \pi^{-1}(D) < F_n^2$. Now, $(x,y) \in \Delta$ if and only if x and y project to the same element in Q. But Q has unsolvable word problem so we cannot decide if $(x,y) \in \Delta$.

It remains to check that Δ is finitely generated. Let R be a finite set of relations for Q. We claim that

$$\mathcal{S} := \{(g,g)\colon g \in S\} \cup \{(r,1)\colon r \in R\} \cup \{(1,r)\colon r \in R\}$$

is a generating set. The issue is that R only normally generates the kernel of the map $F_n \twoheadrightarrow Q$, however, if $w \in F_n$ and $r \in R$ we have

$$(wrw^{-1}, 1) = (w,w)(r,1)(w^{-1},w^{-1})$$

of which each term is in $\mathcal{S}$. □

Corollary I.B.1.6. *$F_2 \times F_2$ has inseparable subgroups.*

I.B.2. LERF groups. We say a group G is *extended residually finite* (ERF) if every subgroup of G is separable. We say G is *locally extended residually finite* (LERF) if every finitely generated subgroup of G is separable.

Two groups G and H are *commensurable* if there exists finite index subgroups $K \leqslant G$ and $L \leqslant H$ such that $K \cong L$.

Exercise I.B.2.1. Let G be a finitely generated LERF group. If H is a group commensurable with G, then H is LERF.

Examples I.B.2.2. We record a few examples:

(1) finite groups are trivially LERF;
(2) $\mathbb{Z}$ is also trivially seen to be LERF;
(3) $\mathbb{Z}^n$ is LERF.
Proof. Let $L \leqslant \mathbb{Z}^n$ and let $g \in \mathbb{Z}^n \backslash L$. We may assume L has infinite index since all finite index subgroups are separable by definition. Since $L \lhd \mathbb{Z}^n$ we may consider the quotient $\pi_L\colon \mathbb{Z}^n \twoheadrightarrow A = \mathbb{Z}^n/L$. Note that $\pi_L(g) \neq 0$. Now, A is a finitely generated abelian group and hence residually finite. Thus, there exists a finite (abelian) quotient $\alpha\colon A \twoheadrightarrow Q$ where $\alpha(\pi_L(g)) \neq 0 \in Q$. Moreover, $\ker(\alpha \circ \pi_L)$ contains L. Hence, we have separated L from g in Q. ◆
(4) It follows from Corollary I.B.1.6 that $F_2 \times F_2$ is not LERF.

A particularly famous result due to Marshall Hall Jr. is that free groups are LERF [**Hal49**]. We divert the proof to the next section.

THEOREM I.B.2.3 (Marshall Hall Jr.'s Theorem). *Free groups are LERF*

It follows from Theorem I.B.2.3 and Corollary I.B.1.6 that direct products of LERF groups are not necessarily LERF.

I.B.3. The connection with covering theory. Separability of subgroups has a particularly powerful interpretation when viewed from a topological lens. The following somewhat miraculous result of Scott allows one to promote immersions to embeddings, at the cost of taking a finite cover, when in the presence of subgroup separability.

Let X and Y be CW complexes. Recall that an *immersion* is map $f\colon X \looparrowright Y$ such that f is a locally injective combinatorial map. The map f map is *combinatorial* if it maps the interior of each cell homeomorphically onto its image.

THEOREM I.B.3.1. [**Sco78**] *Let G be a finitely generated group and let X be a CW complex with $\pi_1 X = G$. Let $H \leqslant G$ and let $Y \to X$ be the cover corresponding to H. Then, H is separable if and only if for every finite subcomplex $K \subseteq Y$, there exists an intermediate finite sheeted cover $Y \to Z \to X$ such that K embeds as a subcomplex of Z.*

PROOF. Suppose the geometric condition holds for some subgroup H with cover $Y \to X$ and let $g \in G \backslash H$. Pick a basepoint $x \in \widetilde{X}$ in the universal cover of X and let $K = \pi(x \cup gx) \subseteq Y$, where $\pi\colon \widetilde{X} \to Y$ is the universal covering map. By hypothesis we obtain a finite covering $Z \to X$ with corresponding finite index subgroup G' such that $K \subseteq Z$. But, then $g \notin G'$. Hence, H is separable.

Now, suppose H is a separable subgroup of G with corresponding cover $Y \to X$ and let K be a finite subcomplex of Y. Let $\pi\colon \widetilde{X} \to Y$ be the universal cover and let $C = \pi^{-1}K$. Pick a finite subcomplex D of C such that $\pi(D) = K$ and note that $S := \{g \in G \colon gD \cap D \neq \varnothing\}$ is finite. Since H is separable we can find a finite index subgroup G' of G such that $H \leqslant G$ and $G' \cap S \subseteq H$. The cover $\widetilde{X}/G' \to X$ is then the required intermediate finite cover. □

We will also need the following easy lemma.

Lemma I.B.3.2. *Let Δ and Γ be finite graphs and let $f\colon \Delta \looparrowright \Gamma$ be an immersion. Then extends f to a finite-sheeted covering $\hat{\Gamma} \to \Gamma$ such that Δ embeds in $\hat{\Gamma}$.*

PROOF. Fix an orientation and a colouring on the edges of Γ. This lifts to an orientation and a colouring on Δ. A combinatorial map is an immersion if and only if at each vertex, we see each colour arriving exactly once and leaving exactly once. Let n denote the number of vertices of Δ and for each colour c of Γ let n_c be the number of edges of Δ coloured c. There are $n - n_c$ vertices of Δ missing incoming edges with colour c and the same number of vertices missing outgoing edges with colour c. Pick any bijection between these sets and glue in $n - n_c$ edges coloured c. Repeating this for each colour we eventually obtain a covering space $\hat{\Gamma}$ of Γ such that $\Delta \subset \hat{\Gamma}$. □

We now give Stalling's proof of the Marshall Hall Theorem.

PROOF OF THEOREM I.B.2.3. Now, let Γ be a finite rose and H a finitely generated subgroup of $F = \pi_1\Gamma$. Let $\Delta \to \Gamma$ be the covering corresponding to H, and consider a finite subcomplex $K \subset \Delta$. Since H is finitely generated, we may enlarge K to ensure that K is connected and that $\pi_1 K = H$. But $\Delta \to \Gamma$ is an immersion, so can be completed to a finite-sheeted covering $\hat{\Gamma} \to \Gamma$ by Lemma I.B.3.2. The result now follows from Theorem I.B.3.1. □

I.C. Pro topologies on groups

I.C.1. The definition.

Definition I.C.1.1 (Profinite topology)**.** Let $\mathcal{N}$ be a non-empty collection of finite index normal subgroups of a group G. We say $\mathcal{N}$ is *filtered from below* if whenever $N_1, N_2 \in \mathcal{N}$, there exists $N \in \mathcal{N}$ such that $N \leqslant N_1 \cap N_2$. We make G into a topological group by taking $\mathcal{N}$ as a basis of open neighbourhoods of the identity the collection. We refer to the corresponding topology as a *pro topology* on G.

Let $H_1, \ldots, H_n$ be groups. A *subdirect product* of $H = \prod_{i=1}^{n} H_i$ is any subgroup G of H such that the projections $\pi_i \colon G \to H_i$ are surjective.

Definition I.C.1.2 (Formation)**.** A class of finite groups $\mathcal{C}$ is called a *formation* if $\mathcal{C}$ is closed under taking quotients and subdirect products.

Examples I.C.1.3. The following are examples of formations:

(1) the class of all finite groups;
(2) the class of all finite abelian groups;
(3) the class of all finite nilpotent groups;
(4) the class of all finite soluble groups;
(5) the class of all finite p-groups for a fixed prime p.

Definition I.C.1.4 (Pro-$\mathcal{C}$ topology)**.** Let $\mathcal{C}$ be a formation of finite groups. The *(full) pro-$\mathcal{C}$ topology* on G is the topology $\tau_{\mathcal{C}}(G)$ on G given by taking as a basis of open neighbourhoods of the identity the collection

$$\mathcal{N}_{\mathcal{C}}(G) := \{N \trianglelefteq G \mid G/N \in \mathcal{C}\}.$$

When $\mathcal{C}$ is the class of all finite groups, we denote the topology by τ_G and refer to it as the *profinite topology* on G.

Lemma I.C.1.5. *Let $\mathcal{C}$ be a formation of finite groups. A group G is residually $\mathcal{C}$ if and only if its pro-$\mathcal{C}$ topology is Hausdorff.*

Proof. Suppose G is residually-$\mathcal{C}$. Let $g \neq h$ be elements of G. Since, gh^{-1} is non-trivial, there exists a finite index subgroup $N < G$ such that $gh^{-1} \notin N$. Hence, $gN \cap hN = \varnothing$. This proves the topology is Hausdorff. Reversing the argument yields the converse. □

Definition I.C.1.6 (Separable subset)**.** Let S be subset of G. We say S is *$\mathcal{C}$-separable* if S is closed in pro-$\mathcal{C}$ topology on G.

Lemma I.C.1.7. *Let $\mathcal{C}$ be a formation of finite groups and let G be a group. Suppose $H \leqslant G$. Then, H is closed in the pro-$\mathcal{C}$ topology on G if and only if H is the intersection of open subgroups of G.*

Proof. An open subgroup in the topology $\tau_{\mathcal{C}}(G)$ has finite index so it is also a closed subgroup. Hence, any intersection of open subgroups is closed.

For the converse suppose H is closed in $\tau_{\mathcal{C}}(G)$ and let $g \in G \backslash H$. There exists some $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $xN \cap H = \varnothing$. Thus, $x \notin HN$. It follows that $H = \bigcap N \in \mathcal{N}_{\mathcal{C}}(G) HN$. But each HN is open, whence the lemma. □

Exercise I.C.1.8. Let $\mathcal{C}$ be a formation of finite groups and let G be a group. Suppose H is a finite index subgroup of G. Then, H is open in the pro-$\mathcal{C}$ topology on G if and only if $G/\operatorname{Core}_G(H)$ is in $\mathcal{C}$.

I.C.2. Virtual retracts.

Definition I.C.2.1 (Virtual retract)**.** Recall that $H \leqslant G$ is a *retract* if the inclusion $i\colon H \rightarrowtail G$ has a left inverse $r\colon G \twoheadrightarrow H$, that is $r \circ i = \mathrm{id}_H$. Similarly, we call H a *virtual retract*, written $H \leqslant_{\mathrm{vr}} G$, if H is a retract of a finite index subgroup of G.

The following useful lemma is due to Hsu and Wise [**HW99**, Lemma 3.9].

Lemma I.C.2.2. *Let G be a residually finite group and let $H \leqslant G$. If $r\colon G \twoheadrightarrow H$ is retract, then*

1. *H is closed in the profinite topology on G;*
2. *if K is closed in profinite topology on H, then K is closed in the profinite topology on G;*
3. *the inclusion map $H \to G$ induces a homemorphism onto its image of profinite topologies.*

Proof. We first prove (1). Let $N = \ker r$ and note that $G = NH$ and $N \cap H = 1$ so we may express any element of G uniquely as a product nh with $n \in N$ and $h \in H$. Since G is residually finite we may pick a chain (G_i) of finite index normal subgroups of G such that $\bigcap_i G_i = 1$. Let $N_i = G_i \cap N$ and note that $|G : N_iH| = |NH : N_iH| = |N : N_i| \leqslant |G : G_i|$. Hence, (N_iH) is a sequence of finite index subgroups of H whose intersection is exactly H.

We now prove (2). Let K be a closed subgroup of H. Since $r\colon G \to H$ is continuous with respect to the profinite topologies, the preimage $r^{-1}(K)$ is closed in G. Now, $K = H \cap r^{-1}(K)$ and so is the intersection of closed subgroups and hence closed in G.

The claim (3) follows from (2). □

Lemma I.C.2.3. *Let G be a finitely generated residually finite group and let $H \leqslant G$. If $H \leqslant_{\mathrm{vr}} G$, then H is closed in the profinite topology on G.*

Proof. Let $K \leqslant G$ be a finite index subgroup admitting a retract $r\colon K \twoheadrightarrow H$. By Lemma I.C.2.2 we see that H is closed in the profinite topology on K. But K is finite index in G, so every closed subset of K is closed in G. Hence, H ic closed in the profinite topology on G. □

We state some elementary properties of virtual retracts first collected by Minasyan [**Min21**, Lemma 3.2].

Exercise I.C.2.4. Suppose that G and G' are groups.

1. Let $H \leqslant_{\mathrm{vr}} G$ and $A \leqslant G$ such that $H \leqslant A$. Then, $H \leqslant_{\mathrm{vr}} A$.
2. Suppose $H \leqslant G$ and that there exists a homomorphism $\phi\colon G \to G'$ such that $\phi|_H$ is injective. If $\phi(H) \leqslant_{\mathrm{vr}} G'$, then $H \leqslant_{\mathrm{vr}} G$.
3. If $H \leqslant_{\mathrm{vr}} G$ and $\alpha \in \operatorname{Aut}(G)$, then $\alpha(H) \leqslant_{\mathrm{vr}} G$.
4. If $H \leqslant_{\mathrm{vr}} G$ and $A \leqslant_{\mathrm{vr}} H$, then $A \leqslant_{\mathrm{vr}} G$.
5. If $H \leqslant_{\mathrm{vr}} G$ and $H' \leqslant_{\mathrm{vr}} G'$, then $H \times H' \leqslant_{\mathrm{vr}} G \times G'$.
6. If G is finitely generated and $H \leqslant_{\mathrm{vr}} G$, then H is undistorted in G.

Let G be finitely generated by a set S and $H \leqslant G$ be finitely generated by a set T. We say H is *undistorted* if there exists a $C > 0$ such that for all $h \in H$ we have $|h|_T \leqslant C \cdot |h|_S$. Where $|\cdot|_S$ and $|\cdot|_T$ are the word metrics in G and H with respect to T and S respectively.

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